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STRUCTURE OF CERTAIN PERIODIC RINGS AND NEAR RINGS

Abstract. In the present paper we establish a decomposition theorem for rings satisfying either of the properties
\[ xy = x^m f(yx)x^n \] or
\[ xy = x^m f(xy)x^n , \]
where \( m = m(x, y) \geq 0, n = n(x, y) \geq 0 \) are integers and \( f(X) \) is a polynomial in \( X^2Z[X] \) varying with the pair of elements \( x, y \), and further deduce the commutativity of such rings. Finally, related results are obtained for near rings.

1. Introduction

A recent result by Searcoid and MacHale [17] establishes commutativity of rings in which all products of two elements are potent. More recently, using this result Ligh and Luh [13] pointed out that such rings are direct sum of J-rings and zero rings. Further Bell and Ligh [8] obtained direct sum decomposition of rings satisfying the property
\[ xy = (xy)^2 f(x, y), \]
where \( f(X, Y) \in Z(X, Y) \), the ring of polynomials in two non-commuting indeterminates. In the present paper we begin with the rings satisfying either of the following properties and establish a decomposition theorem, which in turn allows us to determine the commutativity of such rings.

(P1) For every pair of elements \( x, y \) in a ring \( R \), there exist integers \( m = m(x, y) \geq 0, n = n(x, y) \geq 0 \) and a polynomial \( f(X) \) in \( X^2Z[X] \) such that \( xy = x^m f(yx)x^n \).

(P2) For every pair of elements \( x, y \) in a ring \( R \), there exist integers \( m = m(x, y) \geq 0, n = n(x, y) \geq 0 \) and a polynomial \( f(X) \) in \( X^2Z[X] \) such that \( xy = x^m f(xy)x^n \).

2. A decomposition theorem for rings

Throughout this section, \( R \) will denote an associative ring (maybe without unity 1), and \( N \) the set of nilpotent elements of \( R \). A ring \( R \) is called zero-commutative if for all \( x, y \) in \( R \) \( xy = 0 \) implies that \( yx = 0 \). An element \( x \) in \( R \) with the property \( x = x^{n(x)} \) for some integer \( n(x) > 1 \) will be called potent. The set of potent elements will be denoted by \( P \). If \( P = R \), we shall call \( R \) a J-ring. By well known \( "x = x^{n(x)} \) theorem" of Jacobson [11], J-rings are necessarily commutative. A sufficient condition for \( R \) to be periodic is Chacron's criterion: for each \( x \) in \( R \) there exists integer \( m = m(x) > 1 \) and a polynomial \( f(X) \) in \( Z[X] \) such that \( x^m = x^{m+1} f(x) \) ([9]). It is shown in [3] that if \( R \) is periodic, then every element \( x \) in \( R \) can be written in the form \( x = a + u \), where \( a \in P \) and \( u \in N \).
In a very surprising structural result (signified as Theorem B in sequel) Bell [4] remarked that if in a periodic ring \( R \) every element has a unique representation as above, then \( P \) and \( N \) both are ideals and \( R = P \oplus N \). In this section we shall obtain a decomposition theorem for rings satisfying either of the properties \( (P_1) \) or \( (P_2) \). In fact we shall prove the following:

**Theorem 2.1.** Let \( R \) be a ring satisfying either of the properties \( (P_1) \) or \( (P_2) \). Then \( R \) is a direct sum of a J-ring and a nil ring.

**Proof.** Notice that if \( y \) is replaced by \( x \) in either of the above properties \( (P_1) \) or \( (P_2) \), \( R \) satisfies Chacron's criterion for periodicity and hence the ring satisfying either of the properties \( (P_1) \) or \( (P_2) \) is necessarily periodic. We shall prove the result for the property \( (P_1) \) and the proof for \( (P_2) \) follows similarly.

It is easy to see that ring satisfying \( (P_1) \) is zero-commutative. Indeed, if \( xy = 0 \), then there exist \( m' = m(y,x) \geq 0 \), \( n' = n(y,x) \geq 0 \) and \( g(X) \in X^2\mathbb{Z}[X] \) such that \( yx = y^{m'}g(xy)y^{n'} = 0 \). Now replace \( y \) by \( x \) in \( (P_1) \), to get

\[
(2.1) \quad x^2 = x^p h(x), \quad \text{for} \quad h(X) \in \mathbb{Z}[X] \quad \text{and} \quad p = p(x) > 2.
\]

If \( u \in N \) and \( x \in R \), then chose integers \( m_1 = m(u,x) \geq 0 \), \( n_1 = n(u,x) \geq 0 \) and polynomial \( f_1(X) \) in \( X^2\mathbb{Z}[X] \) such that

\[
(2.2) \quad ux = u^{m_1}f_1(xu)u^{n_1}.
\]

In view of \( (2.1) \), it can be easily seen that \( u^2 = 0 \), and hence \( xu^2 = (xu) = 0 \). Now zero-commutativity in \( R \) yields that \( u(xu) = 0 \) i.e. \( (xu)^2 = 0 \). This together with \( (2.2) \) implies that \( ux = 0 \) and again zero-commutativity in \( R \) gives that \( xu = 0 \) for all \( x \) in \( R \) and \( u \) in \( N \). This yields that

\[
(2.3) \quad RN = NR = \{0\}.
\]

Since \( R \) is periodic every element \( x \) in \( R \) can be written in the form \( x = a + u \), where \( a \in P \) and \( u \in N \). In view of Theorem B, it is sufficient to show that the above representation is unique. Indeed, if \( a + u = b + v \) for some \( a, b \in P \) and \( u, v \in N \), then

\[
(2.4) \quad a - b = v - u.
\]

Since \( a, b \in P \) there exist integers \( p = p(a) > 1 \) and \( q = q(a) > 1 \) such that \( a^p = a \) and \( b^q = b \). Let \( k = (p - 1)q - (p - 2) = (q - 1)p - (q - 2) \). Then it is clear that \( a^k = a \) and \( b^k = b \). Note that \( e = a^{k-1} \) and \( f = b^{k-1} \) are idempotents with \( ea = a \) and \( fb = b \). Multiply \( (2.4) \) by \( a \) and \( b \) from both the sides and use \( (2.3) \), to get \( a^2 = ab = ba \) and \( b^2 = ab = ba \). This yields that \( a^2 = b^2 \) and hence \( e = f \). Again, multiply \( (2.4) \) from left by \( e \) to get \( a = b \). This completes the proof of our theorem.
REMARK 2.1. In view of (2.3), we conclude that the nilpotent elements of $R$ annihilate $R$ on both sides, and hence central. Since $J$-rings are commutative, the above theorem at once yields the following result, which generalizes main result proved in [17] and [18, Theorem 2].

COROLLARY 2.1. Let $R$ be a ring satisfying either of the properties $(P_1)$ or $(P_2)$. Then $R$ is commutative.

3. Related results for near rings

For the purposes of this section, $R$ will denote a left near ring with multiplicative centre $Z$.

It is natural to question whether the analogous hypotheses yield the direct sum decomposition in case of near rings. The following example due to Clay (cf. [14, Example H-29, page 342]) shows that it is not possible to obtain such decomposition in the case of near rings.

EXAMPLE 3.1. Let $R = \{0, a, b, c, u, v\}$ with addition and multiplication tables defined as follows:

Then $(R, +, \cdot)$ is a commutative near ring satisfying $xy = yx = (xy)^2$ for all $x, y$ in $R$. However, the set $P = \{0, a\}$ is not an ideal of $R$.

Hence, following [8], we define a weaker notion of orthogonal sum. Specifically, a near ring $R$ is an orthogonal sum of sub-near rings $A$ and $B$ – denoted $R = A + B$ – if $AB = BA = \{0\}$, and each element of $R$ is uniquely representable in the form $a + b$ with $a \in A$ and $b \in B$.

In the present section we shall investigate structure of near rings satisfying either of the following properties:

$(P_1)^*$ For every pair of elements $x, y$ in $R$ there exist integers $m = m(x, y) \geq 0$, $n = n(x, y) \geq 0$, $p = p(x, y) > 1$ such that $xy = x^m(yx)^p x^n$.

$(P_2)^*$ For every pair of elements $x, y$ in $R$ there exist integers $m = m(x, y) \geq 0$, $n = n(x, y) \geq 0$, $p = p(x, y) > 1$ such that $xy = x^m(xy)^p x^n$.

Before stating our main theorem of this section, we present the following known results which are essentially proved in [2], [7] and [8] respectively.
Lemma 3.1. Let $R$ be a zero-commutative near ring. Then the set $N$ of nilpotent elements is an ideal if and only if $N$ is a subgroup of the additive group $(R,+)$. 

Lemma 3.2. Let $R$ be a periodic near ring with multiplicative identity. If $N \subseteq Z$, then $(R, +)$ is abelian.

Lemma 3.3. Let $R$ be a near ring in which idempotents are multiplicatively central. If $e$ and $f$ are any idempotents, then there exists an idempotent $g$ in $R$ such that $ge = e$ and $gf = f$.

Theorem 3.1. Let $R$ be a near ring satisfying $(P_1)^*$. Moreover, if idempotent elements of $R$ are multiplicatively central, then $P$ is a sub-near ring with $(P, +)$ abelian and $N$ is a sub-near ring with trivial multiplication and $R = N + P$.

Proof. Notice that a near ring satisfying $(P_1)^*$ is necessarily zero-symmetric – i.e. $R$ has the property $0x = 0$ for all $x \in R$, and hence zero-commutative. Let $u \in N$ and $x \in R$. Then there exist integers $m_1 = m(x, u) \geq 0$, $n_1 = n(x, u) \geq 0$, $p_1 = p(x, u) > 1$, such that

$$xu = x^{m_1} (ux)^{p_1} x^{n_1}.$$  

Further choose integers $m' = m(x) \geq 0$, $n' = n(x) \geq 0$, $p' = p(x) > 1$, such that

$$x^2 = x^{m'+n'+2p'},$$

for all $x$ in $R$ and $m' + n' + 2p' \geq 4$.

The above equation yields that $u^2 = 0$ for any $u \in N$. Thus, we find that $u(ux) = u^2x = 0$ and the zero-commutativity in $R$ implies that $(ux)u = 0$ – i.e. $(ux)^2 = 0$. Hence, in view of (3.1), we get $xu = 0$ and again zero-commutativity in $R$ yields that

$$xu = x^{m_1} (ux)^{p_1} x^{n_1} = 0,$$

This shows that the nilpotent elements of $R$ annihilate $R$ on both sides and hence in particular $N^2 = \{0\}$ and $N \subseteq Z$. Now, if $u, v \in N$, then $(u - v)^2 = 0$. This yields that $N$ is a subgroup of the additive group $(R, +)$ and application of Lemma 3.1. shows that $N$ forms an ideal. In view of (3.2), we also have $x(x - x^{m'+n'+2p'} - 1) = 0$. But since $R$ is zero-commutative, the last equation implies that $(x - x^{m'+n'+2p'} - 1)x = 0$, and hence $(x - x^{m'+n'+2p'} - 1)x^{m'+n'+2p'} - 1 = 0$. Now, a simple computation yields that $(x - x^{m'+n'+2p'} - 1)^2 = 0$ and $x - x^{m'+n'+2p'} - 1 \in N$. We can write $x = x - x^{m'+n'+2p'} - 1 + x^{m'+n'+2p'} - 1$ and also observe that

$$(x^{m'+n'+2p'} - 1)(m'+n'+2p'-1) = x^{(m'+n'+2p'-1)(m'+n'+2p'-1)}$$

$$= x^{(m'+n'+2p'-2)(m'+n'+2p') + 1}$$

$$= (x^{m'+n'+2p'})(m'+n'+2p'-2x)$$

$$= (x^{m'+n'+2p'} - 2x).$$
Since $x^{m'+n'+2p'-2}$ is idempotent, the above yields that $x^{(m'+n'+2p'-1)m'+n'+2p'-1} = x^{m'+n'+2p'-1}$ with $m' + n' + 2p' \geq 4$ and $x^{m'+n'+2p'-1} \in P$. Thus $R = N + P$. Now, we shall show that $P$ is a sub-near ring with $(P, +)$ abelian. Let $a, b \in P$. Then there exist integers $q = q(a) > 1$ and $r = r(b) > 1$ such that $a^q = a$ and $b^r = b$. If $s = (q-1)r - (q-2) = (r-1)q - (r-2) > 1$, then it is straightforward to see that $a^s = a$ and $b^s = b$. Also, since $e = a^{s-1}$ and $f = b^{s-1}$ are central idempotents with $ea = a$ and $fb = b$, it follows that

$$ab = eafb = (ef)(ab) = (ef)^m(abe)^p(ef)^n$$

for some integers $m = m(ef, ab) \geq 0$, $n = n(ef, ab) \geq 0$, $p = p(ef, ab) > 1$. This implies that $ab = ef(ab) = (EF)^{p} - i.e. ab \in P$. Since, $R/N$ has $x^{t} = x$ property, we have integer $j > 1$ such that

$$(3.4) \quad (a - b)^{j} = a - b + u; \quad \text{where } a, b \in P \text{ and } u \in N.$$  

Moreover, $e$ and $f$ are central idempotents in $R$ and hence in view of Lemma 3.3 we can choose an idempotent $g$ in $R$ such that $ge = e$ and $gf = f$. This yields that $ga = a$ and $gb = b$. Now, multiply $(3.4)$ by $g$, to get $(a - b)^{j} = a - b$ i.e. $a - b \in P$. Since, $gR$ is a periodic near ring with multiplicative identity element in which nilpotent elements are multiplicatively central. Hence, by Lemma 3.2, $(gR, +)$ is abelian. Therefore $ga + gb = gb + ga - i.e. a + b = b + a$, and hence $(P, +)$ is abelian. Now, it remains only to show that each element of $R$ has atmost one representation in the form $a + u; a \in P$ and $u \in N$. Accordingly, suppose that $a + u = b + v$, where $a, b \in P$ and $u, v \in N$. Then, we find that $-b + a = v - u \in P \cap N = \{0\}, -i.e. a = b$ and $u = v$. This completes the proof of our theorem.

Remark 3.1. The following example (cf. [14, Example E-14, page 340]) justifies the centrality of idempotents in the hypotheses of the above theorem.

Example 3.2. Let $R = \{0, a, b, c\}$ with addition and multiplication tables defined as follows:

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<th>0</th>
<th>a</th>
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<th>c</th>
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<tr>
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</table>

It can be easily verified that $(R, +, \cdot)$ is a near ring satisfying the condition $xy = x(yz)^{2}x$ for any $x, y$ in $R$. However, the set $P = \{0, a, c\}$ is not a sub-near ring of $R$. 

REMARK 3.2. If \( R \) is a near ring satisfying the condition \((P_2)^*\), then it can be easily seen that \( R \) need not be zero-commutative. However, a zero-symmetric near ring satisfying \((P_2)^*\) is necessarily zero-commutative. Hence, by using similar arguments as used to prove Theorem 3.1, with necessary variations, we can prove the following:

**Theorem 3.2.** Let \( R \) be a zero-symmetric near ring satisfying \((P_2)^*\). Moreover, if idempotent elements of \( R \) are multiplicatively central, then \( P \) is a sub-near ring with \((P, +)\) abelian, \( N \) is a sub-near ring with trivial multiplication and \( R = N + P \).

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REFERENCES


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