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GEOMETRICAL ASPECTS OF THE DYNAMICS
OF NON-HOLONOMIC SYSTEMS*

Abstract. Dynamics of non-holonomic mechanical systems is interpreted as a
submanifold of $TT^*Q$ where $Q$ is the configuration manifold. Integrability of
dynamics is discussed for linear and non-linear constraints. The case of constrained
godesics of a Riemannian manifold studied by Synge is also considered. Local
cordinate representations are used. An example of an ideal non-linear non-holonomic
constraint is proposed.

1. First order equations

A first order (differential) equation on a manifold $M$ is a submanifold $D$ of the
tangent bundle $TM$. A first order equation is said to be integrable if for each $v \in D$ there
exists a differentiable curve $\gamma: I \rightarrow M$ such that $\gamma(0) = v$ and $\gamma(t) \in D$ for each $t \in I$,
where $\dot{\gamma}: I \rightarrow TM$ is the tangent curve to $\gamma$ and $I$ is an open real interval containing $0$.
Such a curve is called an integral curve of $D$ based on $v \in D$. It is possible to extend
this definition to the case in which $D$ is a submanifold with boundary or a subset of $TM$.
If $D$ is a non-integrable first order equation, then the integrable part of $D$ is the maximal
subset of $D$ which is integrable according to the definition above.

If $(x^A)$ are local coordinates of $M$, then we denote by $(x^A, \dot{x}^B)$ the corresponding
fibered coordinates on $TM$. A first order equation $D$ is locally described by a system of
equations

$$D^a(x^A, \dot{x}^B) = 0$$

($a, b = 1, \ldots, m; m = \dim(M); a = 1, \ldots, l; l = \dim(D)$). An integral curve has a
local representation $x^A = \gamma^A(t)$ such that for each $t$

$$D^a(\gamma^A(t), \dot{\gamma}^B(t)) = 0$$

*This paper was delivered at the Université de Savoie in 1987 and published in the proceedings of the
"Journées Relativistes 1987", Chambéry, May 14-16, 1987. As this volume is not easily accessible
and some of the contributors to the present collection refer to this paper, it was thought to be useful
 to reprint it here.
where $D\gamma^A$ is the derivative of the real function $\gamma^A$.

**EXAMPLES.** (1) A subbundle $D$ of $TM$, i.e., a regular distribution on $M$, is an integrable first order equation on $M$. For each $v \in D$ there exists an integral curve of $D$ based on $v$, but it is not unique.

(2) Let $X: M \to TM$ be a differentiable section of the tangent bundle $\tau_M: TM \to M$, i.e., a differentiable vector field on $M$. The image $D = X(M)$ of $X$ is an integrable first order equation. In this case the uniqueness property holds (Cauchy theorem): if $\gamma: I \to M$ and $\gamma': I' \to M$ are integral curves based on $v \in D$, then they coincide in the intersection $I \cap I'$ of the intervals of definition.

2. Dynamics of holonomic systems

A fundamental example of first order equation is given by the dynamics of holonomic mechanical systems.

Let $Q$ be the configuration manifold of a holonomic mechanical system with $n$ degrees of freedom. Let $(q^i)$ be local coordinates on $Q$ (i.e., Lagrangian coordinates of the mechanical system). We denote by $(q^i, \dot{q}^i)$, $(q^i, p_j)$ and $(q^i, p_j, \dot{q}^h, \dot{p}_k)$ the corresponding fibered coordinates on $TQ$, $T^*Q$ and $TT^*Q$ respectively. In the following discussion Latin indices $i, j, h, k \ldots$ run from 1 to $n = \dim(Q)$. The manifolds $TQ$ and $T^*Q$ represent the velocity space and the phase space of the mechanical system.

The dynamics of the mechanical system is the submanifold $D$ of $TT^*Q$ locally defined by equations

$$ p_i - \frac{\partial L}{\partial q^i} = 0, \quad \dot{p}_i - \frac{\partial L}{\partial \dot{q}^i} = 0, $$

or by equations

$$ \dot{q}^i - \frac{\partial H}{\partial p_i} = 0, \quad \dot{p}_i + \frac{\partial H}{\partial q^i} = 0, $$

where $L: TQ \to \mathbb{R}$ is the Lagrangian function locally represented by a function of the coordinates $(q^i, \dot{q}^i)$, and $H: T^*Q \to \mathbb{R}$ is the Hamiltonian function locally represented by a function of the coordinates $(q^i, p_j)$. We call (1) and (2) the Lagrangian representation and the Hamiltonian representation of the dynamics $D$ respectively. Equations (1) follow from the d'Alembert-Lagrange principle. Equations (2) follow from equations (1) under the regularity condition

$$ \det \left( \frac{\partial^2 L}{\partial q^i \partial \dot{q}^j} \right) \neq 0. $$

The Hamiltonian representation $D$ shows that the dynamics $D$ is the image of a vector field $X$ on the phase space $T^*Q$. Hence, $D$ is integrable with the uniqueness
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property. The local expression of the vector field \( X \) is

\[
X = X^i \frac{\partial}{\partial q^i} + X_j \frac{\partial}{\partial p_j},
\]

where

\[
X^i = \frac{\partial H}{\partial p_i}, \quad X_j = -\frac{\partial H}{\partial q^j}.
\]

The vector field is globally defined by equation

\[
i_X d\theta = -dH,
\]

where \( \theta = p_i dq^i \) is the fundamental 1-form of \( T^*Q \) (the Liouville form).

3. Dynamics with non-holonomic linear constraints

We assume that further constraints are imposed on the holonomic system. The possible kinematical states of the system are represented by vectors \( v \in TQ \) which belong to a subset \( K \) of \( T \). In most of the applications \( K \) is a subbundle of \( TQ \), i.e., a regular distribution on \( Q \). If \( K \) is not completely integrable, then the constraints are called non-holonomic linear constraints.

The distribution \( K \) can be represented by local equations

\[
K^a_i q^i = 0 \quad (a = 1, \ldots, l),
\]

where \( K^a_i \) are functions on the domain of the coordinates \( (q^i) \) forming a matrix of maximal rank:

\[
\text{rank}(K^a_i) = l.
\]

It follows from the D'Alembert-Lagrange principle that the dynamics \( D \) is the subset of \( TT^*Q \) locally defined by equations

\[
p_i - \frac{\partial L}{\partial \dot{q}^i} = 0, \quad \dot{p}_i - \frac{\partial L}{\partial q^i} = \lambda_a K^a_i, \quad K^a_i q^i = 0,
\]

or by equations

\[
\dot{q}_i - \frac{\partial H}{\partial p_i} = 0, \quad \dot{p}_i + \frac{\partial H}{\partial q^i} = \lambda_a K^a_i, \quad K^a_i q^i = 0,
\]

where \( (\lambda_a) \) are the Lagrange multipliers. A point of \( TT^*Q \) belongs to \( D \) if and only if its coordinates satisfy equations (3) or (4) with some values of the parameters \( (\lambda_a) \). The terms

\[
R_i = \lambda_a K^a_i
\]

represent the possible reaction forces of the constraints.
The Lagrangian representation (3) of the dynamics $D$ is suitable for proving that $D$ is a submanifold. The Hamiltonian representation (4) is suitable for discussing the integrability of $D$. The integrability will be discussed in the next section.

The distribution $K$ can be represented by parametric equations of the kind

\begin{equation}
q^i = X^i, \quad \dot{q}^i = F^i_\alpha (x^j) \omega^\alpha \quad (\alpha = l + 1, \ldots, n),
\end{equation}

where the functions $F^i_\alpha$ form a matrix of maximal rank:

\begin{equation}
\text{rank}(F^i_\alpha) = n - l.
\end{equation}

The parameters $(x^i = q^i, \omega^\alpha)$ can be interpreted as coordinates on $K$ and equations (6) as representing a local immersion of $K$ into $TQ$. Coordinates $(\omega^\alpha)$ are known in the classical literature as "pseudo-velocities". The equivalence of the representations (1) and (6) implies that

\begin{equation}
F^i_\alpha K^\alpha_i = 0.
\end{equation}

The substitution of (6) into the first two sets of the Lagrange equations (3) yields equations of the kind

\begin{equation}
\dot{p}_i = f_i(x^j, \omega^\alpha); \quad \ddot{\lambda}_a = g_i(x^j, \omega^\alpha) + \lambda_a K^a_i (x^j).
\end{equation}

The system of equations (6) and (9) gives the parametric representation of the dynamics $D$. The $2n$ parameters $(x^i, \lambda_a, \omega^\alpha)$ can be interpreted as coordinates on $D$ and equations (6) and (9) as representing a local immersion of $D$ into $TT^*Q$. Indeed, a straightforward calculation shows that the Jacobian matrix of the functions at the left sides of (6) and (9), with respect to the variables $(x^i, \lambda_a, \omega^\alpha)$, has maximal rank. This proves

**Proposition 1.** Under the regularity conditions (3, §2) and (2) the subset $D \subset TT^*Q$ defined by equations (3) is a submanifold of dimension $2n$.

4. The integrability theorem and the elimination of the Lagrangian multipliers

For the sake of simplicity we shall use the following notation:

\[ H_i = \frac{\partial H}{\partial q^i}, \quad H^i = \frac{\partial H}{\partial p_i}, \quad \text{etc.} \]

The image by the tangent fibration $\tau_{T^*Q}: TT^*Q \to T^*Q$ of the first order equation $D$ considered in Section 3 is the subset $C \subset T^*Q$ locally defined by equations

\begin{equation}
C^a = K^a_i H^i = 0
\end{equation}

which are obtained by combining the first and the third set of equations (4, §3).

The regularity condition (3, §2) is equivalent to

\begin{equation}
\det (H^{ij}) \neq 0.
\end{equation}
The regularity conditions (2.3) and (2) imply that the \( l \) functions at the left side of (1) are independent. Hence, the subset \( C \) is a submanifold of \( T^*Q \) of dimension \( 2n - l \). The first set of the Lagrange equations (3.3) can be interpreted as the local definition of a fibre bundle isomorphism \( \Lambda: TQ \to T^*Q \), i.e., of the Legendre transformation. The first set of the Hamilton equations can be interpreted as the local definition of \( \Lambda^{-1} \). Hence, \( C = \Lambda(K) \).

Let \( \gamma: I \to T^*Q \) be an integral curve of \( \mathcal{D} \). The image \( \gamma(I) \) is contained in \( C \). Hence, the image \( \hat{\gamma}(I) \) of the tangent curve \( \hat{\gamma}: I \to TT^*Q \) is contained in \( TC \). It follows that the integrable part of \( D \) is contained in the intersection \( D \cap TC \). The submanifold \( TC \) is defined by equations (1) and equations (3)

\[
\dot{q}^i \left( K_{ij} \dot{H}^i + K_{ij}^a H^j \right) + \dot{p}_j K_{ij}^a H^{ij} = 0,
\]

obtained from (1) by formal derivation. Since the submanifold \( D \) is defined by equations (4.3), the intersection \( D \cap TC \) is characterized by equations

\[
K_{ij}^a H^i H^j + K_{ij}^a \left( H_j^i H^j - H^ij H_j \right) + K_{ij}^a K_{ij}^b H^{ij} \lambda_b = 0.
\]

Under the condition

\[
\det \left( H^{ij} K_{ij}^a K_{ij}^b \right) \neq 0,
\]

we can solve equations (4) with respect to the multipliers \( (\lambda_a) \). We obtain

\[
\lambda_a = G_{ab} L^b,
\]

where

\[
\begin{align*}
\left\{ L^b = \{H, C^b\} = K_{ij}^b \left( H^{ij} H_j - H_j^i H^j \right) - K_{ij}^b H^i H^j, \\
(G_{ab}) = (G_{ab})^{-1}, \quad G^{ab} = H^{ij} K_{ij}^a K_{ij}^b.
\end{align*}
\]

In the first equation \( \{\cdot,\cdot\} \) denotes the canonical Poisson bracket of functions on the cotangent bundle \( T^*Q \). Hence, for each point \( p \in C \) there exists one and only one element of \( D \) belonging to the intersection \( D \cap T_p C \). This means that \( D \cap TC \) is the image of a section \( X: C \to TC \) of the tangent fibration \( \pi \circ TC \to C \), i.e., the image of a vector field \( X \) on \( C \). Hence, the intersection \( D \cap TC \) coincides with the integrable part of \( D \). This proves:

**Proposition 2.** If the regularity conditions (2.3), (2) and (5) are satisfied, then the integrable part of the first order differential equation \( \mathcal{D} \) defined by equations (4.3) is the image of a vector field \( X \) in the submanifold \( C \subset T^*Q \) defined by equations (1).

**Remark 1.** (i) If the quadratic form defined by the matrix \( (H^{ij}) \) is positive-definite, then the regularity condition (5) is a consequence of condition (2.3). This is the case of the ordinary holonomic mechanical systems. (ii) The explicit form (6)–(7) of the Lagrangian multipliers has been derived by Eden [7] by a different method.
The intersection $D \cap TC$ is defined by equations

$$q^i - H^i = 0, \quad p_i + H_i = G_{ab} L^a K_i^b, \quad K_i^b q^i = 0.$$  

Hence, the vector field $X$ can be interpreted as the restriction to $C$ of the vector field $\overline{X}$ on $T^*Q$ whose components are

$$\overline{X}^i = H^i, \quad \overline{X}_i = -H_i + G_{ab} L^a K_i^b.$$  

This proves

**Proposition 3.** The integral curves of the first order equation $D$ are the integral curves of the vector field $\overline{X}$ based on the points of $C$, i.e., the solutions of the following system of differential equations

$$\frac{dq^i}{dt} = H^i, \quad \frac{dp_i}{dt} = -H_i + G_{ab} L^a K_i^b$$  

whose initial conditions satisfy equations (1).

**Remark 2.** According to the classical terminology, equations (1) are **invariant relations** of the differential system (10). Analogous results, but within the Lagrangian formalism, have been obtained by Synge [13] (see also Agostinelli [1]) for quadratic Hamiltonians, as we shall see in the next section. By the Legendre transformation $\Lambda^{-1}: T^*Q \to TQ$ the vector field $X: C \to TC$ is transformed into a vector field $Y: K \to TK$ over the subbundle $K \subset TQ$. This vector field is the geometric representation of the Gibbs-Appel equations or the Maggi-Volterra equations. The image $Y(K)$ of the vector field $Y$, which is a submanifold of $TK$, is locally represented by equations involving the coordinates $(q^i, \omega^a, \dot{q}^i, \omega^a)$ of $TK$.

**Remark 3.** [Let us introduce on $T^*Q$ the vertical 1-form

$$\phi = G_{ab} L^a K_i^b dq^i.$$  

Then the vector field $\overline{X}$ is intrinsically defined by equation

$$i_{\overline{X}} d\theta = -dH + \phi,$$

where $\theta$ is the Liouville 1-form.]

**Remark 4.** Any other extension can be chosen for finding the integral curves of $D$. To the vector field $\overline{X}$ defined above we can add any arbitrary (smooth) vector field vanishing on the submanifold $C$.

**Remark 5.** The Hamilton-Jacobi method for integrating first order equations can be applied only to Hamiltonian vector fields on cotangent bundles, i.e., to vector fields $Z$ such that $i_Z d\theta$ is an exact form. It follows from Remarks 3 and 4 that if we know a submanifold $J \subset C$ such that (i) $X$ is tangent to $J$, (ii) the pull-back of $\phi$ to $J$ is closed, then we can find a local integral $F :: J \to \mathbb{R}$ of $\phi|J$ and a local extension $\overline{F}$ of this
function on \( T^*Q \). The Hamiltonian vector field \( Z \) generated by \( H' = H - F \) is such that 
\[ Z|J = X|J. \]
Hence, we can apply the Hamilton-Jacobi method to the vector field \( Z \) for 
finding the integral curves of \( X|J \), i.e., the motions of the non-holonomic system lying on 
\( J \).

6. Dynamics with non-linear non-holonomic constraints

When the kinematical constraints are represented by a submanifold \( K \) of \( TQ \) defined 
by local equations
\[ K^a(q^i, \dot{q}^i) = 0 \quad (a = 1, \ldots, l), \]
it is assumed that the dynamics is the first order equation \( D \subset TT^*Q \) locally defined by 
equations
\[ \dot{p}_i - \frac{\partial L}{\partial \dot{q}^i} = 0, \quad \dot{p}_i - \frac{\partial L}{\partial \dot{q}^i} = \lambda_a \frac{\partial K^a}{\partial q^i}, \quad K^a = 0. \]
or by equations
\[ \dot{q}_i - \frac{\partial H}{\partial \dot{q}_i} = 0, \quad \dot{p}_i + \frac{\partial H}{\partial q_i} = \lambda_a \frac{\partial K^a}{\partial \dot{q}_i}, \quad K^a = 0, \]
where \( (\lambda_a) \) are the Lagrange multipliers. Equations (2) follow from the Gauss principle (see 
Prange [11]). If the constraints are linear, then equations (2) and (3) reduce to equations (3) 
and (4) of §3 respectively. We mention here the recent articles by Vershik [14] and Weber 
[15] on geometrical approaches to non-linear constraints which generalize this assumption.

Let \( U \) be the domain of the coordinates \( (q^i) \). Then in the open subset \( T^*U \subset T^*Q \) 
we can define the following functions:
\[ A_i^a = \frac{\partial K^a}{\partial q^i} \bigg|_*, \quad B_i^a = \frac{\partial K^a}{\partial \dot{q}^i} \bigg|_*, \quad C^a = K^a|_*, \quad G^{ab} = A_i^a A_j^b H_{ij}, \]
\[ L^a = A_i^a \left( H^{ij} H_j - H_j^i H^j \right) - B_i^a H^i = \{H, C^a\}, \]
where the symbol \( |_* \) denotes the substitution \( \dot{q}^i = H^i \). By a procedure analogous to that 
of §3 and §4, it can be shown that

**Proposition 4.** If the regularity conditions
\[ \det (H^{ij}) \neq 0, \quad \text{rank}(A^a_i) = l, \quad \det (G^{ab}) \neq 0, \]
are satisfied, then: (i) the subset \( D \subset TT^*Q \) defined by equations (2) is a submanifold of 
dimension \( 2n \); (ii) the subset \( C = \tau_{T^*Q}(D) \subset T^*Q \) locally defined by equations
\[ C^a = 0, \]
is a submanifold of dimension $2n - l$; (iii) the integrable part of $D$ is the image of a vector field $X$ on $C$; (iv) the vector field $X$ is the restriction of a vector field $\bar{X}$ on $T^*U$ with components

\begin{equation}
\bar{X}^i = H^i, \quad \bar{X}_i = -H_i + G_{ab} L^a A^b_i,
\end{equation}

where $(G_{ab})$ is the inverse matrix of $(G^{ab})$.

Remarks analogous to those of §4 hold for non-linear constraints.

A non-holonomic constraint is called homogeneous if $v \in K$ implies $rv \in K$ for each real number $r$. The equations (1) can be chosen to be homogeneous in the coordinates $(q^i)$. Caratheodory [2] pointed out that if the constraints are homogeneous then no work is done by the reaction forces (see also Saletan and Cromer [12]). Then the constraints are said to be ideal.

[For homogeneous (non-linear) constraints the regularity conditions (6) are not fulfilled for $q^1 = 0$, i.e., when the mechanical system is at rest (see the example below). The corresponding singular points of $D$ should be analyzed more closely: in general the integrability is preserved but not the uniqueness.]

It seems that no mechanical system with ideal non-linear non-holonomic constraints is known other than that of Appell (see Fufaev and Neimark [9]). This example, however, suffers of some defects (for criticisms and discussions we refer to Delassus [6], Castoldi [3], Fufaev-Neimark [9] and Pironnau [10]). Castoldi proposed a different example, but its construction seems to be rather complicated. In fact, Hertz pointed out that non-linear constraints can be realized as a limit of linear constraints, when certain masses and distances become negligible. We propose here a simple example, leaving to further investigation the question as to whether it is realistic or not and whether it confirms the theory or not.

**Example.** Two identical rods $r_1$ and $r_2$ move on a plane in such a way that the rods and the velocities $v_1$ and $v_2$ of the midpoints $P_1$ and $P_2$ remain parallel. This constraint can be produced by installing a sharp wheel or a sharp blade (as in an ice skate) at the center of each rod. To guarantee that the two "skates" $r_1$ and $r_2$ remain parallel we constrain four points $(A_1, B_1, C_1, D_1)$ of $r_1$ and corresponding four points $(A_2, B_2, C_2, D_2)$ of $r_2$ to slide without friction along four rigid bars $(a, b, c, d)$ respectively. These four bars can pivot without friction around a common point $P$ which moves freely in the plane. At each configuration the two skates $r_1$ and $r_2$ are in a symmetrical position with respect to the point $P$. The use of four bars instead of three avoids a certain singularity in the construction, which arises when one of the bars is orthogonal to the skates. If we consider on each skate an heavy small body, whose centers of masses $P_1^1$ and $P_2^2$ can move along the skates slightly from the midpoints $P_1$ and $P_2$ respectively, in order that their velocities and those of the skates remain parallel, and we disregard the masses of all the components of the device, then we have constructed a system of two material points $P_1$ and $P_2$ which move in a plane and are constrained to have parallel velocities. This constraint is non-linear and homogenous. It is represented by equation $v_1 \times v_2 = 0$, i.e., by a (single) scalar
homogeneous quadratic equation in the components of the velocities. Unfortunately, the regularity condition (6) is not satisfied for $v_1 = v_2 = 0$. In fact, this singularity is first of all due to the construction. If we leave the two points at rest in a configuration, then we do not know the behaviour of the system without specifying the initial directions of the skates. But this information must be "a priori" ignored because of our assumption of disregarding (all the remaining parts) of the device.

REFERENCES


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