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THE CLASSIFICATION OF SURFACES IN $\mathbb{P}^5$ HAVING FEW TRISECANTS

1. Introduction

The aim of this paper is to give a complete classification of smooth, non degenerate complex surfaces $Y \subset \mathbb{P}^5$, which are not scrolls, such that the embedded trisecant variety $\text{Trisec}(Y)$, (i.e. the union of all lines contained in Y with the lines in $\mathbb{P}^5$ intersecting Y in a 0-dimensional subscheme of length at least three (cf. e.g. [3])) has dimension at most two. Moreover we will describe exactly their trisecant locus.

For a smooth surface $Y \subset \mathbb{P}^5$ the expected dimension of $\text{Trisec}(Y)$ is three and there is up to now only one known example of a smooth Enriques surface Y in $\mathbb{P}^5$ which has a four-dimensional trisecant variety (cf. [7], [8]). This seems to be quite an exceptional case and it would be interesting to find all the surfaces in $\mathbb{P}^5$ having a four-dimensional trisecant variety.

We will prove the following result:

THEOREM 1 (MAIN THEOREM). *Let $Y \subset \mathbb{P}^5$ be a smooth, non-degenerate, connected complex surface, which is a not a scroll. Moreover, let H be a hyperplane section of Y, K a canonical divisor, n the degree of Y and π the sectional genus of Y. Then the dimension of the embedded trisecant variety* $\text{Trisec}(Y)$ *of Y is at most two if and only if Y is one of the following surfaces:*

1. $Y = \mathbb{P}^2$, $|H| = |2L|$;
2. $Y = \hat{\mathbb{P}}^2(x_1, \dots, x_4)$, *where $x_1, \dots, x_4$ are four points in $\mathbb{P}^2$ such that $|H| = |3L - \sum x_i|$ is very ample;*
3. $Y = \hat{\mathbb{P}}^2(x_0, \dots, x_6)$, *where $x_0, \dots, x_6$ are seven (possibly infinitely near) points in $\mathbb{P}^2$ such that $|H| = |4L - 2x_0 - \sum x_i|$ is very ample;*
4. $Y = \hat{\mathbb{P}}^2(x_1, \dots, x_9)$, *where $x_1, \dots, x_9$ are nine points in $\mathbb{P}^2$ such that $|H| = |4L - \sum x_i|$ is very ample and moreover $x_1, \dots, x_9$ do not lie on a pencil of cubics;*
5. $Y = \hat{\mathbb{P}}^2(x_1, \dots, x_6, y_1, \dots, y_4)$, *where $x_1, \dots, x_6, y_1, \dots, y_4$ are ten (possibly infinitely near) points in $\mathbb{P}^2$ such that $|H| = |6L - \sum 2x_i - \sum y_j|$ is very ample;*
6. $Y \subset \mathbb{P}^5$ *is a complete intersection of three independent quadrics Q_1, Q_2, Q_3;*

7. *$Y = \hat{\mathbb{P}}^2(x_1, \ldots, x_{10})$, where $x_1, \ldots, x_{10}$ are ten points in $\mathbb{P}^2$ such that $|H| = |7L - \sum 2x_i|$ is very ample;*

8. *$Y \subset \mathbb{P}^5$ is a minimal Enriques surface of degree 10 with $\pi = 6$, such that $\phi_{|H+K|}(Y)$ is not contained in a smooth quadric $\subset \mathbb{P}^5$.*

In a previous paper (cf. [2], [3]) we have shown the following result:

THEOREM 2 (PROJECTION THEOREM). *Let Y be a connected, smooth, non-degenerate algebraic surface in $\mathbb{P}^5$ with* $\mathrm{Trisec}(Y) \cap Y \neq Y$ *(i.e., Y admits an inner projection). Then the degree of Y is at most 10.*

Moreover, in the above paper we also give a complete list of all the families in $\mathbb{P}^5$ whose general member has $\mathrm{Trisec}(Y) \cap Y \neq Y$.

It is known ([2], Prop.(2.6)) that in the situation of the projection theorem $\dim \mathrm{Trisec}(Y) \leq 2$: therefore the fine classification of the surfaces with $\dim \mathrm{Trisec}(Y) \leq 2$ is a priori equivalent to the classification of the individual surfaces with $\mathrm{Trisec}(Y) \cap Y \neq Y$, and to the determination of the geometrically ruled surfaces which are an irreducible component of their trisecant variety. Here we will restrict ourselves to the first case; the case, when $Y \subset \mathbb{P}^5$ is a scroll will be treated separately in the appendix. Our paper is organized as follows:

First of all we show that for a surface Y (not being a scroll) with $\dim \mathrm{Trisec}(Y) \leq 2$ the union of all the trisecants consists of all the (finitely many) lines contained in the surface and a finite number of planes $\subset \mathbb{P}^5$, intersecting Y in a plane curve of degree ≥ 3.

In the second paragraph we prove the main result. This is done by the following consideration: if we take a family of smooth surfaces in $\mathbb{P}^5$, of which we know that a generic member admits an inner projection or equivalently has $\dim \mathrm{Trisec}\, Y \leq 2$, then it can happen that a special surface in this family has a trisecant variety of too high dimension and therefore is not projectable. We are going to classify completely these exceptions, treating each single family in our list separately. The surfaces admitting an inner projection are $K3$-surfaces, Enriques surfaces or rational surfaces. The $K3$-surfaces can be understood via a direct geometric argument, the phenomena occuring for the Enriques-surface are discussed in [7] resp. [8], therefore we only formulate the result.

For the remaining rational surfaces we have to solve the following problem: Let $Y = \hat{\mathbb{P}}^2(x_1, \ldots, x_r)$ be embedded as a smooth surface in $\mathbb{P}^5$ by $|H| = |aL - \sum b_i x_i|$. Under which conditions for the points $x_1, \ldots, x_r$ there exists a point $x_{r+1} \in Y$, s.th. $H' = aL - \sum b_i x_i - x_{r+1}$ is very ample on $Y' = \hat{\mathbb{P}}^2(x_1, \ldots, x_r, x_{r+1})$?

For this it is very important to have numerical criteria for "very ampleness" as for example Reider's criterion (cf. [17]). In fact, in two of our cases this can be applied and has been carried out by Weinfurtner in his thesis (cf. [19]).

For the remaining two families of rational surfaces of degree 8 resp. 9 it is not longer possible to apply Reider's method. Nevertheless it can be proved that also in these cases one can always find a point $x_{r+1} \in Y$ such that H' is very ample. This is done by restricting H' to appropriate curves $C \subset Y'$ and verifying, that H' is very ample on C. Here we use a numerical criterion for divisors on (not necessarily irreducible) curves to be very ample. First such a criterion was given in the author's thesis (cf. [2], [4]) which was improved substantially in [6]).

We will not give a proof of these results here and refer to the above cited thesis or to [6], where the original proof was substantially simplified.

In the last paragraph we describe the trisecant locus of the projectable surfaces (i.e. Trisec(Y), resp. Trisec $(Y) \cap Y$). In paragraph one we have shown that in general for a projectable surface Y the union of all trisecants consists of all the lines contained in the surface and a finite number of planes $\subset \mathbb{P}^5$, intersecting Y in a plane curve of degree ≥ 3. Moreover, we can calculate this number from the invariants of the surface. Then we determine for each single surface of our list the lines contained in it as well as the plane curves of degree ≥ 3.

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2. Trisecant locus of surfaces $Y \subset \mathbb{P}^5$ (I)

For the rest of the paper we will make the following

MAIN ASSUMPTION (*): Let $Y \subset \mathbb{P}^5 = \mathbb{P}^5(\mathbb{C})$ be a smooth, non degenerate, connected complex surface, which is not a scroll.

Moreover, we assume that the dimension of Trisec(Y) is at most two.

We first want to prove that if we assume that the dimension of Trisec(Y) is at most two, then the trisecant variety is of a very simple form, namely the union of a finite number of lines contained in Y and a finite number of planes.

PROPOSITION 1. *Let Y be as above. Then we have:* $\text{Trisec}(Y) = \bigcup_{i=1}^{r} L_i \cup \bigcup_{j=1}^{s} E_j$, *where $L_1, \ldots, L_r$ are the lines contained in Y and $E_1, \ldots, E_s \subset \mathbb{P}^5$ are planes.*

Proof. Let $\text{Hilb}^3\mathbb{P}^5$ (resp. $\text{Hilb}^3 Y$) be the Hilbert scheme of 0-dimensional subschemes of $\mathbb{P}^5$ (resp. of Y) of length 3 ("triples") and let $\text{Hilb}^3_c\mathbb{P}^5$ be the open set in $\text{Hilb}^3\mathbb{P}^5$ consisting of the triples, which are (locally around each point of their support) contained in a smooth curve in $\mathbb{P}^5$. Then $\text{Hilb}^3_c Y$ is defined to be the pull back of $\text{Hilb}^3 Y$ to

$\mathrm{Hilb}_c^3\mathbb{P}^5$. Moreover, $\mathrm{Al}^3\mathbb{P}^5$ is the subscheme of $\mathrm{Hilb}_c^3\mathbb{P}^5$ consisting of triples lying on a line. Then we consider the cartesian diagram:

$$\begin{array}{ccc} \mathrm{Al}^3Y := \mathrm{Al}^3\mathbb{P}^5 \times_{\mathrm{Hilb}_c^3\mathbb{P}^5} \mathrm{Hilb}_c^3Y & \hookrightarrow & \mathrm{Hilb}_c^3Y \\ \downarrow & & \downarrow \\ \mathrm{Al}^3\mathbb{P}^5 & \hookrightarrow & \mathrm{Hilb}_c^3\mathbb{P}^5. \end{array}$$

Hilb_c^3Y, $\mathrm{Hilb}_c^3\mathbb{P}^5$, $\mathrm{Al}^3\mathbb{P}^5$ are smooth of respective dimension 6,15,11 and thus it follows that either Al^3Y is empty or every irreducible component of Al^3Y has dimension bigger or equal to $11-(15-6)=2$.

We consider now the commutative diagram

$$\begin{array}{ccc} \mathrm{Al}^3Y & \overset{i}{\hookrightarrow} & \mathrm{Al}^3\mathbb{P}^5 \\ & \searrow & \downarrow a \\ Z := a(\mathrm{Al}^3Y) & \subset & \mathbb{G}(1,\mathbb{P}^5). \end{array}$$

Let L belong to Z, then we have the following two cases:

1. L is a line contained in Y and then it holds obviously: $(a|_{\mathrm{Al}^3Y})^{-1}(L) = \mathrm{Hilb}^3\mathbb{P}^1 \simeq \mathbb{P}^3$, or
2. $L \not\subset Y$ and then $\left(a|_{\mathrm{Al}^3Y}\right)^{-1}(L)$ consists of finitely many points (since $\#(L \cap Y) < \infty$).

Since for all $\xi \in (a|_{\mathrm{Al}^3Y})^{-1}(L)$: $\dim_L Z \geq \dim_\xi \mathrm{Al}^3Y - \dim_\xi (a|_{\mathrm{Al}^3Y})^{-1}(L)$, it follows in case 2 that $\dim_L Z = \dim_\xi \mathrm{Al}^3Y \geq 2$.

Thus it holds: $Z = \bigcup_{i=1}^r \{L_i\} \cup \bigcup_{j=1}^s Z_j$, where $L_1, \ldots, L_r$ are the lines contained in Y ($\{L_i\}$ being the corresponding points in $\mathbb{G}(1,\mathbb{P}^5)$) and $Z_1, \ldots, Z_s$ are irreducible components of Z of dimension ≥ 2.

We now consider the incidence correspondence $\mathcal{F} := \{(x, L) \in \mathbb{P}^5 \times \mathbb{G}(1,\mathbb{P}^5) : x \in L\}$ with the two projections $p : \mathcal{F} \longrightarrow \mathbb{G}(1,\mathbb{P}^5)$ and $q : \mathcal{F} \longrightarrow \mathbb{P}^5$. Then it holds: $\mathrm{Trisec}\,(Y) = q(p^{-1}(Z)) = \bigcup_{i=1}^r q(p^{-1}(\{L_i\})) \cup \bigcup_{j=1}^s q(p^{-1}(Z_j)) = \bigcup_{i=1}^r L_i \cup \bigcup_{j=1}^s q(p^{-1}(Z_j))$.

Since $p : \mathcal{F} \longrightarrow Z$ is a $\mathbb{P}^1$-bundle, $p^{-1}(Z_j)$ is irreducible and has dimension equal to $\dim Z_j + 1$. Because $p^{-1}(Z_j)$ is irreducible we see from the following diagram:

$$\begin{array}{ccccc} (q|(p^{-1}(Z_j))^{-1}(x) & \subset & p^{-1}(Z_j) & \subset & \mathcal{F} \\ \downarrow & & \downarrow & & \downarrow q \\ x & \in & q(p^{-1}(Z_j)) & \subset & \mathbb{P}^5, \end{array}$$

that it holds:

$$\begin{aligned}\dim(q(p^{-1}(Z_j))) &= \dim(p^{-1}(Z_j)) \\ &\quad - \min\{\dim(q|(p^{-1}(Z_j))^{-1}(x) : x \in q(p^{-1}(Z_j))\} \\ &= \dim Z_j + 1 \\ &\quad - \min\{\dim(q|(p^{-1}(Z_j))^{-1}(x) : x \in q(p^{-1}(Z_j))\} \\ &\geq 3 - \min\{\dim(q|(p^{-1}(Z_j))^{-1}(x)\}.\end{aligned}$$

Since by assumption $\dim \mathrm{Trisec}(Y) \leq 2$, we have: $\dim q(p^{-1}(Z_j)) \leq 2$ and therefore $\min\{\dim(q \,|\, (p^{-1}(Z_j))^{-1}(x) : x \in q\,(p^{-1}(Z_j))\} \geq 1$. Since $(q\,|(p^{-1}(Z_j))^{-1}(x) = \{(x, L) : L \in Z_j \text{ and } x \in L\}$, this means that for all $x \in q\,(p^{-1}(Z_j))$ there exists (at least) a 1-parameter family of lines $L_t \subset q\,(p^{-1}(Z_j))$ with $x \in L_t$. Let $x \in q\,(p^{-1}(Z_j))$ be a smooth point and let $(L_t)_{t \in T}$ be a 1-parameter family of lines in $q\,(p^{-1}(Z_j))$ with $x \in L_t$. Obviously, each of these lines is contained in the (projective) tangent space $\mathcal{T}(\simeq \mathbb{P}^2)$ of $q\,(p^{-1}(Z_j))$ in the point x and therefore the closure of $\cup_t L_t$ is equal to the tangent plane $\mathcal{T}$.

Hence $\mathcal{T}$ is contained in $q(p^{-1}(Z_j))$ and by the irreducibility of $q(p^{-1}(Z_j))$ we obtain: $q(p^{-1}(Z_j)) = \mathcal{T}$.

□

COROLLARY 1. *Let Y be a smooth, non degenerate surface in $\mathbb{P}^5$, which is not a scroll. Then* $\dim \mathrm{Trisec}(Y) \leq 2$ *if and only if* $\mathrm{Trisec}(Y) \cap Y \neq Y$.

Proof. One direction is obvious, since Y is not a scroll, the other follows from [2], Prop. (2.6).

□

3. Proof of the main theorem

The aim of this paragraph is to prove the main result of this paper:

THEOREM 3. *Let $Y \subset \mathbb{P}^5$ be as in the main assumption (*). Moreover, let H be a hyperplane section of Y, K a canonical divisor, n the degree of Y and π the sectional genus of Y. Then the dimension of the embedded trisecant variety* $\mathrm{Trisec}(Y)$ *of Y is at most two if and only if Y is one of the following surfaces:*

1. *$Y = \mathbb{P}^2$, $|H| = |2L|$;*

2. *$Y = \hat{\mathbb{P}}^2(x_1, \ldots, x_4)$, where $x_1, \ldots, x_4$ are four points in $\mathbb{P}^2$ such that $|H| = |3L - \sum x_i|$ is very ample;*

3. *$Y = \hat{\mathbb{P}}^2(x_0, \ldots, x_6)$, where $x_0, \ldots, x_6$ are seven (possibly infinitely near) points in $\mathbb{P}^2$ such that $|H| = |4L - 2x_0 - \sum x_i|$ is very ample;*

4. $Y = \hat{\mathbb{P}}^2(x_1, \ldots, x_9)$, *where* $x_1, \ldots, x_9$ *are nine points in* $\mathbb{P}^2$ *such that* $|H| = |4L - \sum x_i|$ *is very ample and moreover* $x_1, \ldots, x_9$ *do not lie on a pencil of cubics;*

5. $Y = \hat{\mathbb{P}}^2(x_1, \ldots, x_6, y_1, \ldots, y_4)$, *where* $x_1, \ldots, x_6, y_1, \ldots, y_4$ *are ten (possibly infinitely near) points in* $\mathbb{P}^2$ *such that* $|H| = |6L - \sum 2x_i - \sum y_j|$ *is very ample;*

6. $Y \subset \mathbb{P}^5$ *is a complete intersection of three independent quadrics* Q_1, Q_2, Q_3*;*

7. $Y = \hat{\mathbb{P}}^2(x_1, \ldots, x_{10})$, *where* $x_1, \ldots, x_{10}$ *are ten points in* $\mathbb{P}^2$ *such that* $|H| = |7L - \sum 2x_i|$ *is very ample;*

8. $Y \subset \mathbb{P}^5$ *is a minimal Enriques surface of degree 10 with* $\pi = 6$, *such that* $\phi_{|H+K|}(Y)$ *is not contained in a smooth quadric* $\subset \mathbb{P}^5$.

First of all we recall the following result:

THEOREM 4. *Let* $Y \subset \mathbb{P}^5$ *be a smooth, non-degenerate, connected surface,* H *a hyperplane section of* Y, K *a canonical divisor,* n *the degree of* Y *and* π *the sectional genus of* Y. *If* Y *admits an inner projection, i.e.* $\text{Trisec}(Y) \cap Y \neq Y$, *then it is a member of one of the following families:*

	n	π	$\chi(Y)$	K^2	$H.K$	kod	structure of Y
(I)	4	0	1	9	-6	-1	$\mathbb{P}^2$, $\lvert H\rvert = \lvert 2L\rvert$
(II)	5	1	1	5	-5	-1	$Y = \hat{\mathbb{P}}^2(x_1, \ldots, x_4)$, $\lvert H\rvert = \lvert 3L - \sum x_i\rvert$
(III)	6	2	1	2	-4	-1	$\hat{\mathbb{P}}^2(x_0, \ldots, x_6)$, $\lvert H\rvert = \lvert 4L - 2x_0 - \sum x_i\rvert$
(IV)	7	3	1	0	-3	-1	$\hat{\mathbb{P}}^2(x_1, \ldots, x_9)$, $\lvert H\rvert = \lvert 4L - \sum x_i\rvert$
(V)	8	4	1	-1	-2	-1	$\hat{\mathbb{P}}^2(x_1, \ldots, x_6, y_1, \ldots, y_4)$, $\lvert H\rvert = \lvert 6L - \sum 2x_i - \sum y_j\rvert$
(VI)	8	5	2	0	0	0	$Y \subset \mathbb{P}^5$, minimal $K3$-surface
(VII)	9	5	1	-1	-1	-1	$\hat{\mathbb{P}}^2(x_1, \ldots, x_{10})$, $\lvert H\rvert = \lvert 7L - \sum 2x_i\rvert$
(VIII)	10	6	1	0	0	0	$Y \subset \mathbb{P}^5$, minimal Enriques surface.

Proof. Cf. [2], thm. (3.1). □

We have given a list of families of surfaces $Y \subset \mathbb{P}^5$ such that the generic member is projectable to a smooth surface in $\mathbb{P}^4$. More precisely, we know that there exists a member of the family which admits an inner projection. But since the dimension of the trisecant variety is upper semi-continous this implies that a generic member (of an irreducible component) of the families given above is projectable. Though for a special surface in a family it can happen that the dimension of the trisecant variety is bigger than 2 and so the surface does not admit an inner projection. We are going to describe these exceptional cases in the following, treating each case separately.

(I). $Y = \mathbb{P}^2$, $|H| = |2L|$.

REMARK 1. Let $Y = \mathbb{P}^2$ be embedded in $\mathbb{P}^5$ by $|H| = |2L|$. Then Y does not contain lines and also does not have "real" trisecants, i.e. trisecants which are not contained in Y. Therefore the projection from any point $y \in Y$ induces a closed embedding $\pi_y : \hat{Y}(y) \hookrightarrow \mathbb{P}^4$.

(II). $Y = \hat{\mathbb{P}}^2(x_1, \dots, x_4)$, $|H| = |3L - \sum E_i|$.

LEMMA 1. *Let be* $Y = \hat{\mathbb{P}}^2(x_1, \dots, x_4)$ *(resp.* $Y = \hat{\mathbb{P}}^2(x_1, \dots, x_5)$*),* $|H| = \left|3L - \sum_{i=1}^4 E_i\right|$ *(resp.* $= \left|3L - \sum_{i=1}^4 E_i\right|$*). Then it holds:*

H is very ample $\Leftrightarrow$ *two of the four (resp. five points) are never infinitely near and three of the four (resp. five) points are never collinear.*

The proof is wellknown and easy (cf. [9] V, thm. 4.6).

COROLLARY 2. *Let* $Y = \hat{\mathbb{P}}^2(x_1, \dots, x_4)$ *be embedded in* $\mathbb{P}^5$ *by* $|H| = \left|3L - \sum_{i=1}^4 E_i\right|$ *and let* $x \in Y$ *be a point such that* $h^0(L - E_i - E_j - x)$, $h^0(E_i - x) = 0$ *for* $i, j \in \{1, \dots, 4\}$.

Then $\pi_x : \hat{Y}(x) \to \mathbb{P}^4$ *is a closed embedding.*

Also in this case each element of the family in $\mathbb{P}^5$ is projectable to a smooth surface in $\mathbb{P}^4$.

(III). $Y = \hat{\mathbb{P}}^2(x_0, \dots, x_6)$, $|H| = |4L - 2E_0 - \sum E_i|$.

This case can be treated by using Reider's method and we get the following result, which has been proved by R. Weinfurtner in his thesis (cf. [19]).

PROPOSITION 2. *Let* $Y' = \hat{\mathbb{P}}^2(x_0, \dots, x_7)$ *be the blow-up of the projective plane in eight points and* $|H'| = \left|4L - 2E_0 - \sum_{i=1}^7 E_i\right|$. *Then* H' *is very ample if and only if the following four conditions are fulfilled:*

1. $h^0(L - E_0 - E_i - E_j) = 0$ *for any* $i \neq j \in \{1, \dots, 7\}$,

2. $h^0(L - E_i - E_j - E_k - E_l) = 0$ *for four different elements* $i, j, k, l \in \{0, \dots, 7\}$,

3. $h^0\left(2L - \sum_{\substack{i=0\\ i\neq j}}^{7} E_i\right) = 0$ *for* $j \in \{1, \ldots, 7\}$,

4. $h^0\left(E_0 - \sum_{i\in\Delta} E_i\right) = 0$ *for* $\#\Delta \geq 2$, $h^0(E_i - E_j) = 0$ *for* $i \neq j \geq 1$.

For a proof of this result we refer to [2], (3.22).

REMARK 2. Let be $Y = \hat{\mathbb{P}}^2(x_0, \ldots, x_6)$ and we assume $|H| = \left|4L - 2E_0 - \sum_{i=1}^{6} E_i\right|$ to be very ample, then it follows:

(*a*) $h^0(L - E_0 - E_i - E_j) = 0$ for arbitrary $i \neq j \in \{1, \ldots, 6\}$,

(*b*) $h^0(L - E_i - E_j - E_k - E_l) = 0$ for four different elements $i, j, k, l \in \{0, \ldots, 6\}$,

(*c*) $h^0\left(2L - \sum_{i=0}^{6} E_i\right) = 0$,

(*d*) $h^0\left(E_0 - \sum_{i\in\Delta} E_i\right) = 0$ for $\#\Delta \geq 2$, $h^0(E_i - E_j) = 0$ for $i \neq j \geq 1$.

Proof. We see immediately that $H.(L-E_0-E_i-E_j)$, $H.(L-E_i-E_j-E_k-E_l), \ldots \leq 0$ and therefore these systems cannot be effective.

□

PROPOSITION 3. *Let be* $Y = \hat{\mathbb{P}}^2(x_0, \ldots, x_6)$ *and* $|H| = \left|4L - 2E_0 - \sum_{i=1}^{6} E_i\right|$ *be very ample (i.e. H embeds Y as a smooth surface in* $\mathbb{P}^5$*). Then there exists a point* $y \in Y$*, such that* $\pi_y : \hat{Y}(y) \to \mathbb{P}^4$ *is a closed embedding.*

Proof. Since H is very ample it follows from Remark 2, that $h^0(L-E_0-E_i-E_j) = 0$ for arbitrary $i \neq j \in \{1, \ldots, 6\}$, $h^0(L - E_i - E_j - E_k - E_l) = 0$ for four different elements $i, j, k, l \in \{0, \ldots, 6\}$, $h^0\left(2L - \sum_{i=0}^{6} E_i\right) = 0$, $h^0\left(E_0 - \sum_{i\in\Delta} E_i\right) = 0$ for $\#\Delta \geq 2$, $h^0(E_i - E_j) = 0$ for $i, j \geq 1$. Therefore there exists a point $y \in Y$, such that $x_0, \ldots, x_6, y$ fulfill the conditions 1)-4) of Proposition 2 and thus $H' = 4L - 2E_0 - \sum_{i=1}^{6} E_i - E$ is very ample.

□

(IV). $Y = \hat{\mathbb{P}}^2(x_1, \ldots, x_9)$, $|H| = |4L - \sum x_i|$.

In this case we will see that the dimension of the trisecant variety jumps, i.e., for general members of this family the dimension of the trisecant variety is two (cf. Prop. 7), but there also exist surfaces in this family, which have a three dimensional trisecant variety.

PROPOSITION 4. *Let* $Y = \hat{\mathbb{P}}^2(x_1, \ldots, x_9)$ *be the projective plane blown-up in nine points such that* $|H| = |4L - \sum E_i|$ *is very ample (hence embeds Y as a smooth surface in* $\mathbb{P}^5$*). Then Y admits an inner projection iff* $h^0\left(3L - \sum_{i=1}^{9} E_i\right) = 1$.

Proof. This is an immediate consequence of ([19], Korollar 1.6.(a)).

□

In the case where $x_1, \dots, x_9$ lie on a pencil of cubics, $Y = \hat{\mathbb{P}}^2(x_1, \dots, x_9)$ does not admit an inner projection to $\mathbb{P}^4$. In this situation we calculate Trisec(Y).

We consider $Y \subset \mathbb{P}^2 \times \mathbb{P}^1$ given by $Y = \{(x, y) \in \mathbb{P}^2 \times \mathbb{P}^1 : G_0(x)y_1 - G_1(x)y_0 = 0\} \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^1$, where G_0, G_1 are two independent cubics of the pencil. Y is non singular, if the pencil generated by G_0, G_1 does not have a double base point.

$\psi : Y \hookrightarrow \mathbb{P}^5$ is given by the composition of $i : Y \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^1$ with the Segre embedding $\rho : \mathbb{P}^2 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^5$, $\rho(x, y) = (x_i y_j) \in \mathbb{P}(M(3, 2))$.

ψ induces on $\mathbb{P}^2$ the rational map $\mathbb{P}^2 - - \to \mathbb{P}^5$, $(x) \longmapsto (x_i G_i(x))$, given by $|4L - \sum P_i|$, where $\{P_1, \dots, P_9\} = \{G_0 = G_1 = 0\}$.

Let (x', y') and (x'', y'') be two different points on Y, and let us consider the secant through $\psi(x', y')$ and $\psi(x'', y'')$.

1. If $x' = x''$, then x' is a base point of the pencil, therefore the line through $\psi(x', y')$ and $\psi(x'', y'')$ is an exceptional line $\subset Y$.
2. If $y' = y''$, then $\psi(\mathbb{P}^2 \times \{y'\})$ is a $\mathbb{P}^2$ and $\psi((\mathbb{P}^2 \times \{y'\}) \cap Y)$ is a plane curve of degree 3. Therefore each secant is a trisecant.
3. If $x' \neq x''$, $y' \neq y''$, then we can assume that $x' = (1, 0, 0)$, $x'' = (0, 1, 0)$, $y' = (1, 0)$ and $y'' = (0, 1)$. Then $\psi(x', y') = (1, 0, 0, 0, 0, 0)$, and $\psi(x'', y'') = (0, 0, 0, 1, 0, 0)$. Thus $\psi(x, y)$ lies on the line through $\psi(x', y')$ and $\psi(x'', y'')$ iff $x_3 = 0$, $x_2 y_1 = x_1 y_2 = 0$.

   Therefore either $x_1 = y_1 = 0$, and $(x, y) = (x', y')$ or $x_2 = y_2 = 0$, i.e. $(x, y) = (x'', y'')$.

Thus we have shown: Trisec(Y) $= \mathbb{P}^2 \times \mathbb{P}^1$.

(V). $Y \subset \mathbb{P}^5$, **$K3$-surface,** $\deg Y = 8$.

Let now $Y \subset \mathbb{P}^5$ be a $K3$-surface of degree 8, and sectional genus $\pi(Y) = 5$. By the adjunction formula we get: $H.K = 2g(H) - 2 - 8 = 10 - 2 - 8 = 0$ and therefore Y is minimal. Furthermore it holds: $\chi(mH) = h_0(mH) = \frac{1}{2} mH.(mH - K) + \chi(\mathcal{O}_Y) = 4m^2 + \chi(\mathcal{O}_Y) = 4m^2 + 2$, and so $h^0(H) = 6$, $h^0(2H) = 18$ and we see that $\dim(\ker(S^2(H^0(H)) \longrightarrow\!\!\!\!\!\!\!\to H^0(2H))) = 3$. Therefore Y is contained in the intersection X of three independent quadrics Q_1, Q_2, Q_3 in $\mathbb{P}^5$. Then there are two cases:

1) $\dim X = 2$.

In this case Y does not have "real" trisecants and contains only finitely many lines (cf. Prop. 6). Therefore Y admits an inner projection.

2) $\dim X = 3$.

Then $\deg X = 3$ and by the classification of the projective varieties of degree 3 (cf. [18]) it follows that $X = \mathbb{P}^2 \times \mathbb{P}^1$, embedded in $\mathbb{P}^5$ by $|H| = |L + h|$, where

$L = p_2^*(\mathcal{O}_{\mathbb{P}^2}(1))$ and $h = p_1^*(\mathcal{O}_{\mathbb{P}^1}(1))$. We recall that $\mathrm{Pic}(\mathbb{P}^2 \times \mathbb{P}^1) = \mathbb{Z}L \oplus \mathbb{Z}h$, with $L^3 = 0, h^2 = 0, L^2.h = 1$ and $K_{\mathbb{P}^2 \times \mathbb{P}^1} = -3L - 2h$.

Since Y is minimal, K is trivial, hence by the adjunction formula $[Y] + K_{\mathbb{P}^2 \times \mathbb{P}^1} \equiv 0$. Therefore $[Y] = 3L + 2h$, in particular Y is ruled by plane cubics and does not admit an inner projection.

THEOREM 5 (VIII). *Let $Y = \hat{\mathbb{P}}^2(x_1, \dots, x_6, y_1, \dots, y_4)$ be embedded in $\mathbb{P}^5$ as a smooth surface by $|H| = |6L - \sum 2E_i - \sum E'_j|$. Then there exists a $y_5 \in Y$, s.th. $\pi_{y_5} : \hat{Y}(y_5) \longrightarrow \mathbb{P}^4$ is a closed embedding.*

Proof. (cf. [2] Kap. (3.2) or [6]).

□

THEOREM 6 (IX). *Let $Y = \hat{\mathbb{P}}^2(x_1, \dots, x_{10})$ be embedded as a smooth surface in $\mathbb{P}^5$ by $|H| = \left|7L - \sum_{i=1}^{10} E_i\right|$. Then there is a point $x_0 \in Y$, s.th. $\pi_{x_0} : \hat{Y}(x_0) \longrightarrow \mathbb{P}^4$ is a closed embedding.*

Proof. (cf. [2] Kap. (3.2) or [6]).

□

THEOREM 7. *([7] (2.12), (3.14), [8] Thm. 4). Let $Y \subset \mathbb{P}^5$ be an Enriques surface of degree 10, $\pi = 6$, H a hyperplane section and K a canonical divisor. Then it holds:*

1. *If $\phi_{|H+K|}(Y)$ is not contained in a smooth quadric $\subset \mathbb{P}^5$, then* Trisec(Y) *is the union of twenty planes $\pi_1, \dots, \pi_{20}$ and $\pi_j \cap Y$ is a plane cubic.*

   In particular Y admits an inner projection to $\mathbb{P}^4$.

2. *If $\phi_{|H+K|}(Y)$ is contained in a smooth quadric $\subset \mathbb{P}^5$, then* Trisec(Y) *has dimension four (more precisely: the trisecant cycle in the Grassmannian is isomorphic to $\hat{\mathbb{P}}^3(x_1, \dots, x_{20})$ and* Trisec(Y) *is a determinantal hypersurface of degree 4 in $\mathbb{P}^5$). Hence $Y \subset$* Trisec(Y) *and does not admit an inner projection.*

Combining the results of the Theorems 4 - 7 we have proven our main theorem.

4. Trisecant locus of surfaces $Y \subset \mathbb{P}^5$ (II)

In the last paragraph we return to the problem of describing the trisecant locus of surfaces in $\mathbb{P}^5$ which have a twodimensional trisecant variety.

First of all we prove:

PROPOSITION 5. *Let Y be as in (*) and we assume that the dimension of* Trisec(Y) *is at most two. By Proposition 1 we know that* Trisec$(Y) = \bigcup_{i=1}^r L_i \cup \bigcup_{j=1}^s E_j$. *Then:*

1. *$E_j \cap Y$ is a (purely dimensional) subscheme of Y of dimension one.*
2. *For a generic smooth hyperplane section H of Y it holds: $H \subset \mathbb{P}^4$ has only finitely many trisecants. After Le Barz (cf. [12] Theoreme 1) it follows therefore that the number $\theta(H)$ of trisecants of H given is by*

$$\theta(H) = \binom{n-2}{3} - g(H)(n-4),$$

where $g(H)$ is the genus of H. Furthermore it holds: $s \leq \theta(H)$.

Proof. We use the same notation as in the proof of Proposition 1.

Let Z_j be an irreducible component of Z with $q(p^{-1}(Z_j)) = E_j$. Since Z_j has dimension two, it follows that each line in E_j is a trisecant of Y. This implies that the dimension of $E_j \cap Y$ is one. Let now $\mathcal{H} \subset \mathbb{P}^5$ be a hyperplane transversal to E_j and L_j such that $\mathcal{H} \cap Y =: H$ is smooth. Then the trisecants of H are exactly the lines $\mathcal{H} \cap E_j$, whence H has only a finite number of trisecants. Since $\theta(H)$ counts the trisecants of H with their multiplicities, we obtain: $s \leq \theta(H)$.

Let H' be a further smooth hyperplane section of Y, transversal to E_j and L_j. Then H' has again s different trisecants (namely the lines $\mathcal{H} \cap E_j$) and the multiplicitiy m_j of $\mathcal{H} \cap E_j$ equals the multiplicity m'_j of $\mathcal{H}' \cap E_j$. This follows, since m_j is given by the length of $\mathcal{O}_{\mathrm{Al}^3 H \cap \mathrm{Hilb}^3 H, \mathrm{Hilb}^3 \mathbb{P}^4}$ (this holds by [Fu], Proposition 8.2, since $\mathrm{Al}^3 H \times \mathrm{Hilb}^3 H$ is smooth, in particular Cohen-Macaulay along $\mathrm{Al}^3 H \cap \mathrm{Hilb}^3 H$) and the length is semicontinous. Because the sum of the m'_j's equals the sum of the m_j's ($= \theta(H)$) it follows: $m'_j = m_j$.

This also implies that $E_j \cap Y$ has no zero dimensional component $\{p\}$, since otherwise for a smooth hyperplane section (transversal to E_j and L_j) containing p, the multiplicity m_j of the trisecant $\mathcal{H} \cap E_j$ would be strictly bigger than m'_j, where m'_j is the multiplicity of a trisecant $\mathcal{H}' \cap E_j$ of H', with H' a smooth hyperplane section of Y transversal to E_j and L_j, not containing p.

□

Therefore, in order to describe the trisecant locus of a surface with $\dim \mathrm{Trisec}(Y) \leq 2$ we have to find the lines contained in $Y \subset \mathbb{P}^5$ as well as the plane curves.

4.1. Lines contained in the (projectable) surfaces

We recall that since Y is not a scroll, the Hilbert scheme of lines contained in Y is finite. The following propositions describe the linear equivalence classes of these lines.

First of all we want to give an easy lemma which simplifies the calculations for rational surfaces.

LEMMA 2. *Let $Y \subset \mathbb{P}^5$ be a smooth rational surface, embedded by $|H| = |kL - \sum n_i E_i|$ ($k \geq 1, n_i \geq 0$) and $C \subset Y$ a line. Then: $C \subset E_i$ or $C \in |a_C L - \sum b_i E_i|$ with $a_C \leq k-2$.*

Proof. Since C is effective it holds: $C \subset E_i$ or $C \in |a_C L - \sum b_i E_i|$ with $a_C \geq 1$ and $b_i \geq 0$. Since C is a line it follows that $|H - C| = |a'L - \sum(n_i - b_i)E_i|$ has projective dimension three. From $a_C > k-2$ follows $k - a_C < 2$ and $\dim|H - C| \leq \dim|(k - a_C)L| \leq \dim|L| = 2$. This is a contradiction, hence $a_C \leq k-2$.

□

We will now consider the above list of surfaces case by case and determine the lines contained in the respective surface.

(I) $Y = \mathbb{P}^2, |H| = |2L|$.

Obviously Y does not contain a line, since for a curve $C \subset Y$ it holds: $C.H \equiv 0 \mod 2$.

(II) $Y = \hat{\mathbb{P}}^2(x_1, \dots, x_4), |H| = |3L - \sum E_i|$.

First we remark that each of the exceptional divisors E_i is irreducible, since $(E_i - E_j).H = 0$, hence $|E_i - E_j|$ cannot be effective.

Let now C be a line in Y, then it holds: $C = E_i$ or $C \in |aL - \sum b_i E_i|$ with $a \geq 1$ and $b_i \geq 0$. Therefore it follows from Lemma 2 that $a = 1$. Moreover we have:

$C.H = 3a - \sum_{i=1}^4 b_i = 1$ and

$C^2 + C.K = a^2 - \sum_{i=1}^4 b_i^2 - 3a + \sum_{i=1}^4 b_i = 2p(C) - 2 = -2$, which implies:

$\sum_{i=1}^4 b_i^2 = \sum_{i=1}^4 b_i = 2$, and it follows:

$C \in |L - E_i - E_j|$, where $i \neq j \in \{1, \dots, 4\}$.

Therefore we have shown the following.

LEMMA 3. *Let $Y = \hat{\mathbb{P}}^2(x_1, \dots, x_4)$ be embedded as a smooth surface in $\mathbb{P}^5$ by the linear system $|H| = |3L - \sum x_i|$ and let $C \subset Y$ be a line. Then it holds:*

$$C = E_i,\ 1 \leq i \leq 4 \text{ or } C \in |L - E_i - E_j|, \text{ where } i \neq j \in \{1, \dots, 4\}.$$

In particular there are exactly ten lines contained in Y.

By the same method as before we obtain in the remaining cases:

LEMMA 4 (III). *Let $Y = \hat{\mathbb{P}}^2(x_0, \dots, x_6)$ be smoothly embedded in $\mathbb{P}^5$ by $|H'| = |4L - 2E_0 - \sum E_i|$ and let $C \subset Y$ be a line, then it holds:*

$C = E_i$, $1 \leq i \leq 6$, or

$C \in |E_0 - E_i|$, where $i \in \{1, \dots, 6\}$, or

$C \in |L - E_0 - E_i|$, where $i \in \{1, \dots, 6\}$, or

$C \in |L - E_i - E_j - E_k|$, where i, j, k are three different elements of $\{1, \dots, 6\}$, or

$$C \in \left|2L - E_0 - \sum_{i \neq j} E_i\right| \text{ for } 1 \leq j \leq 6.$$

Since all the above linear systems have (projective) dimension at most zero, there are only finitely many lines contained in Y.

REMARK 3. We want to point out that the last two linear systems are not effective for $x_0, \dots, x_6$ in general position. Therefore the lemma (as well as the following ones) is to be understood in the following way: if $|E_0 - E_i|$, $|L - E_i - E_j - E_k|$ resp. $\left|2L - E_0 - \sum_{i \neq j} E_i\right|$ is effective, then $C \in |E_0 - E_i|$, $|L - E_i - E_j - E_k|$ resp. $\left|2L - E_0 - \sum_{i \neq j} E_i\right|$ is a line contained in Y.

LEMMA 5 (IV). *Let $Y = \hat{\mathbb{P}}^2(x_1, \dots, x_9)$ be smoothly embedded in $\mathbb{P}^5$ by $|H| = |4L - \sum x_i|$ and let $C \subset Y$ be a line. Then it holds:*

$C = E_i$, $1 \leq i \leq 9$, or

$C \in |L - E_i - E_j - E_k|$, where i, j, k are three different elements of $\{1, \dots, 9\}$, or

$C \in \left|2L - \sum_{i \neq j,k} E_i\right|$, where $j \neq k \in \{1, \dots, 9\}$.

(The last two linear systems are in general not effective.)

The projective dimension of the above linear systems is at most zero, hence there exist only finitely many lines in Y.

LEMMA 6 (V). *Let $Y = \hat{\mathbb{P}}^2(x_1, \dots, x_6, y_1, \dots, y_4)$ be smoothly embedded in $\mathbb{P}^5$ by $|H| = |6L - \sum 2E_i - \sum E'_j|$ and let C be a line in Y. Then it holds:*

$C = E'_j$, $1 \leq j \leq 4$, or (if the respective linear system is effective which is in general not the case)

$C \in |E_i - E'_j|$, where $i \in \{1, \dots, 6\}$ and $j \in \{1, \dots, 4\}$, or

$C \in |L - E_i - E_j - E'_k|$, where $i \neq j \in \{1, \dots, 6\}$ and $k \in \{1, \dots, 4\}$, or

$C \in \left|L - E_i - \sum_{j \neq k} E'_j\right|$, where $i \in \{1, \dots, 6\}$ and $k \in \{1, \dots, 4\}$, or

$C \in \left|2L - \sum_{i \neq k} E_i - E'_j\right|$, where $k \in \{1, \dots, 6\}$ and $j \in \{1, \dots, 4\}$, or

$C \in \left|2L - \sum_{i \neq k,l} E_i - \sum_{j \neq m} E'_j\right|$, where $k \neq l \in \{1, \dots, 6\}$ and $m \in \{1, \dots, 4\}$, or

$C \in \left|3L - 2E_k - \sum_{i \neq k} E_i - \sum_{j \neq l} E'_j\right|$ with $1 \leq k \leq 6$ and $l \in \{1, \dots, 4\}$.

In particular there exist only finitely many lines in Y, since the above linear systems have projective dimension at most zero.

LEMMA 7 (VII). *Let $Y = \hat{\mathbb{P}}^2(x_1, \dots, x_{10})$ be embedded as a smooth surface in $\mathbb{P}^5$ by $|H| = |7L - \sum 2E_i|$. For $x_1, \dots, x_{10}$ in general position the only lines in Y are the exceptional lines. More precisely we have: if $C \subset Y$ is a line, then:*

$C = E_i$, $1 \leq i \leq 10$, or

$C \in |L - E_i - E_j - E_k|$, where i, j, k are three different elements of $\{1, \dots, 9\}$, or

$C \in |3L - 2E_k - \sum_{i \neq k,j} E_i|$, where j, k are two different elements of $\{1, \dots, 10\}$.

(VI). $Y \subset \mathbb{P}^5$**, minimal** $K3$**-surface,** $\deg Y = 8$**.**

There are only finitely many lines contained in Y which is a consequence of the following lemma.

LEMMA 8. *Let* $Y \subset \mathbb{P}^n$ *be a minimal* $K3$*-surface, then there exists only a finite number of lines in* Y.

Proof. Let $C \subset Y$ be a line, then $C^2 = 2p_a(C) - 2 - C.K = -2$. Let $G(Y) \subset \mathbb{G}(1, \mathbb{P}^n)$ be the closed subset of $\mathbb{G}(1, \mathbb{P}^n)$ given by $\{L \in \mathbb{G}(1, \mathbb{P}^n) : L \subset Y\}$ and let G_i, $1 \le i \le m$, be the (finitely many) irreducible components of $G(Y)$. Since $C^2 = -2$ for each line $C \subset Y$, it follows that the dimension of G_i is equal to zero, therefore $G_i = \{C_i\}$ and so we obtain the claim.

□

REMARK 4. For the description of the trisecant locus of the remaining Enriques surface we refer to [7] (cf. also Theorem 7).

In order to determine the trisecant locus of a (in our sense) projectable surface $Y \subset \mathbb{P}^5$ completely it remains by the Propositions 1, 5 to find the plane curves on Y.

4.2. Plane curves contained in the projectable surfaces

We first treat the cases without "real" trisecants, i.e. L is a trisecant of Y iff L is a line contained in Y.

PROPOSITION 6. *Let* $Y \subset \mathbb{P}^5$ *be one of the following smooth surfaces:*

1. $Y = \mathbb{P}^2$, *embedded in* $\mathbb{P}^5$ *by* $|H| = |2L|$,
2. $Y = \hat{\mathbb{P}}^2(x_1, \dots, x_4)$, *embedded in* $\mathbb{P}^5$ *by* $|H| = |3L - \sum E_i|$,
3. $Y = \hat{\mathbb{P}}^2(x_0, \dots, x_6)$, *embedded in* $\mathbb{P}^5$ *by* $|H| = \left|4L - 2x_0 - \sum_{i=1}^{6} x_i\right|$,
4. $Y \subset \mathbb{P}^5$, *minimal* $K3$*-surface,* $\deg Y = 8$.

We assume the points in $\mathbb{P}^2$ *in 2), 3) to be chosen that* Y *admits an inner projection and in case 4) we assume that* Y *is projectable.*

Then Y *does not have "real" trisecants (i.e. if* $L \subset \mathbb{P}^5$ *is a trisecant of* Y*, then* $L \subset Y$*).*

Proof. We see that in all these cases $\theta(H) = \binom{\deg H - 2}{3} - g(H)(\deg H - 4) = 0$, which by Proposition 5 implies the claim.

□

PROPOSITION 7. *Let $x_1, \dots, x_9 \in \mathbb{P}^2$ be such that $Y = \hat{\mathbb{P}}^2(x_1, \dots, x_9)$ is embedded in $\mathbb{P}^5$ by the linear system $|H| = |4L - \sum E_i|$ and it is projectable to a smooth surface in $\mathbb{P}^4$. Then:*

$$\mathrm{Trisec}(Y) = \bigcup_{i=1}^{r} L_i \cup \pi,$$

where L_i are the lines contained in Y (cf. Lemma 5) and π is a plane in $\mathbb{P}^5$ with $\pi \cap Y = C$, $C \in \left|3L - \sum_{i=1}^{9} E_i\right|$.

Proof. For a hyperplane section H of Y it holds: $\theta(H) = \binom{5}{3} - 3(7-4) = 1$ (since $\deg Y = 7$ and $\pi(Y) = 3$). Therefore it follows by Proposition 1 that $\mathrm{Trisec}(Y) = \{\text{lines} \subset Y\} \cup \pi$, where π is a plane in $\mathbb{P}^5$. Thus it is sufficient to show that $C \in \left|3L - \sum_{i=1}^{9} x_i\right|$ is a plane curve of degree ≥ 3 in Y (because then: $\mathrm{Trisec}(C) =$ plane $\subset \mathrm{Trisec}(Y)$ and $\theta(H) = 1$).

But $H = C + L$, $\dim L = 2$, which implies that C is contained in a 2-parameter family of hyperplanes. This shows that C is a plane curve. Furthermore the degree of C is equal to $C.H = 3$.

□

PROPOSITION 8. *Let $x_1, \dots, x_6, y_1, \dots, y_4 \in \mathbb{P}^2$ be such that $Y = \hat{\mathbb{P}}^2(x_1, \dots, x_6, y_1, \dots, y_4)$ is embedded in $\mathbb{P}^5$ by $|H| = |6L - \sum 2E_i - \sum E'_j|$ and it is projectable to a smooth surface in $\mathbb{P}^4$. Then it holds:*

$$\mathrm{Trisec}(Y) = \{\, \textit{lines} \subset Y\} \cup \bigcup_{j=1}^{4} \pi_j,$$

where $\pi_1, \dots, \pi_4 \subset \mathbb{P}^5$ are planes with $\pi_j \cap Y = C_j$, and $C_j \in \left|3L - \sum_{i=1}^{6} E_i - \sum_{k \neq j} E'_k\right|$. (For the lines $\subset Y$ cf. Lemma 6).

Proof. As in Proposition 7.

□

PROPOSITION 9. *Let $x_1, \dots, x_{10} \in \mathbb{P}^2$ be such that $\hat{\mathbb{P}}^2(x_1, \dots, x_{10})$ is embedded in $\mathbb{P}^5$ by $|H| = |7L - \sum 2E_i|$ and it is projectable to a smooth surface in $\mathbb{P}^4$. Then we have:*

$$\mathrm{Trisec}(Y) = \{\, \textit{lines} \subset Y\} \cup \bigcup_{j=1}^{10} \pi_j,$$

where $\pi_1, \dots, \pi_{10} \subset \mathbb{P}^5$ are planes with $\pi_j \cap Y = C_j$, where $C_j \in \left|3L - \sum_{i \neq j} E_i\right|$. (For the lines $\subset Y$ cf. Lemma 7).

Proof. As in Proposition 7. □

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Appendix: trisecants of scrolls

In this appendix we will classify the smooth, non degenerate scrolls $Y \subset \mathbb{P}^5$, such that the embedded trisecant variety Trisec(Y) has dimension at most two, and describe their trisecant locus. It will turn out that there are only three types of scrolls $Y \subset \mathbb{P}^5$, which have two dimensional trisecant variety, namely two rational quartic scrolls (i.e. $Y = \mathbb{F}_0, \mathbb{F}_2$) and the non-trivial elliptic quintic scroll.

We will first prove this result and then calculate the trisecant variety of these scrolls.

This result together with the previous part of the article gives a complete classification of smooth, non-degenerate, connected complex surfaces $Y \subset \mathbb{P}^5$ having an at most two dimensional triseant variety.

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A.1. Scrolls with few trisecants

Let $Y \subset \mathbb{P}^5$ be a smooth, non-degenerate, connected scroll having a two dimensional trisecant variety. Then, obviously, Y is an irreducible component of Trisec(Y).

We show that the other irreducible components of Trisec(Y) are very simple.

PROPOSITION 10. *Let $Y \subset \mathbb{P}^5$ be a smooth, non-degenerate, connected scroll with* $\dim \text{Trisec}(Y) = 2$.

Then $\text{Trisec}(Y) = Y \cup T^*$, *where T^* is the Zariski closure of* $\text{Trisec}(Y) - Y$.

Furthermore it holds:

1. $T^* = \bigcup_{j=1}^{s} E_j$, *where $E_1, \ldots, E_s \subset \mathbb{P}^5$ are planes,*
2. *$E_j \cap Y$ is a (purely dimensional) subscheme of Y of dimension one,*
3. $s \leq \theta(H) := \binom{n-2}{3} - g(H)(n-4)$, *where H is a generic smooth hyperplane section H of Y, $g(H)$ is the genus of H and n is the degree of Y.*

Proof. The argument is exactly the same as the proof of the Propositions 1 and 5. □

We are now ready to state the following result:

THEOREM 8. *Let $Y \subset \mathbb{P}^5$ be a smooth, non-degenerate, connected scroll with* $\dim \mathrm{Trisec}(Y) = 2$. *Then Y has degree smaller or equal to six.*

Proof. Let $Y \subset \mathbb{P}^5$ be a smooth, non-degenerate, connected scroll of degree n with $\dim \mathrm{Trisec}(Y) = 2$. Then Y is not contained in T^* and we choose a point $y \in Y - T^*$. Moreover, we suppose that y is not an element of the section of C in $Y = \mathbb{P}_C(\mathcal{E})$. Performing an elementary transformation on Y (starting to blow up y and then blowing down the strict transform of the fibre of $\mathbb{P}_C(\mathcal{E})$, which contains y, we obtain a smooth scroll Y' in $\mathbb{P}^4$ of degree $n-1$. But this implies that $n - 1 \leq 5$ (cf. [13]).

□

Since there are only two families of smooth scrolls in $\mathbb{P}^4$, we obtain the following stronger result.

THEOREM 9. *Let $Y \subset \mathbb{P}^5$ be a smooth, non-degenerate, connected scroll with* $\dim \mathrm{Trisec}(Y) = 2$. *Moreover let H be a hyperplane section of Y. Then (Y, H) is one of the following:*

1. $Y = \mathbb{P}^1 \times \mathbb{P}^1$, $H = (1,2)$ *(resp. (2,1)) ("rational normal scroll"),*
2. $Y = \mathbb{F}_2$, $|H| = |C_0 + 3F|$, *where C_0 is the section of $C = \mathbb{P}^1$ in Y and F is a fibre ("rational quartic scroll"),*
3. $Y = \mathbb{P}_C(\mathcal{E})$, *where C is an elliptic curve, $\mathcal{E}$ is a normalized indecomposable vector bundle of rank two,* $\deg(\det \mathcal{E}) = 0$ *and* $|H| = |C_0 + \pi^*(b)|$, *where C_0 is the section of C in Y and b is a divisor on C of degree 3 ("elliptic sextic scroll").*
4. $Y = \mathbb{P}_C(\mathcal{E})$, *where C is an elliptic curve, $\mathcal{E} = \mathcal{O} \oplus \mathcal{C}$, where $\mathcal{C}$ is a non-trivial line bundle on C with* $\deg \mathcal{C} = 0$ *and* $|H| = |C_0 + \pi^*(b)|$, *and again b is a divisor on C of degree 3.*

(For the notation we refer to [9], chap. V, §2).

Proof. Let $Y \subset \mathbb{P}^5$ be a smooth, non-degenerate, connected scroll with $\dim \mathrm{Trisec}(Y) = 2$. Performing the elementary transformation as in the proof of Theorem 8 we get a smooth scroll Y' in $\mathbb{P}^4$ of degree $n-1$. But (cf. eg. [13]) the only smooth scrolls in $\mathbb{P}^4$ are the rational cubic scroll (i.e. $Y = \mathbb{F}_1$, embedded by $|H| = |C_0 + 2F|$) and the quintic elliptic scroll (i.e. $Y = \mathbb{P}_C(\mathcal{E})$, where C is an elliptic curve, $\mathcal{E}$ is a normalized vector bundle of rank two, $\deg(\det \mathcal{E}) = 1$ and $|H| = |C_0 + bF|$, where C_0 is the section of C in Y, F a fibre and b is a divisor on C of degree 2. Since Y is obtained from Y' by one elementary transformation we see that Y is of the form 1,2,3 or 4. We note that all the other possibilities for $\mathcal{E}$ (except the ones in 3), 4)) are not realizable as smooth surfaces in $\mathbb{P}^5$ or they have trisecant variety of dimension three (cf. also Remark 5).

For the existence of the above surfaces in $\mathbb{P}^5$ we refer to [9], chapter V, 2.

□

A.2. Trisecant locus of scrolls

Let $Y \subset \mathbb{P}^5$ be a smooth, non-degenerate, connected scroll with $\dim \text{Trisec}(Y) = 2$. In this section we will describe the trisecant variety of these scrolls, i.e. the cases 1), 2), 3) and 4) of Theorem 9.

Since $\dim \text{Trisec}(Y) = 2$, Y is an irreducible component of $\text{Trisec}(Y)$ and $\text{Trisec}(Y) = Y \cup T^*$, where T^* is the Zariski closure of $\text{Trisec}(Y) - Y$. Therefore it is enough to determine T^*. By Proposition 10 $T^* = \bigcup_{j=1}^{s} E_j$, where $E_1, \dots, E_s \subset \mathbb{P}^5$ are planes, hence it suffices to find the plane curves of degree at least three in Y.

In the first two cases of Theorem 9 (using the explicit description of $\text{Pic}(Y)$), we have obviously the following result:

LEMMA 9. *Let (Y, H) be equal to $(Y = \mathbb{P}^1 \times \mathbb{P}^1, H = (1, 2))$, resp. equal to $(Y = \mathbb{F}_2, |H| = |C_0 + 3F|)$ (cf. Theorem 9), then Y does not contain a plane curve of degree at least three. Hence* $\text{Trisec}(Y) = Y$.

For the rest of the section, let Y be a sextic elliptic scroll in $\mathbb{P}^5$, i.e. $Y = \mathbb{P}_C(\mathcal{E})$, as in Theorem 9, 3) or 4).

PROPOSITION 11. *Let Y be as in Theorem 9, 3). Then C_0 is the only plane curve of degree ≥ 3 on Y and therefore* $\text{Trisec}(Y) = Y \cup < C_0 >$, *where $< C_0 >$ is the plane generated by C_0.*

Proof. Let Δ be a plane curve of degree ≥ 3 on Y. Then it follows using $H.\Delta \geq 3$ and $\dim |H - \Delta| = 2$, that $\Delta \in |H - cF|$, where c is the pull-back of a divisor of degree 3 on C. But this is effective if and only if $c = b$ and then $\dim |H - cF| = 0$, whence $\Delta = C_0$.

□

This settles the case when $\mathcal{E}$ is indecomposable.

REMARK 5. 1. If $\mathcal{E}$ is indecomposable, then assuming $\mathcal{E}$ to be normalized, we have that the invariant e of Y is equal to 0 or -1 (cf. [9], chapter V, (2.15)). But $|H| = |C_0 + \pi^*(b)|$, with $\deg b = 3$ is very ample if and only if $e = 0$ ([9], chap. V, Exercise 2.12).

2. If $\mathcal{E}$ is trivial, then $Y = \mathbb{P}^1 \times C$ and the embedding of Y into $\mathbb{P}^5$ factors through the Segre variety $\mathbb{P}^1 \times \mathbb{P}^2$, and therefore $\text{Trisec}(Y) = \mathbb{P}^1 \times \mathbb{P}^2$.

3. If $\mathcal{E}$ is decomposable, then we can assume that $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(a)$, where a has degree ≤ 0. Then the invariant e of $Y = \mathbb{P}_C(\mathcal{E})$ is equal to $-\deg(a)$. By [9], chap. V, Exercise 2.12 $|H| = |C_0 + bF|$ (where C_0 is the section of C in Y, F a

fibre and b is a divisor on C of degree 3) is very ample if and only if $e = 0$, i.e. $\deg(a) = 0$.

Hence (Y, H) is a scroll in $\mathbb{P}^5$ iff (Y, H) is of the form 3) or 4).

PROPOSITION 12. *Let Y be as in Theorem 9, 4). Then* $\mathrm{Trisec}(Y) = Y \cup < \Delta > \cup < \Delta' >$, *where Δ and Δ' are plane curves on Y of degree 3.*

Proof. As in the proof of Proposition 11 (for purely numerical reasons) we see that any plane curve of degree ≥ 3 has to be linearly equivalent to $H - dF$, where d is a divisor of degree 3 on C. But $H - dF$ is effective (and of dimension 0) if and only if $H - dF$ is linearly equivalent to a section C_0 of Y with $C_0^2 = 0$. But there are exactly two such sections Δ and Δ' (cf. [9], V, 2.11.2). Then obviously Δ and Δ' are plane cubics. □

REMARK 6. Let Y be as in Proposition 11 and let H be a generic hyperplane section. Then $\theta(H) = 2$. Therefore it follows from Proposition 1 that the component $< C_0 >$ of $\mathrm{Trisec}(Y)$ has multiplicity 2.

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