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# RIGID GEOMETRY AND GALOIS EXTENSIONS OF FUNCTION FIELDS IN ONE VARIABLE

**Abstract.** In this paper we review some recent results in the theory of Galois coverings of curves obtained by rigid geometric methods. Our starting point is a $p$-adic analogue of the Inverse Galois Problem. We then introduce rigid analytic spaces and show how they may be used as a tool to construct coverings.

My purpose is to give an overview on some recent results on Galois extensions of function fields obtained with formal and rigid analytic methods.

Rigid geometry allows to give meaning to global geometric constructions over $p$-adic fields, an extremely useful tool to define finite coverings of curves.

This theory attracted the attention of number theorists for the first time in 1987, when appealing to it, or rather to its formal counterpart, Harbater [8] showed that any finite group occurs as Galois group over the field $\mathbb{Q}_p$. We will follow here this historical guideline and use this approach to Galois Inverse Problem as a pretext for the introduction of rigid analytic spaces.

It is worth mentioning that, although this theory is perfectly suited for general local fields, so far almost all the results obtained are of *geometric* nature, that is, relative to ground fields which are algebraically closed or freely extended when needed. This paper may also be considered as an invitation to arithmeticians to use their skills to *descend* these results to local fields suitable for number theoretical applications.

## 1. Inverse Galois problem

The Inverse Galois Problem in Number Theory is the one to characterize the finite groups which are Galois groups of some extension of $\mathbb{Q}$, or of a more general number field; for a complete introduction we refer to Serre [27], and to [19] for the ultimate results.

According to Hilbert's Irreducibility Theorem, a finite group $G$ occurs as Galois group over a number field $K$ if it occurs as Galois group of an extension of the function field in one variable $K(T)$. From an algebraic geometer's point of view, such an extension is equivalent to a finite covering of $\mathbb{P}^1_K$, the projective line over $K$. The idea for this criterion is to look for $K$-rational points of $\mathbb{P}^1_K$ which remain inert in this covering: specialization to these places provides the required extension of $K$.

From this point of view, the Inverse Galois Problem becomes a problem of (branched) coverings of algebraic curves.

Of course, one is interested in *regular* extensions $E$ of $K(T)$, i.e. such that $K$ is algebraically closed in $E$. In terms of algebraic geometry, one looks for *geometrically connected* (branched) coverings of the projective line over $K$.

Similar arguments hold also for extensions of function fields of several variables and for the corresponding higher dimensional varieties.

See Serre [27], chapt. IV for a number of examples of extensions of $\mathbb{Q}$ constructed in this way.

In order to have more freedom than that allowed by the Zariski topology, a classical argument consists in embedding the field $K$ into $\mathbb{C}$ and, using the complex topology, perform a finite (analytical) branched covering of the Riemann sphere $\mathbb{P}^1(\mathbb{C})$.

Now, a branched covering of the Riemann sphere is just a topological covering of the open complement of its ramification locus. This is a consequence of Riemann's Existence Theorem: any topological covering of a pointed disc extends to a finite branched covering of the whole disc.

A well-known result of algebraic topology states that the topological fundamental group of a compact connected Riemann surface of genus $g$ with $s$ points deleted is the group generated by $2g+s$ elements $a_1, \ldots, a_g, b_1, \ldots, b_g, c_1, \ldots, c_s$ and the relation

$$[a_1; b_1] \ldots [a_g; b_g] c_1 \ldots c_s = 1.$$

Therefore, if $s > 0$, it is a free group with $2g+s-1$ generators .

Hence, by taking $s$ large enough, any finite group can be realized as Galois group of a branched covering of the Riemann sphere.

Such a finite covering is defined by a coherent sheaf of finite $\mathcal{O}^{an}_{\mathbb{P}^1}$-algebras. In view of the *GAGA* principle, this is equivalent to a coherent algebraic sheaf $\mathcal{O}_{\mathbb{P}^1}$-algebras, so this covering is algebraic.

Now the problem becomes that of descending this covering from $\mathbb{C}$ to $K$.

This is of course a rather delicate (and still unsolved) problem. Although one has enough freedom to build extensions of $\mathbb{C}(T)$ descending to $\bar{\mathbb{Q}}(T)$ and hence to $L(T)$, for some finite extension $L/K$, the descent to $K$ cannot rely only upon the geometric construction but requires some arithmetical argument.

Let $K_v$ be the completion $K$ at some finite place $v$. One could try to apply the method outlined above. This can be done only if one has at his disposal a local and global $v$-adic analytic geometry over $K_v$, compatible with algebraic geometry (e.g. satisfying the GAGA-principles).

The theory of *rigid analytic spaces* provides the convenient setting in which these programs can be realized.

In the sequel we want to briefly introduce these spaces and sketch how they were used by Harbater ([8]) to establish a $p$-adic analogue of the geometric approach to Inverse Galois Problem. Then we will review a couple of recent results in the theory of coverings of curves over $p$-adic or characteristic $p$ fields obtained by rigid methods.

## 2. Formal and rigid geometry

Rigid analytic spaces were first introduced by Tate in [28]. He needed them for $p$-adic uniformization of elliptic curves (the famous Tate curve). The $p$-adic uniformization was soon extended to higher genus curves and abelian varieties (Mumford, Raynaud, van der Put). On the other hand, rigid analytic spaces were developed for their own interest in analogy with the theory of complex analytic spaces, especially by the german school (Kiehl, Grauert and Remmert).

The connection with formal geometry was established by Raynaud [21], thus allowing one to apply the powerful EGA-machine. Many expected results were easily recovered as for example the GAGA principle, first established by Kiehl.

Since then, the theory has expanded dramatically; it is today of constant use in nearly all domains of $p$-adic algebraic and arithmetic geometry.

It is not our intention to give here a full introduction to formal and rigid geometry. We merely give some definitions and recall some facts that are needed for constructions of Galois coverings of algebraic curves. In particular, we will often restrict ourselves to one-dimensional spaces. However, from the formal point of view, we are dealing with relative curves over complete discrete valuation rings, hence two-dimensional spaces.

Apart from the "analytic spaces"point of view, explained in full details in the treatise of Bosch Güntzer and Remmert [3], there seems to be no definitive textbook on the subject.

To the interested reader, we suggest as a first reading the book of Fresnel and van der Put [7]. There are also handwritten notes by Fresnel, based on a graduate course given at the University of Bordeaux. The formal-rigid connection is explained in the short notes [21] and in the papers of Bosch and Lütkebohmert [14] and Lütkebohmert [4] in more details.

There are also (reasonably) accessible references to some more advanced topics, such as $p$-adic uniformization of curves (Gerritzen-van der Put [13]) and cohomology, in connection with other $p$-adic cohomology theories (Berthelot [1]).

Due to this lack of general references, quite often research papers include sections recalling definitions and basic facts. Hence, if one is interested in some specific result, with some faith and a bit of imagination, he can hope to make it through directly.

Let $K$ be a complete local field, with respect to some discrete valuation, $R$ its ring of integers, $\pi$ a generator of the maximal ideal and $k$ the residue field, which we will assume of characteristic $p > 0$.

A *Tate algebra* over $K$ is a topological $K$-algebra isomorphic to a quotient of a restricted power series algebra $K\{T_1, \dots, T_n\}$. Recall that a formal power series $F = \sum_I a_I \underline{X}_I$ is said to be restricted if $a_I \to 0$ as $|I| \to \infty$.

Two basic one-dimensional examples are the algebra $K\{X\}$, of series converging on the open unit disc of $K$ and the algebra $K\{X, Y\}/(XY - \pi^n)$ of formal power series converging in the annulus $C(|\pi^n|) = \{x \in K : |\pi^n| \leq |x| \leq 1\}$.

If $f_0, \dots, f_n$ is a set of elements of a Tate algebra $A$ generating the unit ideal, let $B = A\{Y_1, \dots, Y_n\}/\langle f_1 - f_0 Y_1, \dots, f_n - f_0 Y_n\rangle$. The inclusion morphism $A \to B$ defines a map between the *maximal spectra* $\mathrm{Spm}\, B \to \mathrm{Spm}\, A$ of these Tate algebras, sending $\mathrm{Spm}\, B$ onto the subset $U = \{x \in \mathrm{Spm}\, A : |f_i(x)| \leq |f_0(x)|\}$ of $\mathrm{Spm}\, A$.

The subsets of $\mathrm{Spm}\, A$ of the type just described are called *standard open subsets* of $\mathrm{Spm}\, A$. The subsets of $\mathrm{Spm}\, A$ that can be covered by a finite number of standard open subsets are the open sets of a Grothendieck topology over $\mathrm{Spm}\, A$. An open covering of $\mathrm{Spm}\, A$ is said to be *admissible* if it may be refined by a finite standard open covering.

This topological space $X = \mathrm{Spm}\, A$ may be equipped with a sheaf of rings $\mathcal{O}_X$ such that for any standard open subset $U = \mathrm{Spm}\, B$ of $X$, $\Gamma(U, \mathcal{O}_X) = B$.

The ringed space $(X, \mathcal{O}_X)$ is called an *affinoid rigid analytic space.*

One defines the category of affinoid rigid analytic spaces in an obvious manner and many usual properties of affine schemes translate immediately.

The spaces $D = \mathrm{Spm}\, K\{X\}$ and $C(r) = \mathrm{Spm}\, K\{X, Y\}/(XY - \pi^n)$ are called the closed unit disc and the closed annulus of radius $r = |\pi^n|$.

The link between affinoid rigid analytic spaces and affine formal $R$-schemes is very simple. A noetherian affine formal scheme is just the formal spectrum $\mathfrak{X} = \mathrm{Spf}\, \mathcal{A}$ of a noetherian complete $R$-algebra topologically of finite type. This means that $\mathcal{A}$ is isomorphic to a quotient of a restricted power series algebra $R\{T_1, \dots, T_r\}$ modulo an ideal $\mathcal{I}$. To the formal scheme $\mathfrak{X} = \mathrm{Spf}\, \mathcal{A}$ we can thus associate the affinoid rigid analytic space $X_K = \mathrm{Spm}\, (\mathcal{A} \otimes_R K)$, i.e. the maximal spectrum of the Tate algebra $K\{T_1, \dots, T_r\}/(\mathcal{I} \otimes K)$.

An example of a formal model for the disc $D$ above, is the formal disc $\mathfrak{D} = \mathrm{Spf}\, R\{X\}$, which is the $\pi$-adic completion of the affine line $\mathbf{A}^1_R$ along its special fibre $\mathbf{A}^1_k$.

The idea of general rigid space is less intuitive, and can be reached from two different (but luckily equivalent) approaches.

Roughly speaking, a rigid analytic space over $K$ should be the *generic fibre* of a formal scheme over $R$, just as the scheme $X \times_R K$ is the generic fibre of a scheme over $R$. Conversely, a formal scheme should be an integral model of a rigid space.

One possible way to introduce general rigid spaces is to crudely define the category of separated quasi-compact rigid spaces as the *localization* of the category of separated formal $R$-schemes of finite type and complete (with respect to the $\pi$-adic topology) by making *admissible blowing-up* invertible. By definition, the blowing-up of a coherent sheaf of ideals $\mathcal{I}$ is said to be admissible if $\mathcal{I}$ contains some power of $\pi$.

If we do this, just by definition, any (separated, complete, of finite type) formal scheme will define a rigid analytic space. A morphism of rigid analytic spaces whose formal representatives are $\mathfrak{X}_1$ and $\mathfrak{X}_2$ is given by a triple $(\mathfrak{Y}, s, f)$ where $\mathfrak{Y}$ is a formal scheme, $f : \mathfrak{Y} \to \mathfrak{X}_1$ a morphism of formal schemes and $s : \mathfrak{Y} \to \mathfrak{X}_2$ an admissible blowing-up.

The idea for such a definition is clearly that a generic fibre (whatever this means) should not be affected by a blowing-up in the special fibre. A more serious reason is that it provides a link between usual open coverings (i.e. coverings by the complements of the zero locus of sections of a line bundle). The choice of an integral model is done by chasing denominators and this leads to admissible sheaves of ideals that must be blown up in order to make them invertible.

A much more concrete definition is obtained by associating to a formal scheme $\mathfrak{X}$ a topological space $X_K$ with a sheaf of rings which are defined by gluing the affinoid pieces Spm $(\mathcal{A}_i \otimes K)$, where $\mathfrak{X} = \bigcup_i \mathrm{Spf}\, \mathcal{A}_i$ is a finite affine formal covering. If $\mathcal{I}$ is an admissible ideal of $\mathcal{O}_{\mathfrak{X}}$, (i.e. containing some power of $\pi$) then $\mathcal{I} \otimes K$ is the unit ideal of $\mathcal{O}_{X_K}$ and its blowing-up has no effect on $X_K$.

Hence any separated formal $R$-scheme of finite type canonically defines a separated quasi-compact rigid $K$-space (its generic fibre). Conversely, any separated quasi compact rigid space is the generic fibre of several formal schemes (its $R$-models), differing by an admissible blowing-up. Note that in this way, affinoid rigid spaces have non-affine models (see the example at the end of this section).

Any coherent sheaf on a formal scheme defines a coherent sheaf on its generic fibre (by tensorization with $K$) and any coherent sheaf on a rigid space comes from any of its models (by chasing denominators).

Proper rigid spaces correspond to proper formal schemes, and we recover the usual theorems about finiteness of coherent cohomology. If a formal scheme $\mathfrak{X}$ is *algebraisable* (i.e. $\mathfrak{X} = \hat{X}$ is the separated completion of an $R$-scheme $X$ along its special fibre $X_k = X \times_R k$) then the associated rigid space $X_K$ is isomorphic to the $\pi$-adic analytic space $X_K^{an}$ underlying the generic fibre $X \times_R K$ of the scheme $X$.

As a consequence, for any proper rigid space, the choice of a model allows one to apply the "formal GAGA"theorems of EGA III. This fact deserves the name of *rigid GAGA principle.*

As a rigid space $X_K$ comes with some model(s) $\mathfrak{X}$ attached to it, it comes also with a *special fibre* $X_k$, namely the special fibre of the (chosen) model and with a *morphism of specialization* $X_K \to X_k$. This is a very important feature of rigid spaces and it is what really makes them different from the analogous complex analytic spaces.

The geometry of the rigid space is strongly related to its associated special fibres. As a matter of fact, any choice of an open covering of the rigid space will define a special fibre of some model of the space.

This is best shown with an example. Let $D = \{z \in K, |z| \leq 1\}$ be the standard unit disc. $D$ may be viewed as the generic fibre of the affine formal scheme $\mathfrak{D} = \mathrm{Spf}\, R\{Z\}$ and in this case its special fibre is the affine line $\mathbf{A}_k^1$. This corresponds to the trivial open cover. $D$ may be viewed as the union of the annulus $C(1) = \{z \in K, |\pi| \leq |z| \leq 1\}$ and of the smaller disc $D(1) = \{z \in K, |z| \leq |\pi|\}$. A model for $C(1)$ is $\mathfrak{C} = \mathrm{Spf}\, R\{X, Y\}/(XY - \pi)$ and a model for $D(1)$ is $\mathfrak{D}(1) = \mathrm{Spf}\, R\{T\}$. If we glue these models (with the convention that $T = \pi X$), we get a (non-affine!) formal scheme whose special fibre consists of our old affine line (of coordinate $X$) which meets at the origin a projective line (of coordinate $T = 1/Y$). This projective line is nothing but the result of an (admissible) blowing-up at the origin.

### 3. First application to coverings: a theorem of Harbater

The idea of using rigid (in fact formal) geometry for the construction of Galois covering of curves first appeared in the paper of Harbater [8], where he proved the following

THEOREM. *Any finite group is Galois group of an extension of* $\mathbb{Q}_p$.

We briefly sketch the proof of this result. Harbater's argument is very easy to follow, so we encourage the interested reader to look at the original proof. There is also a detailed presentation of Q. Liu, which has now been published in [19].

It is not possible to appeal to the structure of the fundamental group of $\mathbb{P}^1_{\mathbb{Q}_p}$ minus a finite number of points, as we did in the complex analogue situation, since $\mathbb{Q}_p$ is not algebraically closed. Since its algebraic closure $\mathbb{Q}_p^{\mathrm{alg}}$ is not complete, we could extend the base to $\mathbb{C}_p = \hat{\mathbb{Q}}_p^{\mathrm{alg}}$, but then we would have almost no hope to descent a covering from this huge extension down to $\mathbb{Q}_p$.

The proof proceeds in two steps.

First we construct a cyclic *slit* covering of $\mathbb{P}^1_{\mathbb{Q}_p}$ with prescribed ramification (i.e. an irreducible covering which becomes trivial when restricted to the open complement of small discs centered at the ramification points).

One easily reduces to cyclic coverings of prime order $q$ (with $q$ possibly equal to $p$). They are easy to construct over $\mathbb{Q}_p[\zeta_q]$ by Kummer theory (for instance $y^q = t^{q-1}(t-p)$ is branched at $t=0$ and $t=p$ and trivial over the complement of $\{t \in \mathbb{Q}_p, |t|_p < 1\}$). With some more work, one can find clever modification of these equations that descend to slit coverings of $\mathbb{P}^1_{\mathbb{Q}_p}$.

For a general group, choose generators $g_1, \ldots, g_n$, take the corresponding cyclic subgroups $C_1, \ldots, C_n$ and build cyclic slit coverings of $\mathbb{P}^1_{\mathbb{Q}_p}$. We can freely choose the ramification locus, so we can assume that points ramified in coverings given by different generators are ($p$-adically) distant (i.e. that the small discs over which the coverings are not trivial are disjoint).

If $H$ is a subgroup of $G$ and if $Y/X$ is a covering of group $H$ (of varieties, schemes, rigid spaces etc.), the *induced covering* $Ind_H^G Y$ of $X$ is the disjoint union of copies of $Y$ indexed by a family $\Gamma$ of representatives of left cosets of $H$ in $G$. $Ind_H^G Y$ is made into a (reducible) $G$-Galois covering if we agree that for any $g \in G$ and $\alpha \in \Gamma$, $g$ sends the component $Y_\alpha$ to the component $Y_\beta$ *via* the automorphism $h$, where $h \in H$ and $\beta \in \Gamma$ are uniquely determined by the equation $g\alpha = h\beta$.

Now take the $G$-coverings defined by induction from $C_i$ to $G$ by the cyclic slit coverings constructed above. Since all these are trivial coverings on the complement of all the disjoint small discs containing the ramification points, they can be glued along this admissible open subspace to get an irreducible covering of $\mathbb{P}^1_{\mathbb{Q}_p}$ of group $G$. The rigid GAGA principle insures that this covering is indeed algebraic.

I would like to stress that this kind of patching technique is the common point to all recent results on Galois coverings, and that it is this feature that makes rigid geometry so successful for this kind problems.

## 4. Specialization of coverings over a local field

From now on, unless otherwise stated, we will denote by $K$ a local field, complete with respect to some discrete valuation, by $R$ its ring of integers and by $k$ the residue field. We are concerned here with geometric properties of coverings, hence we will assume that

$k$ is algebraically closed. We need not assume that $K$ is algebraically closed, provided we accept to replace it with some finite extension from times to times. The degree of this extension is in many situations explicitly boundable.

The geometry of a rigid analytic space over a non-archimedean field $K$ is strongly related to that of its reductions modulo the maximal ideal of $R$, the link being established via some formal $R$-model.

This correspondence works in both senses. It is possible to establish some properties of a $K$ space by taking some model and changing it conveniently by blowing-up and contractions so that it behaves nicely (for instance with respect to the Galois action, in the case of coverings). On the other hand, one can start with some problem concerning schemes over $k$, find some way to lift it to a problem of formal $R$-schemes and then work it on the generic fibre (for instance with "cut and glue"techniques of the type we have described).

For the reader's convenience we have separated here the two points of view. In this section we recall some old and new results on the reduction of coverings of $R$-schemes and formal $R$-schemes (mostly obtained without explicitly appealing to rigid geometry). In the next section we will review some recent results on covering of $k$-curves that were obtained with this method.

However we feel that this distinction is unnatural and that the two points of view should be kept in mind at the same time[1].

Let then

$$Y_K \longrightarrow X_K$$

be a finite separable connected Galois covering of projective smooth $K$-curves (algebraic or rigid, it makes no difference in view of GAGA).

If $X$ is some normal $R$-model of $X_K$, we can take the normalization of $X$ in the function field of $Y_K$ to get a finite Galois covering

$$Y \longrightarrow X$$

of proper normal $R$-curves.

First, assume that $X_K$ admits a smooth model $X$. Then, since $X$ is regular and $Y$ normal, we can appeal to the *purity theorem* of Zariski-Nagata (cfr. SGA 1, exp. X, 3.1), stating that the branch locus of the covering is a closed subscheme of $Y$ purely of codimension one.

This means that, possibly after finite extension of $R$, this branch locus consists of the scheme-theoretical closure in $Y$ of the branched points of $Y_K$ (these are *horizontal*

[1] I guess I am subscribing here to some kind of *formal taoism*.

curves in the morphism $Y \to \operatorname{Spec} R$) plus a certain number of irreducible components of the special fibre $Y_k$ which are inseparable over $X_k$ (*vertical* curves).

If the Galois group $G$ of the covering has order prime to the characteristic of $k$, there is no vertical ramification. Moreover, if $Y_K \to X_K$ is étale (i.e. locally an unramified extension of Dedekind rings), there cannot be horizontal ramification either. Hence $Y \to X$ is étale and $Y_K$ also admits a smooth model (over some finite extension of $R$).

If $Y_K \to X_K$ is ramified and the order of $G$ is prime to char $k$, then $Y_k \to X_k$ is tamely ramified and one can show that $Y$ is also smooth.

We can translate these statements in terms of fundamental groups.

Recall from SGA 1 that if $S$ is a noetherian connected scheme, there is (up to the choice of a "base point") a profinite group (i.e. a topological group, profinite limit of finite groups with discrete topology) $\pi_1(S)$ classifying finite étale coverings of $S$. It can be constructed by taking the projective limit of the Galois groups of all finite Galois coverings of $S$. If $S$ is the spectrum of a field $F$, $\pi_1(\operatorname{Spec} F) = \operatorname{Gal}(F^{\mathrm{sep}}/F)$. If $S$ is a complex variety its (algebraic) fundamental group is just the profinite completion of the topological fundamental group of the underlying complex analytic space.

If $S$ is a proper scheme over $R$ with geometrically connected fibres, there is a *morphism of specialization* $\pi_1(S_K) \to \pi_1(S_k)$ which is shown to be surjective.

The above results can be restated as follows: if $X$ is a smooth $R$-curve the morphism of specialization $\pi_1(X_{K^{\mathrm{alg}}}) \to \pi_1(X_k)$ induces an isomorphism on the maximal prime-to-$p$ quotients of these groups (we had to replace $K$ with its algebraic closure $K^{\mathrm{alg}}$).

If $\{x_1, \ldots, x_s\}$ is a finite number of integral points of $X(R)$ (i.e. horizontal curves, that we assume disjoint), a finite covering of $X_K$ (resp. $X_k$) ramified at the $x_{i/K}$ (resp. $x_{i/k}$) is equivalent to an unramified covering of $X_K - \{x_{i/K}\}_i$ (resp. of $X_k - \{x_{i/k}\}_i$). Hence we have also that the morphism of specialization $\pi_1(X_{K^{\mathrm{alg}}} - \{x_{i/K}\}_i) \to \pi_1(X_k - \{x_{i/k}\}_i)$ induces an isomorphism on the maximal prime-to-$p$ quotients.

In particular, when $K$ is of characteristic zero, this shows that the prime-to-$p$ quotient of the fundamental group (usually called the tame fundamental group) of a smooth projective curve (with some points deleted) over an algebraically closed field of characteristic $p$ behaves just as the one of a smooth projective curve of the same genus (and with the same number of points deleted) over an algebraically closed field of characteristic zero.

This is false for the $p$-part of the fundamental group, which is still a mysterious object. However, thanks to recent results of Raynaud and Harbater, today we know all of its finite quotients (i.e. all the finite groups that occur as Galois groups over a function field in one variable in characteristic $p$).

These results (Abhyankar Conjecture) are the most spectacular achievement of rigid geometry on coverings of curves and will be the goal of Section 5.

The above results were established by Grothendieck in the late '50 (cfr. SGA 1). We will review in the next section some more recent results after recalling some facts about models of curves over discrete valuation rings.

In the general case, even for an unramified covering $Y_K \to X_K$ of smooth curves with nontrivial $p$-part, one cannot expect $Y_K$ to have a smooth model.

Given an algebraic curve $C_K$ over $K$, the main tool for studying its $R$-models is the Semi-Stable Reduction Theorem. A semi-stable curve over a field $k$ is a reduced connected curve with only ordinary double points as singularities and such that every irreducible component isomorphic to $\mathbb{P}^1_k$ meets other components in at least 2 points (even 3, if $C_K$ is of genus at least 2, this prevents the special fibre from having automorphisms). The Semi-Stable Reduction Theorem states that, if $C_K$ has genus at least 1, after possible finite separable extension of $K$, $C_K$ admits a regular semi-stable model, i.e. a model whose special fibre is semi-stable.[2]

An equivalent statement is the Semi-Abelian Reduction Theorem for abelian varieties over $K$: after possible finite separable extension $K'/K$, any abelian variety over $K$ extends to a semi-abelian scheme over $R'$, i.e. a smooth commutative group scheme, extension of an abelian variety by a torus. The equivalence of the statements is of course established via jacobians and their Néron models.

There are essentially three approaches to the proof of these theorems.

Grothendieck proved the semi-abelian reduction theorem via his Monodromy Theorem. He reduced the problem to studying the action of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ on the $\ell$-adic Tate module of the abelian variety for $\ell \neq p$. This method also provides bounds for the degree $[K' : K]$.

Artin and Winters proved the semi-stable reduction theorem using the resolution of singularities of two-dimensional schemes.

Later Bosch and Lütkebohmert [4], [5] found a new proof of both theorems using rigid geometry. The main step in their proof is a description of the set of points of $C_K$ (viewed as a rigid space) that reduce to an isolated singular point of the special fibre $C_k$, in terms of the nature of the singularity.

Let $f$ be a local coordinate on $C_K$ centered at some point $P$. Then $f$ defines a map between the points of $C_K$ having the same reduction as $P$ and the open unit disc $\{z \in K, |z| < 1\}$.

[2]Indeed, this theorem may be viewed as the valuative criterion of properness for the moduli space of stable curves of genus at least two.

If the reduction of $P$ is a smooth point of $C_k$, $f$ is an isomorphism. If $P$ specializes to a singular point $\bar{P}$, let $\bar{P}_1, \ldots, \bar{P}_b$ be the points that lie above $\bar{P}$ in the normalization $\tilde{C}_k$ of $C_k$, and let $\delta_i = \dim_k \mathcal{O}_{\tilde{C}_k, \bar{P}_i} / \mathcal{O}_{C_k, \bar{P}}$. Bosch and Lütkebohmert show that if $\alpha \in |K^\times|$ is close to 1, then $f^{-1}\{z \in K, \alpha \leq |z| < 1\}$ is isomorphic to a disjoint union of $b$ annuli $\{t \in K, \alpha^{\delta_i} \leq |t| < 1\}$.

Therefore we can build a complete smooth $K$-curve by gluing $f^{-1}\{z \in K, |z| < 1\}$ with $b$ discs $\{t \in \mathbb{P}^1_K, \alpha^{\delta_i} \leq |t| \leq \infty\}$. The special fibre of this curve is a rational curve with only one singular point of the same type as $\bar{P}$. Hence the genus of this new curve only depends on the number $b$ of branches through $\bar{P}$ and on the "orders"$\delta_i$ of the cuspidal parts of the $\bar{P}_i$'s.

This allows one to define a "local genus"attached to a singularity, and Bosch-Lütkebohmert's proof works on descending induction on this number.

A combinatorial tool, often used when dealing with semi-stable reductions, is the *semi-stable reduction graph*. It is the dual graph of the special fibre of the curve. Its vertices are the irreducible components of the curve, and there are as many edges joining two vertices as points in the intersection of the corresponding components.

We come back to the study of reduction of coverings of $K$-curves.

Let $X_K$ be a smooth $K$-curve, and let $Y_K \to X_K$ be an unramified covering with Galois group a $p$-group. Assume moreover that $X_K$ admits a smooth $R$-model. Under these hypotheses, Raynaud [22] showed that, after possible finite extension of $K$, $Y_K$ admits a semi-stable model such that the dual graph of its special fibre is a tree and that the jacobian $J(Y_K)$ has potentially good reduction. He proved the same for a nilpotent Galois group, admitting tame ramification along disjoint horizontal curves.

If $X_K$ is no longer assumed to have a smooth model, we can of course take regular semi-stable models of both $X_K$ and $Y_K$, but little is known in general on the behavior of the covering at singular points of the special fibres.

So far, the best result is a theorem of Saïdi [24], who gave a description of this behavior in terms of currents on the dual graph $\Gamma_X$ of the special fibre $X_k$ in the case of an abelian Galois group of order prime-to-$p$. In terms of the morphism of specialization of the fundamental groups (see Section 4), this is expressed by the exact sequence:

$$0 \longrightarrow H^1(\Gamma_X, \prod_{\ell \neq p} \mathbb{Z}_\ell) \longrightarrow \pi_1^{ab}(X_{K^{\mathrm{alg}}})^{(p)} \longrightarrow \pi_1^{ab}(X_k)^{(p)} \longrightarrow 0$$

The superscript $^{(p)}$ means "maximal prime-to-$p$ quotient". Saïdi's proof makes use of rigid geometry and has recourse to Bosch-Lütkebohmert's argument described above.

## 5. Galois coverings of curves in positive characteristic

We hope to have already convinced the reader of the usefulness of rigid geometry as a tool for constructing and studying (Galois) coverings of curves. Of course, the best results are obtained when, being interested in a covering of curves over $k$, one builds coverings on the generic fibre by rigid methods (usually patching arguments). The GAGA principle ensures then an algebraic result. In many situations, if the coverings are cleverly constructed, it is possible to keep control of the specialization and obtain the desired covering in this way.

The most striking example of the power of this method is the recent proof of Abhyankar's Conjecture. For an historical introduction we refer to Serre's Bourbaki talk [26] (only a few months prior to the announcement of the complete proof of the conjecture).

The algebraic fundamental group of a connected smooth projective curve over an algebraically closed field $F$ of characteristic zero, with $s \geq 0$ points deleted is well-known. If $F = \mathbb{C}$, it is isomorphic to the profinite completion of the topological fundamental group of the underlying complex manifold, which may be computed thanks to Riemann Existence theorem (see Section 1). This can be extended to an arbitrary algebraically closed field $F$ of characteristic zero by the Lefschetz principle. It is worth mentioning that no purely algebraic proof of the computation of the fundamental group is known.

A rigid analytic version of Riemann Existence theorem for a non-archimedean algebraically closed field of characteristic zero was proved by Lütkebohmert [15]. The analogous statement is false in positive characteristic.

In particular, the structure of this group only depends on the value of the integer $2g + s - 1$ and not on the curve, nor on the field.

This fails in positive characteristic. For instance if $E \subsetneq F$ are algebraically closed fields of characteristic $p$, for $\alpha \in F - E$ the Artin-Schreier equations $t^p - t - \alpha = 0$ define $\mathbb{Z}/p\mathbb{Z}$-covering of the affine line over $F$ that are not isomorphic to any covering of the affine line over $E$.

This is due to the bad behavior of the $p$-part of the fundamental groups. In Section 4 we saw that the prime-to-$p$ quotient of the fundamental group of a curve over an algebraically closed field of characteristic $p$ is the same as the one of a curve of the same type over a $\mathbb{C}$. It is therefore a free group on $2g + s - 1$ generators, if $s > 0$.

This implies that if $G$ occurs as the Galois group of a covering of a curve $C$ branched above $s > 0$ points, the maximal prime-to-$p$ quotient of $G$ can be generated by $2g + s - 1$ elements. Recall that if $p(G)$ denotes the subgroup generated by all the Sylow $p$-subgroups of $G$, the maximal prime-to-$p$ quotient of $G$ is $G/p(G)$.

In 1957 Abhyankar suggested that this condition should also be sufficient:

ABHYANKAR'S CONJECTURE. *A finite group $G$ can occur as Galois group of a connected covering of a smooth projective curve $C$ over an algebraically closed field $k$ of characteristic $p$, of genus $g$, branched above a finite number $s > 0$ of points if and only if its maximal prime-to-p quotient $G/p(G)$ is generated by $2g + s - 1$ elements.*

Note in particular that the condition does not depend on the field and on the curve.

After partial results of Abhyankar and Nori and a fundamental step of Serre [25] (who showed that any extension of a group occurring over the affine line $\mathbf{A}^1_k$ by a solvable group also occurs over $\mathbf{A}^1_k$) the conjecture was proven in 1992 by Raynaud [23] for the affine line and soon thereafter by Harbater [10] in the general case. Harbater's proof shows even more, namely that it is possible to realize such a group as Galois group of a branched covering with wild ramification above only one of the prescribed $s$ points.

It would be beyond the scope of this overview to even sketch the proof of these results, but we would like to present a few of the arguments of Raynaud's paper, just to give the reader the flavor of it.

In the case of the affine line, $g = 0$ and $s = 1$, so the conjecture says that $G$ should be generated by its Sylow $p$-subgroups.

In his first manuscript, Raynaud established some relations between $G$, its Sylow subgroups and the subgroups of $G$ that occur as Galois groups over the affine line, thus proving several particular cases of the conjecture. Soon thereafter, Thompson explained how, having recourse to the classification of finite simple groups, these partial results were in fact sufficient for proving the full conjecture for the affine line. Finally an argument of Hée allowed to avoid the classification theorem.

We want here to explain how rigid geometry is involved, both in characteristic $p$ (over $k[[t]]$) and zero (over $W(k)$).

A first step in the proof consists in showing that if $G$ is generated by a $p$-subgroup $Q$ and a family of subgroups $\{G_i\}_{i\in I}$ which occur as Galois groups of irreducible unramified coverings of the affine line with inertia group at infinity $Q_i \leq G_i \cap Q$, then $G$ itself occurs as Galois group of such a covering with $Q$ as inertia group at infinity.

This is done by a patching argument. On the affine line over the local field $k((t))$, select a family of disjoint open discs $\{\Delta_i\}_{i\in I}$ centered at points $P_i$. Assume that these discs have positive distance, so that one can find slightly bigger but still disjoint open discs $\Delta_i^+ \supsetneq \Delta_i$.

If $t_i$ is a local coordinate at $P_i$ (so that $\Delta_i = \{t_i \in k((t)), |t_i| < 1\}$), by hypothesis there is an unramified Galois covering of the affine line $\operatorname{Spec} k[t_i]$ of group $G_i$.

This covering can be lifted to an unramified covering of $\Delta_i = \operatorname{Spm} k((t))\{t_i\}$ because unramified coverings of curves admit (essentially unique) infinitesimal deformation;

we will return on this point later.

By an approximation argument, these coverings of the $\Delta_i$'s can be extended a little bit and, by shrinking the $\Delta_i^+$ we may assume they extend to those bigger discs. Since the points of the open annuli $\Delta_i^+ - \Delta_i$ all specialize to the point at infinity (of the projective line $\mathbb{P}^1_{k(t_i)}$) these extended coverings split over $\Delta_i^+ - \Delta_i$ into a finite sum of coverings of group $Q_i$.

Raynaud could then define a connected Galois covering of $\mathbb{P}^1_{k((t))}$ of group $Q$, totally ramified at infinity, possibly ramified at the $P_i$'s and such that, in restriction to $\Delta_i^+ - \Delta_i$ it is isomorphic to the coverings induced from $Q_i$ to $Q$ by the above ones.

Finally, taking induced coverings from $G_i$ to $G$ and from $Q$ to $G$ and gluing along the $\Delta_i^+ - \Delta_i$, one gets a connected rigid covering of $\mathbb{P}^1_{k((t))}$ of Galois group $G$ branched only above infinity with inertia group $Q$. By the GAGA principle, this covering is algebraic and specialization provides the desired covering of $\mathbb{P}^1_k$.

For an arbitrary finite group $G$, we know that it can be realized as Galois group of some connected covering of the projective line over any algebraically closed field of characteristic zero. Let $K$ be a complete local field having $k$ as residue field. After replacing $K$ by some finite extension, we can get a connected branched covering of the projective $K$-line of group $G$:

$$Y_K \longrightarrow \mathbb{P}^1_K$$

According to the Semi-Stable Reduction Theorem (see Section 4), after possible finite extension of $K$, $Y_K$ admits a regular semi-stable model $Y$ over the ring of integers $R$ of $K$. We can take a minimal such model, so that it inherits an action of $G$. The quotient $Y/G$ is a semi-stable model of $\mathbb{P}^1_K$ (cfr. [22], appendix).

Therefore, the group $G$ acts on the dual graph of the special fibre of $Y$ and the quotient graph is the dual graph of the special fibre of $Y/G$.

The further results of Raynaud [23] follow from a careful investigation of this action.

We would like to mention a last result on this circle of problems.

In reviewing some of Raynaud's arguments we mentioned the following fact: if $Y_k \rightarrow X_k$ is an unramified (Galois) covering of smooth curves over $k$, then there exists an étale (Galois) covering of smooth formal $R$-curves $\mathfrak{Y} \rightarrow \mathfrak{X}$ lifting the given one (i.e. reducing to it modulo the maximal ideal of $R$). This holds also for schemes of a more general type (cfr. SGA 1), and is the reason for the surjectivity of the morphism of specialization of the fundamental groups (See. Section 4).

Taking generic fibres, we get an unramified (Galois) covering of rigid spaces $Y_K \rightarrow X_K$ (we used it in the special case $X_k = \mathbf{A}^1_k$, hence $X_K$ is a disc). Of course, if

$X_k$ is projective, by GAGA we get an algebraic lifting $Y \to X$ too.

More generally, if $X_k$ is projective and $Y_k \to X_k$ is a (Galois) branched covering, we would like to know if it can be lifted to a (Galois) branched covering of $R$-curves. In the particular case above we would like to know if a Galois unramified covering of the affine line over $k$ can be lifted to a branched Galois covering of the projective line over $K$, not just of the disc.

This is possible if $Y_k \to X_k$ is tamely ramified, but there are easy examples of wildly ramified coverings that cannot be lifted. For instance, there are curves in characteristic $p$ with more than $84(g-1)$ automorphisms. If $C_k$ is such a curve, it is clear that the covering $C_k \to C_k/Aut(C_k)$ cannot be lifted to any characteristic zero ring.

However ([12]), after possible finite extension of $R$, the covering can be lifted in a weaker sense. Namely, there exists a normal $R$-curve $Y'$ (hence with smooth generic fibre) and a Galois covering $Y' \to X$, such that the special fibre $Y'_k$ is obtained from the original $Y_k$ by introduction of cuspidal singularities at the wildly branched points. In particular, this weaker lifting is a lifting in the topological sense, since $Y_k$ and $Y'_k$ are homeomorphic.

This also can be proved by a patching argument. Namely, lift the restriction of the covering to the open set $U_k$ over which it is unramified, and get a covering of formal curves $\mathfrak{V} \to \mathfrak{U}$. Then take the generic fibres $V_K \to U_K$. The (affinoid) rigid curve $U_K$ is the complement in $X_K$ of a finite number of open discs, and one has to extend the covering over these missing discs.

In general these extension will have more than one ramified point in each disc, and this leads to a singularity in the special fibre.

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