Rend. Sem. Mat. Univ. Pol. Torino
Vol. 53, 4 (1995)
Number Theory

U. Zannier

VANISHING SUMS OF ROOTS OF UNITY

Abstract. In the present survey article we deal with the problem of classifying relations of length k, $\sum_{i=1}^{k} a_i\zeta_i = 0$, where a_i are given complex numbers and ζ_i are roots of unity. We describe results due to several authors and discuss in detail a generalization of a certain theorem due to J. H. Conway and A. J. Jones. The Theorem assumes that the a_i are rational numbers and bounds in a best possible way the primes dividing the common order of the roots ζ_i, provided suitable normalizations have been carried out. The generalization, obtained jointly with R. Dvornicich, deals with arbitrary coefficients. The method of proof differs completely from the one of Conway and Jones, which seems not to extend at once to cover the general case. A crucial role is played by a result in linear algebra which seems to have some independent interest and which has certain features in common with H. B. Mann's theorem on addition of sequences of integers.

The subject to be discussed below originates with the question of classifying equations $\sum_{i=0}^{k-1} \zeta_i = 0$, where ζ_i are roots of unity. Besides some interest of this problem in itself, such relations arise naturally in a number of contexts, as remarked for instance in [3], §7. Also, they constitute the simplest type of the so called *S-unit equations*, whose remarkable applications in Number Theory are well known.

Actually, we shall allow arbitrary nonzero complex coefficients a_i and consider the equation $\sum_{i=0}^{k-1} a_i\zeta_i = 0$; we shall view for the moment k and the a_i's as fixed and try to find out the possibilities for the ζ_i's. On multiplying throughout by ζ_0^{-1}, say, we may normalize the relation. Also, we may write the various relevant roots of 1 as powers of a single one, so our equation becomes

$$a_0 + \sum_{i=1}^{k-1} a_i\zeta^{n_i} = 0 \tag{1}$$

for some root of unity ζ, of exact order Q, say. We may agree that $n_0 = 0$ and choose ζ of order as small as possible, which corresponds to the assumption $(Q, n_1, \ldots, n_{k-1}) = 1$. Our aim is to find a bound for Q in terms of the given data. For such a bound to be possibly true, another normalization has to be carried out, namely we assume (1) to be *irreducible*, i.e. that *no proper nonempty subsum vanishes.* Splitting any relation into a "disjoint union" of irreducible ones shows that this assumption causes no loss of generality; also, examples may be immediately constructed of reducible relations with given length k and given coefficients a_i's, but unbounded Q.[1]

Assuming the relation (1) to be irreducible and the a_i's to be rational numbers, a bound for Q was first produced by H.B.Mann [8] in 1965. By means of a simple but elegant argument, he showed that Q must divide the product $\prod_{p\leq k} p$ of prime numbers up to k. In particular the bound for Q is independent of the coefficients and, by the prime number theorem, takes the form $\log Q \leq k + o(k)$. The irreducibility of the cyclotomic polynomials of prime order proves that each prime $p \leq k$ may in fact occur as a divisor of the common order of the roots, so Mann's theorem is best possible for special values of Q.

For general Q, the result was improved by J.H.Conway and A.J.Jones [3] in 1976; they discussed *trigonometric diophantine equations* from various points of view and proved, among other things, again on Mann's assumptions, that Q is squarefree (which follows from Mann's theorem too) and that (Thm. 5 in [3])

$$\sum_{p|Q}(p-2) \leq k-2. \tag{2}$$

Also, they produced examples proving that the inequality is best possible for every choice of squarefree Q. As remarked in [3], (2) leads to $\log Q \leq C\sqrt{k\log k} + O(1)$ for every $C > 1$, improving on the estimate which comes from Mann's result.

Next, comes the analogous question for coefficients a_i lying in a number field L of degree d, say (we shall see in a moment that the case of general a_i's may be easily reduced to this). After a partial result by J.H.Loxton [7], A.Schinzel [10] proved in 1988 (as a lemma) that there is some bound for Q depending only on k and d. His proof used Van der Waerden's theorem on arithmetic progressions and so led to enormous values for such a bound. Although this was sufficient for Schinzel's purposes, the problem remained open to obtain a more realistic estimate. It is to be remarked that a straightforward adaptation of Mann's method leads to a bound depending on the discriminant of L too, not just on k and d. Before going on we point out two simple but important remarks, stated explicitly in [4], but appearing already in [12]:

Let (1) *be irreducible and set* $L := \mathbb{Q}(a_0, \ldots, a_{k-1})$. *Then there exists an irreducible relation* $a_0^* + \sum_{i=0}^{k-1} a_i^* \zeta^{n_i} = 0$, *where* $a_i^* \in L_0 := \mathbb{Q}(\zeta) \cap L$.

---

[1] if $\Sigma_1 = \Sigma_2 = 0$ are two relations, consider $\Sigma_1 + \xi\Sigma_2$ for any root of unity ξ.

(This shows in particular why we can reduce to algebraic coefficients.) We sketch the proof. Let $w_j, j \in J$ run through a basis for the vector space generated by the a_i's over L_0, and write $a_i = \sum_j a_{i,j} w_j$ with coefficients in L_0. Plugging into (1) and observing that L and $\mathbb{Q}(\zeta)$ are linearly disjoint over L_0 we derive equations $\sum_{i=0}^{k-1} a_{i,j}\zeta^{n_i} = 0$, for $j \in J$. We can form linear combinations of the equations, with coefficients $\xi_j \in L_0$ to produce a new relation $\sum_{i=0}^{k-1}(\sum_j \xi_j a_{i,j})\zeta^{n_i} = 0$, hopefully irreducible. Plainly the set of vectors $(\xi_j)_{j\in J} \in L_0^J$ which correspond to a given possible splitting of the relation, form a vector subspace of L_0^J. The possible splittings are finite in number, so, if each such subspace is proper, their union is strictly contained in L_0^J and we can choose any $(\xi_j)_{j\in J}$ outside the union. If some splitting gave the entire space L_0^J, then the original relation would also split in the same way, a contradiction.

This observation allows us to assume at once $L \subset \mathbb{Q}(\zeta)$. Now, it is possible to show that $p^{a+1}|Q \Rightarrow p^a|d$ (see [12], p. 175 or [4] for two simple different arguments). This allows us to restrict to primes $p||Q$, since we can bound quite well the others.

Combining Mann's method with certain supplementary considerations of algebraic and combinatorial nature, U.Zannier [12] proved in 1989 a result which may be considered as the analogue of Mann's for the case of algebraic a_i and which led in particular to the inequality $\log Q \leq c\frac{\sigma_0(d)d}{\varphi(d)}\log(dk)\frac{k}{\log k}$, where c is a certain absolute constant and $\sigma_0(d)$ is the number of divisors of d.[2] But here I would like to discuss in some detail the analogue of the Conway-Jones's theorem rather than this, so, instead of giving any detail about the proofs in [12], I will recall an assertion that turned out to be a byproduct of those methods, which will be useful later. Namely we have (see Remark 3 in [12]):

If $p\|Q$ then the number of incongruent ones modulo p among the n_i's is at least

$$1 + \frac{p-1}{(p-1,d)}. \tag{3}$$

Observe, by the way, that this is sufficient to produce a bound for Q depending only on k, d; also, as in the case of Mann's theorem, the inequality is the best that we can say taking into account a single prime factor of Q.

We remark that the very ingenious proof of (2) given in [3] does not seem to extend at once to the case of general coefficient field. Together with R.Dvornicich we have obtained, on completely different lines, a result which holds for general L and, in case $L = \mathbb{Q}$, gives an inequality which is even more precise than (2).[3]

---

[2] A treatment of (1) in the case of complex coefficients appears also in Lemma 3.1, p. 40 of [1]. It is proved that, even if the relation is reducible, $\zeta^{n_i - n_j}$ has order bounded in terms of the a_i's, for some $i \neq j$. This follows also from both [10] and [12].

[3] The proof of Thm. 5 in [3] also leads to a result more precise than (2). A third independent proof of (2) has been obtained by W.M.Schmidt (unpublished).

To state it, let $G := \mathrm{Gal}(\mathbb{Q}(\zeta)/L)$ and define $\zeta_i^* : G \to \mathbb{C}$ by $\zeta_i^*(\sigma) = \sigma(\zeta^{n_i})$. Then we have

THEOREM 1. *If* (1) *is irreducible, then*

$$\dim_{\mathbb{C}}\langle \zeta_0^*, \ldots, \zeta_{k-1}^*\rangle \geq 1 + \sum_{p\|Q}\left(\frac{p-1}{(p-1,d)} - 1\right).$$

Here and in the sequel $\langle v_0, \ldots, v_s\rangle$ denotes the vector space generated by the v_i. We remark that it is possible to prove that the left side of the inequality is just the dimension of the numbers ζ^{n_i} over L; however the proof given below leads naturally to the above statement.

The left side is in any case $\leq k-1$, so when $d = 1$ we obtain (2). On the other hand the theorem does not seem to follow formally from (2) even in the case $L = \mathbb{Q}$.

The proof we have found works by induction on the number r of primes $p|Q$. When $r = 1$, a simple argument suffices to derive the result from (3) (see [4]). To operate the induction we use a general result in linear algebra, which seems to have some interest in itself and represents the crucial step. To state it we need a few definitions. For S a set, let $A(S)$ denote the space of complex functions on S. For $f \in A(X), g \in A(Y)$ define $f \star g \in A(X \times Y)$ by $(f \star g)(x,y) = f(x)g(y)$.

Let now X, Y be arbitrary sets and let $f_1, \ldots f_m \in A(X)$ generate a space of dimension a and $g_1, \ldots, g_m \in A(Y)$ generate a space of dimension b. We have

THEOREM 2. *Assume we have an irreducible relation* $\sum_{i=1}^m f_i \star g_i = 0$. *Then* $f_i \star g_i$, $1 \leq i \leq m$, *generate a space of dimension* $\geq a + b - 1$.

In particular we deduce $m \geq a + b$, but this may be proved trivially in a direct way, even without assuming irreducibility. An equivalent, and perhaps better, statement may be obtained letting f_i, g_i be elements of vector spaces V, W resp., and replacing $f_i \star g_i$ with the tensors $f_i \otimes g_i$. The above formulation turns out to be more suitable for our application. Also, one may deduce an analogous result valid for any number of sets of m functions.

Before giving a sketch of the proof of Theorem 2 we show how Theorem 1 follows from it. Let $Q = pQ_1$, where $p \nmid Q_1$ and write $\zeta = \rho\psi$, where ρ, ψ are suitable primitive p-th and Q_1-th roots of unity resp.. Equation (1) reads $\sum_{i=0}^{k-1}(a_i\rho^{n_i})\psi^{n_i} = 0$ and may be considered as an irreducible relation among the ψ^{n_i}, with coefficients in $L(\rho)$. We have remarked that this implies a similar relation, also irreducible, with coefficients in $F := L(\rho) \cap \mathbb{Q}(\psi)$. Let $G_1 = \mathrm{Gal}(\mathbb{Q}(\psi)/F)$ and consider the functions $\psi_i^* : G_1 \to \mathbb{C}$

defined by $\psi_i^*(\sigma) = \sigma(\psi^{n_i})$. Then, by induction, we have

$$\dim_{\mathbb{C}}\langle \psi_0^*, \dots, \psi_{k-1}^* \rangle \geq 1 + \sum_{\ell \| Q_1} \left(\frac{\ell - 1}{(\ell - 1, d_1)} - 1 \right)$$

where $d_1 = [F : \mathbb{Q}]$; simple Galois theory shows that $d_1 | d$, so the inequality holds with d in place of d_1. A similar argument, reversing the roles of ρ, ψ, shows that

$$\dim_{\mathbb{C}}\langle \rho_0^*, \dots, \rho_{k-1}^* \rangle \geq \frac{p-1}{(p-1, d)}.$$

Again, simple Galois theory shows that G contains a subgroup isomorphic to $G_1 \times G_2$ such that $\zeta_i^*(g_1, g_2) = \psi_i^*(g_1)\rho_i^*(g_2)$ and Theorem 2 finally gives what we want.

We shall now sketch a proof of Theorem 2. This is based on the treatment of the following related problem. Suppose $\mathcal{H}, \mathcal{W}$ are subspaces of $A(S)$ of finite dimensions γ, δ. Define $\langle \mathcal{H}\mathcal{W} \rangle$ as the space generated by $\mathcal{H}\mathcal{W} := \{hw : h \in \mathcal{H}, w \in \mathcal{W}\}$ and let ϕ be its dimension. We would like to bound ϕ from below. Here we have an analogy with certain results on *addition of sequences of natural numbers or residue classes*: well known theorems of Cauchy-Davenport-Chowla (for the case of integers modulo m) and H.B.Mann (for the case of infinite sequences) give, under appropriate conditions, a lower bound for the cardinality (or the density, as the case may be) of the *sum* of two sequences in terms of the *sum* of the analogous quantities for the sequences in question. Here the *product* of functions replaces the sum of integers and the *dimension* replaces the cardinality (or the density). In analogy we would like to compare ϕ with $\gamma + \delta$ (in contrast with the upper bound $\gamma\delta$). We can see this analogy also in the proof of Lemma 1 below, which has certain features in common with the proofs of the quoted theorems, as given e.g. in [5].

An obstruction for $\phi \geq \gamma + \delta$ to hold is certainly the existence of many elements $h \in \mathcal{H}$ such that $h\mathcal{W} \subset \mathcal{W}$, for example. In fact we have, somewhat conversely,

LEMMA 1. *Assume that $\mathcal{H}$ contains the function, denoted $\mathbf{1}$, with constant value 1. There exist spaces $\mathcal{H}' \subset \mathcal{H}$, $\mathcal{W}' \supset \mathcal{W}$ such that*

(i) $\mathbf{1} \in \mathcal{H}'$.

(ii) $\mathcal{H}'\mathcal{W}' \subset \mathcal{W}' \subset \langle \mathcal{H}\mathcal{W} \rangle$.

(iii) $\dim \mathcal{H}' + \phi \geq \gamma + \delta$.

We remark that the assumption about $\mathbf{1}$ (which could be relaxed) will be relevant later. The proof proceeds by induction on γ. If $\gamma = 1$ we just choose $\mathcal{W}' = \mathcal{W}$, $\mathcal{H}' = \mathcal{H}$. Assume $\gamma \geq 2$ and the lemma true up to $\gamma - 1$. If $\mathcal{H}\mathcal{W} \subset \mathcal{W}$ we again choose $\mathcal{W}' = \mathcal{W}$, $\mathcal{H}' = \mathcal{H}$. Otherwise, let $t \in \mathcal{W}$ be such that $\mathcal{H}t \not\subset \mathcal{W}$ and define L_t as the linear map

$$L_t : \mathcal{H} \to \frac{\langle \mathcal{H}\mathcal{W} \rangle}{\mathcal{W}}, \qquad L_t(v) = vt + \mathcal{W}.$$

Also, define $\mathcal{H}'' := \ker L_t$, $\mathcal{W}'' := \mathcal{W} + \mathcal{H}t$ and let γ'', δ'' be resp. the dimensions of these spaces. We have, plainly,

$$\gamma'' + \delta'' = \gamma + \delta, \qquad \gamma > \gamma''. \tag{4}$$

Moreover $\mathbf{1} \in \mathcal{H}''$. Observe that

$$\langle \mathcal{H}''\mathcal{W}'' \rangle \subset \langle \mathcal{H}\mathcal{W} \rangle. \tag{5}$$

In fact $\mathcal{H}''\mathcal{W} \subset \mathcal{H}\mathcal{W}$ trivially, while $\mathcal{H}''\mathcal{H}t = \mathcal{H}\mathcal{H}''t \subset \mathcal{H}\mathcal{W}$ in view of the definition of $\mathcal{H}''$. Since $\gamma > \gamma''$ we may apply the induction hypothesis to $\mathcal{H}''$ and $\mathcal{W}''$. It is now an easy matter to conclude, using (4) and (5).

Lemma 1 contains the kernel of the proof of Theorem 2. To complete it we however need another lemma.

LEMMA 2. *Let $S = \{1, 2, \ldots, m\}$ and let $\mathcal{W} \subset \mathcal{F}$ be subspaces of $A(S)$. Define $\mathcal{H} := \{\lambda \in A(S) : \lambda\mathcal{W} \in \mathcal{F}\}$. Then, either* $\dim \mathcal{H} + \dim \mathcal{W} \leq 1 + \dim \mathcal{F}$ *or $\mathcal{H}$ contains the characteristic function of a suitable proper nonempty subset of S.*

For the proof, apply Lemma 1 to $\mathcal{H}$ and $\mathcal{W}$. Observe that $\langle \mathcal{H}\mathcal{W} \rangle \subset \mathcal{F}$ by definition so, if the first alternative is not verified we have that, by (iii) of Lemma 1, $\dim \mathcal{H}' \geq 2$. If $\mathcal{A}$ is the subalgebra of $A(S)$ generated by $\mathcal{H}'$, then $\mathcal{A} \subset \mathcal{H}$. In fact, by iteration of (ii) of Lemma 1, we have $\mathcal{A}\mathcal{W}' \subset \mathcal{W}'$. In particular $\mathcal{A}\mathcal{W} \subset \mathcal{W}' \subset \mathcal{F}$ and the claim follows in view of the present definition of $\mathcal{H}$. Since $\dim \mathcal{H}' \geq 2$, there exists some nonconstant function $\mathbf{f} \in \mathcal{H}'$. Set $\mathbf{f}(S) = \{f_1, \ldots, f_h\}$, so $h \geq 2$. Now, it is immediate to verify that the function $\chi := (\prod_{i<h}(f_h - f_i))^{-1} \prod_{i<h}(\mathbf{f} - f_i \cdot \mathbf{1})$, which belongs to $\mathcal{A}$, is the characteristic function of $\{x \in S : \mathbf{f}(x) = f_h\}$, which is a proper nonempty subset of S.

We can now conclude the proof of Theorem 2. Let $S = \{1, \ldots, m\}$, so $A(S) = \mathbb{C}^m$. Let $\mathcal{H} \subset \mathbb{C}^m$ be the subspace

$$\mathcal{H} := \{(\lambda_1, \ldots \lambda_m\} : \sum_{i=1}^{m} \lambda_i f_i \star g_i = 0\}$$

and set

$$\mathcal{F} := \{(\xi_1, \ldots, \xi_m) : \sum_{i=1}^{m} \xi_i f_i = 0\}.$$

For $y \in Y$ write $g(y) := (g_1(y), \ldots, g_m(y))$. We have

$$\mathcal{H} = \{\lambda = (\lambda_1, \ldots, \lambda_m) : \lambda g(y) \in \mathcal{F}, \forall y \in Y\}.$$

Hence, defining $\mathcal{W}$ as the subspace of $A(S)$ generated by the vectors $g(y), y \in Y$, we are in position to apply Lemma 2. Since the relation $\sum f_i \star g_i = 0$ is irreducible, we see that

$\mathcal{H}$ cannot contain the characteristic function of any nonempty proper subset of S, whence the first alternative of that lemma holds, i.e. $\dim \mathcal{H} + \dim \mathcal{W} \leq 1 + \dim \mathcal{F}$. But it is readily verified that $\dim \mathcal{H} = m - \dim(f_1 \star g_1, \ldots, f_m \star g_m)$, $\dim \mathcal{F} = m - a$, $\dim \mathcal{W} = b$. This completes the proof.

We may mention that Theorem 2 admits an application, analogous to the present one, to the problem of the classification of linear relations among a given number of characteristic functions of arithmetical progressions, a subject having to do with the so-called *covering systems* of congruences, but appearing in other contexts as well (see [4]).

We conclude by briefly mentioning a few other problems involving equations in roots of unity. First, there was a conjecture of Lang saying that, for an irreducible polynomial $f \in \mathbb{C}[x, y]$, either the equation $f(\zeta_1, \zeta_2) = 0$ has finitely many solutions in roots of unity ζ_1, ζ_2, or f is a polynomial of the form $aX^n + bX^m$ or $cX^nY^m + d$. Equivalently, Lang formulates the statement by saying that *if a curve has infinite intersection with the group of torsion points on a torus, then the curve is the translation of a subtorus.*

This appears as Theorem 6.1, p. 201 of [6]; proofs by Ihara, Serre and Tate are quoted and Tate's proof is reproduced. Also, Liardet's proof is given of the more precise Theorem 6.4, p. 203. That theorem makes the above statement explicit, by bounding the maximum order of a torsion-solution of $f(\zeta_1, \zeta_2) = 0$ in terms of the degree of f and the degree of the field L generated by the coefficients (provided of course f is not of the above exceptional type).[4] These problems have been discussed, in more general form, also by Ruppert [9]. Among other things, he gives in certain cases best possible bounds for the number of solutions.

Forgetting the precise form of the mentioned bounds, it is easy to derive such results from what we have discussed previously. In fact, the equation $f(\zeta_1, \zeta_2) = 0$ may be plainly written as a vanishing linear combination of roots of unity, with coefficients in L. This relation may not be irreducible, but a possible nontrivial vanishing subsum corresponds to an equation $g(\zeta_1, \zeta_2) = 0$, where g is a certain polynomial, depending on the subsum, whose terms are a subset of the terms appearing in f. If f is an irreducible polynomial then, by Bezout's theorem, the equations $f = g = 0$ have finitely many common roots, their number depending only on $\deg f$. So we may deal with the reducible vanishing sums. To the remaining ones we may apply e.g. Theorem 1 to complete the argument: the conclusion will be that, if $x^a y^b$ and $x^c y^d$ are any two terms appearing in f then the order of $\zeta_1^{a-c}\zeta_2^{b-d}$ is bounded in terms of $\deg f$ and the degree of L. This will imply a bound for the orders of ζ_1, ζ_2, except when f has the above special form. (Due to the use of Bezout's theorem

[4] Liardet actually proved more general results on torsion points of $\mathbb{C}^*$ modulo a finitely generated subgroup. See [6] for references and description of this work.

the final bound will depend on the degree of f rather than the number of its terms only.)

We remark that the bound so obtained will be effective. We also point out that a similar argument will prove an analogous result valid for polynomials in any number of variables.

More recently S. Zhang [13], [14] and E. Bombieri, U. Zannier [2] studied the distribution of points of small height on subvarieties of $\mathbf{G}_m^n$, where the height is now defined by taking the sum of the Weil height of the coordinates. We do not discuss in detail the results of such papers, but we remark that they are relevant here, since the points of $\mathbf{G}_m^n$ having zero height are precisely the torsion points, those whose coordinates are roots of unity. As special cases of such results, one finds again the above mentioned conjecture of Lang, in a far more general form (an intermediate result was proved in [9]: he studied algebraic subvarieties of $\mathbf{G}_m^n$ such that torsion points are Zarisky dense). Also, Theorem 1 of [2] is completely uniform with respect to the field of definition and yields, as a very special case, the following: *consider an algebraic subvariety X of $\mathbf{G}_m^n$, defined by equations of degree at most d, having algebraic coefficients. Consider the set $\tilde{X}$ defined as the union of translates of nontrivial subtori of $\mathbf{G}_m^n$ contained in X.*[5] *Then the number of torsion points in $X - \tilde{X}$ is bounded, depending only on d and the ambient dimension n.*

As a special case, one can easily see that, given nonzero complex numbers $a_0, \ldots a_{k-1}$, the number of *irreducible* vanishing sums $a_0 + \sum_{i=1}^{k-1} a_i\zeta_i = 0$, where ζ_i are roots of unity, is finite, depending only on k, but *not* on the a_i's.

In fact, consider the subvariety $X \in \mathbf{G}_m^{k-1}$ defined by $a_0 + \sum a_i x_i = 0$, and consider a translate ξT of a nontrivial subtorus contained in X. ξT will be given by parametric equations of the type $x_i = \xi_i \mathbf{u}^{v_i}$, where $\xi_i \in \mathbb{C}^*$, $\mathbf{u} := (u_1, \ldots, u_s)$, $s \geq 1$, and where the matrix of the $v_i \in \mathbb{Z}^s$ has maximal rank s. Moreover we shall have, identically in the u_i's, $a_0 + \sum a_i \xi_i \mathbf{u}^{v_i} = 0$. Plainly, if S denotes the subset of $\{1, \ldots, k-1\}$ consisting of indices i such that $v_i = 0$, then S will be proper (since $s \geq 1$) and we have $a_0 + \sum_{i \in S} a_i x_i = 0$. So, if a point $(\zeta_1, \ldots, \zeta_{k-1}) \in X$ actually lies in $\tilde{X}$, then the equation $a_0 + \sum_{i=1}^{k-1} a_i\zeta_i = 0$ will not be irreducible, which clearly concludes the proof of the claim.

We remark that a different proof of this corollary has been given by H.P.Schlickewei [11], who obtains moreover the explicit bound $2^{3k!}$ for the total number of irreducible vanishing sums (which he calls *nondegenerate solutions*).

[5] It turns out that $\tilde{X}$ is Zariski closed in X.

REFERENCES

[1] ADAMS S., SARNAK P., *Betti numbers of congruence groups*, Israel J. of Math. **88** (1994), 31-70.

[2] BOMBIERI E., ZANNIER U., *Algebraic points on subvarieties of* $\mathbf{G}_m^n$, Int. Math. Res. Notices **7** (1995), 333-347.

[3] CONWAY J. H., JONES A. J., *Trigonometric diophantine equations (On vanishing sums of roots of unity)*, Acta Arith. **30** (1976), 229-240.

[4] DVORNICICH R., ZANNIER U., *Hypergeometric functions, arithmetic progressions and sums of roots of unity*, preprint Univ. Pisa, n. 644, 1992.

[5] HALBERSTAM H., ROTH K. F., *Sequences*, Springer-Verlag, 1983.

[6] LANG S., *Fundamentals of Diophantine Geometry*, Springer-Verlag, 1983.

[7] LOXTON J. H., *On two problems of R.M.Robinson about sums of roots of unity*, Acta Arith. **26** (1974), 159-174.

[8] MANN H. B., *On linear relations between roots of unity*, Mathematika **12** (1965), 107-117.

[9] RUPPERT W. M., *Solving algebraic equations in roots of unity*, J. Reine Angew. Math. **435** (1993), 119-156.

[10] SCHINZEL A., *Reducibility of lacunary polynomials VIII*, Acta Arith. **50** (1988), 91-106.

[11] SCHLICKEWEI H. P., *Linear equations in roots of unity*, to appear in Acta Arithmetica.

[12] ZANNIER U., *On the linear independence of roots of unity over finite extensions of* $\mathbb{Q}$, Acta Arith. **50** (1989), 171-182.

[13] ZHANG S., *Positive line bundles on arithmetic surfaces*, Annals of Math. **136** (1992), 569-587.

[14] ZHANG S., *Positive line bundles on arithmetic varieties*, J. Amer. Math. Soc. **8** (1995), 187-221.

Umberto ZANNIER
Istituto Universitario di Architettura D.C.A.
S. Croce, 191 (Tolentini)
30135 Venezia, Italy.