THE POISSON-NIJENHUIS MANIFOLDS REVISITED

Abstract. We give a short exposition of the basic results of the theory of Poisson-Nijenhuis manifolds developed in [MM] and [KSM], using Lie algebroids and noticing a certain generalization. Then, we consider affine Poisson structures of cotangent bundles $T^*M$, and show that these structures are associated with a Lie algebroid structure of $TM$ and a 2-form of $M$. We examine the case where the affine Poisson structure is compatible with the canonical symplectic structure of $T^*M$ and, thereby, it provides $T^*M$ with a Poisson-Nijenhuis structure.

The Poisson-Nijenhuis structures on differentiable manifolds $M$ were studied by Magri in an effort to get new integrability results for Hamiltonian dynamical systems (see Magri and Morosi [MM] and the references quoted there, particularly [GD]). Then, these structures and similar structures on Lie algebras and on differential Lie algebras (the abstract version of the Lie algebra of the tangent vector fields of $M$, first studied by Herz [H] under the name of Lie pseudoalgebras) were studied by Y. Kosmann-Schwarzbach and Magri [KSM]. (See also the references quoted in [KSM] as well as [KS], [Oz] and [BM]).

However, the geometry of the Poisson-Nijenhuis manifolds is yet to be developed, and, in the present paper, we hope to contribute to such a development by: i) a short exposition of the main known results put in the framework of Lie algebroids, ii) the study of certain examples on cotangent bundles.

While the first part of the paper doesn't bring new results, it indicates, however, a generalization which retains the basic properties of the Poisson-Nijenhuis structures.

1. Poisson-Nijenhuis structures

We are working in the $C^\infty$-category, and we recall that a Lie algebroid $\pi : E \to M$ with anchor map $A : E \to TM$ is a vector bundle $E$ with a bundle homomorphism $A$, 

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such that: i) the space $\Gamma(E)$ of the cross-sections has a Lie algebra bracket $[\ ,\ ]_E$ sent by $A$ to the usual bracket of vector fields, ii) $\forall s_1, s_2 \in \Gamma(E)$ and $\forall f \in C^\infty(M)$ one has

\begin{equation}
[s_1, f s_2]_E = ((As_1)f)s_2 + f[s_1, s_2]_E. \tag{1.1}
\end{equation}

Then, we may look at the bundles $E^*, \wedge E^*, \wedge E$, and obtain operators similar to the classical exterior differential $d$, the Lie derivative $L_X$, and the interior product $i(X)$, with $X \in \Gamma(E)$ (rather than $X \in \Gamma(TM)$), as well as the Schouten-Nijenhuis bracket—extension of $[\ ,\ ]_E$ by the classical formulas of calculus on manifolds, where $E = TM$, if $[\ ,\ ]_E$ is replaced by of $[\ ,\ ]_E$ in these formulas. Detailed developments can be found in [KSM], [Mz], etc.

In particular, it is possible to put various Lie algebraoid structures $[\ ,\ ]'$ on $TM$, and then we shall add the particle pseudo to the name of the structures that have classical definitions but for such nonclassical Lie brackets of vector fields. Here, the important structures are the pseudo-Poisson structures (or bivector fields) $\Pi \in \wedge^2 TM$ characterized by the vanishing of the Schouten-Nijenhuis bracket $[\Pi, \Pi]' = 0$. The classical Poisson structures have $[\Pi, \Pi] = 0$. If $[\Pi, \Pi]' \neq 0$ ($[[\Pi, \Pi]' \neq 0$), $\Pi$ is a (pseudo)-almost Poisson structure.

It is a fundamental fact that the bivector $\Pi$ also defines a skew-symmetric dual bracket of 1-forms $\alpha, \beta \in \wedge^1 M$ [KSM]

\begin{equation}
\{\alpha, \beta\}' = L'_{#\alpha} \beta - L'_{#\beta} \alpha - d'(\Pi(\alpha, \beta)), \tag{1.2}
\end{equation}

where the prime denotes operations associated to $[\ ,\ ]'$ (and it is omitted if $[\ ,\ ]' = [\ ,\ ]$), and $#\alpha$ is the vector field determined by $\gamma(#\alpha) = \Pi(\alpha, \gamma), \gamma \in \wedge^1 M$. (In [KSM], the opposite sign of $\#$, denoted there $\emptyset$, is used). This bracket has the following fundamental properties [KSM]: i) the homomorphism equality

\begin{equation}
#\{\alpha, \beta\}' = [#\alpha, [#\beta]' \tag{1.3}
\end{equation}

holds iff $[\Pi, \Pi]' = 0$; ii) if $[\Pi, \Pi]' = 0$, $[\ ,\ ]'$ satisfies the Jacobi identity, and the converse is also true if $A$ is nonsingular. The proofs are computational.

Formula (1.2) is motivated by the Poisson case, where $\{\alpha, \beta\}$ is the natural extension of the operation $\{df, dg\} = d\{f, g\}_{\Pi}$. In the present general case, (1.2) yields

$$\{d'f, d'g\}' = d'(\Pi(d'f, d'g))$$

Thus, on a pseudo-Poisson manifold, $(T^*M, \{\ ,\ \}')$ is a Lie algebroid of anchor map $#$. Accordingly, there is a corresponding exterior differential $\sigma'$ ($\sigma$ in the Poisson case) acting on $\Gamma(\wedge TM)$ (since $T^*M = TM$), which plays an essential role in calculus on Poisson manifolds [KSM].

If $([\ ,\ ]', A)$ is a Lie algebroid structure on $TM$, a simple coordinate computation using (1.1) shows that

\begin{equation}
[X, Y]' = [X, Y]_A + \Theta(X, Y) \ (X, Y \in \Gamma(TM)), \tag{1.4}
\end{equation}
where

\[(1.5) \quad [X, Y]_A = [AX, Y] + [X, AY] - A[X, Y],\]

and \(\Theta\) is a \(TM\)-valued 2-form. Furthermore, we have

\[(1.6) \quad A\Theta(X, Y) = \mathcal{N}_A(X, Y) \overset{\text{def}}{=} [AX, AY] - A[X, AY] - A[AX, Y] + \Theta(AX, AY),\]

\[(1.7) \quad \sum_{cyc(X, Y, Z)} \{\Theta(\Theta(X, Y), Z) + \Theta([X, Y]_A, Z) \]
\[+ [\Theta(X, Y), Z]_A - \mathcal{N}_A(X, Y), Z] - \mathcal{N}_A([X, Y], Z)\} = 0.\]

The relation (1.6) comes from condition i) of the definition of a Lie algebroid and, if it holds, (1.7) is the Jacobi identity for \([\ , \ ]\). Conversely, (1.4) with (1.5), (1.6), (1.7) is a Lie algebroid structure on \(TM\).

\(\mathcal{N}_A\) is the Nijenhuis torsion of \(A\) [N], and \(A\) is a Nijenhuis tensor if \(\mathcal{N}_A = 0\). If this happens, (1.6) and (1.7) hold for \(\Theta = 0\) and \((TM, [\ , \ ]_A, A)\) will be a Nijenhuis algebroid. Notice also that for any non-degenerate \(A\), \((\cdot [\cdot , \cdot ]_A, A)\) is a Lie algebroid structure with anchor \(A\) on \(TM\), called the \(A\)-algebroid structure. For this structure we have \(\Theta = A^{-1}\mathcal{N}_A\).

With these preparations we can now give

**Definition 1.1.** [KSM] A Poisson-Nijenhuis structure on the manifold \(M\) is a pair \((\Pi, A)\), where \(\Pi\) is a Poisson bivector \((\Pi, \Pi) = 0\), and \(A\) is a Nijenhuis tensor such that the \((2,0)\)-tensor field \(\Pi_1\) defined by \#_{\Pi_1} = \Pi^0 \#_{\Pi} is skew-symmetric, and the dual bracket of the almost-Poisson structure \((\cdot [\cdot , \cdot ], \Pi_1)\) is equal to the dual bracket of the pseudo-almost-Poisson structure \((\cdot [\cdot , \cdot ], A, \Pi)\). Then, the triple \((M, \Pi, A)\) is a Poisson-Nijenhuis manifold.

From the previously given definition of a \#, it follows that the skew-symmetric condition above means

\[(1.8) \quad \Pi_1(\alpha, \beta) \overset{\text{def}}{=} \Pi(\alpha, \beta \circ A) = -\Pi_1(\beta, \alpha) = \Pi(\alpha \circ A, \beta),\]

and the required bracket equality can be written as

\[(1.9) \quad \{\alpha, \beta\}_\Pi = \{\alpha, \beta\}_{\Pi_1} - \{\alpha, \beta\}_{\Pi_1} (\alpha, \beta \in \Lambda^1 M)\].

**Theorem 1.2.** Let \(\Pi\) be a Poisson bivector, and \(A\) a Nijenhuis tensor which satisfies (1.8). Then \((\Pi, A)\) is a Poisson-Nijenhuis structure iff one of the following equivalent conditions is satisfied:
a) [MM] the $T^* M$-valued bivector
\begin{equation}
C(\Pi, A)(\alpha, \beta) = \beta \circ L\#_{\Pi} \alpha A - \alpha \circ L\#_{\Pi} \beta A + d(\Pi(\alpha, \beta)) \circ A - d(\Pi(\alpha \circ A, \beta))
\end{equation}
vansishes;

b) \forall f, g \in C^\infty(M) one has
\begin{equation}
dg \circ LX_f A = i(X_g) d(df \circ A),
\end{equation}
where $X_f = \#_{\Pi} df$, $X_g = \#_{\Pi} dg$ are the $\Pi$-Hamiltonian vector fields of $f$ and $g$;

c) [KSM] the dual brackets of $([\ , \ ] = [\ , \ ]_A, \Pi)$ and $([\ , \ ], \Pi)$ are related by
\begin{equation}
\{\alpha, \beta\}^\Pi = \{\alpha \circ A, \beta\}_\Pi + \{\alpha, \beta \circ A\}_\Pi - \{\alpha, \beta\}_\Pi \circ A.
\end{equation}

Proof. Using the definition of a Lie derivative and the bracket (1.4), we get
\begin{equation}
L_X \alpha = L_A \alpha - \alpha \circ (L_X A + i(X) \Theta),
\end{equation}
where $X \in \Gamma(TM)$, $\alpha \in \Lambda^1 M$. If (1.13) and (1.2) are used to express (1.9), we get
\begin{equation}
C(\Pi, A)(\alpha, \beta) = \alpha \circ i(\#_{\Pi} \beta) \Theta - \beta \circ i(\#_{\Pi} \alpha) \Theta.
\end{equation}
Since in our case $\Theta = 0$, a) is proven. ($C$ is the same tensor as the one denoted by $R$ in [MM]).

Furthermore, if we take $\alpha = df$, $\beta = dg$ in (1.10), we obtain the equivalent condition
\begin{equation}
dg \circ LX_f A - df \circ LX_g A + d\{f, g\}_\Pi \circ A - d(\Pi_1(df, dg)) = 0.
\end{equation}
If we use here
\begin{equation}
df \circ LX_g A = L_g (df \circ A) - (L_g df) \circ A,
\end{equation}
and $L_X g = i(X)dg + di(X_g)$, we get precisely (1.11), which proves b).

For c), it is enough to check the case $\alpha = df$, $\beta = dg$. Using again (1.2) and (1.13) we see by comparing with (1.15) that condition (1.12) is equivalent to
\begin{equation}
2C(\Pi, A)(\alpha, \beta) = \alpha \circ i(\#_{\Pi} \beta) \Theta - \beta \circ i(\#_{\Pi} \alpha) \Theta.
\end{equation}
Since $\Theta = 0$, this means $C = 0$, and we are done. ■

The reason for mentioning $\Theta$ in the above proof, while we knew that $\Theta = 0$, is that it allows us to notice a new notion, that of a generalized Poisson-Nijenhuis manifold. This will be a manifold $M$ endowed with a Poisson structure $\Pi$ and a Lie algebroid structure $([\ , \ ], A)$ on $TM$ such that (1.8) and (1.9) are satisfied and, $\forall \alpha \in \Lambda^1 M$, one has $i(\#_{\Pi} \alpha) \Theta = 0$. Obviously, Theorem 1.2 remains true for these generalized structures.
The main result of the theory of Poisson-Nijenhuis structures is that they admit an iteration process. Namely, if \((\Pi, A)\) is a Poisson-Nijenhuis structure of \(M\), and if we also put \(\Pi = \Pi_0\), the sequence of tensors \(\Pi_k(k = 0, 1, 2, \ldots)\) defined by \(\#\Pi_k = A\#\Pi_{k-1}\) are all skew symmetric (easy induction; if skew-symmetry holds up to \(k - 1\), we get \(\Pi_k(\alpha, \beta) = \Pi_{k-2}(\alpha \circ A, \beta \circ A)\)). Moreover, we may also consider the sequence \(A^p(p = 0, 1, 2, \ldots)\), and what we get is

**THEOREM 1.3.** [MM]. [KSM]. With the notation above, all the pairs \((\Pi_k, A^p)\) are Poisson-Nijenhuis structures, and such that, \(\forall h, k, [\Pi_k, \Pi_h] = 0\).

**Proof.** The Nijenhuis torsion \(N_A\) of (1.6) can be defined equivalently by

\[
i(X)N_A = L_{AX}A - A \circ L_XA,
\]

whence an easy induction yields

\[
L_{A^pX}A = A^{p0}L_XA + \sum_{h=0}^{p-1} i(A^hX)N_A.
\]

Now, if \(N_A = 0\), the following computation holds good

\[
L_{A^pX}A^p = \sum_{h=0}^{p-1} A^{h0}L_{A^pX}A \circ A^{p-h-1} = \sum_{h=0}^{p-1} A^{p+h0}L_XA \circ A^{p-h-1}
\]

\[
= A^{p0}\left\{ \sum_{h=0}^{p-1} A^{h0}L_XA \circ A^{p-h-1} \right\} = A^{p0}L_XA^p.
\]

Therefore,

\[
i(X)N_{A^p} = L_{A^pX}A^p - A^{p0}L_XA^p = 0,
\]

and we see that all \(A^p\) are Nijenhuis tensors if \(A\) is one.

Furthermore, (1.10) leads to

\[
C_{(\Pi, A^p)}(\alpha, \beta) = [C_{(\Pi, A)}(\alpha, \beta)] \circ B + C_{(\Pi, B)}(\alpha \circ A, \beta) + \beta \circ A \circ L_{\#\Pi}B - \beta \circ L_A B,
\]

whenever \(A \circ \#\Pi\) and \(B \circ \#\Pi\) are associated with skew-symmetric \((2,0)\)-tensors, and \(AB = BA\). If (1.21) is applied for the matrices \(A^p\) and \(A\), and (1.19) is used again, we get

\[
C_{(\Pi, A^{p+1})}(\alpha, \beta) = [C_{(\Pi, A^p)}(\alpha, \beta)] \circ A + C_{(\Pi, A)}(\alpha \circ A^p, \beta) - \sum_{h=0}^{p-1} \beta \circ i(A^h \#\Pi A)N_A.
\]

Therefore, we see inductively that all the pairs \((\Pi, A^p)\) are Poisson-Nijenhuis structures.
Now, in order to prove the first part of Theorem 1.3 it is enough to prove that the pairs \((\Pi_k, A)\) are Poisson-Nijenhuis structures. Moreover, it is enough to show that, if \((\Pi_0, A)\) is a Poisson-Nijenhuis structure so is \((\Pi_1, A)\), since the passage from \((\Pi_k, A)\) to \((\Pi_{k+1}, A)\) will be similar.

\(\Pi_0 = \Pi\) satisfies (1.12), and, if we apply \(\#_\Pi\) there, we obtain precisely (1.3) which, by the properties i), ii) of the dual bracket which we recalled earlier [KSM], implies \([\Pi_0, \Pi_0]' = 0\) and the Jacobi identity for \(\{ , \}^\Pi_0\). Then, (1.9) shows that \(\{ , \}^\Pi_1\) also satisfies the Jacobi identity and we may conclude that \([\Pi_1, \Pi_1] = 0\) i.e., \(\Pi_1\) is a Poisson bivector.

Finally, by a technical computation which uses also (1.18), one gets [MM]
\[
C_{(\Pi_1, A)}(\alpha, \beta) = C_{(\Pi_0, A)}(\alpha, \beta \circ A) + \beta \circ i(\#_\Pi_0 \alpha)N_A.
\]
and since \(C_{(\Pi_0, A)} = 0, N_A = 0\), we are done.

Now, for the last assertion of Theorem 1.3, we notice that the equality (1.9) of the two Lie algebroid brackets involved there implies the equality of the corresponding exterior differentials \(\sigma'_\Pi_0 = \sigma_{\Pi_1}\). It is well known in Poisson calculus (e.g., [KSM]) that one has
\[
\sigma'Q = -[\Pi, Q]' \quad (Q \in \Gamma(\wedge TM)).
\]
(Use the definition of \(\sigma'\) and that of the Schouten-Nijenhuis bracket if you want to check (1.24)). Hence \(\sigma'_\Pi_0 = \sigma_{\Pi_1}\) means
\[
[\Pi_0, Q]' = [\Pi_1, Q],
\]
and this allows us to prove by induction that \([\Pi_k, \Pi_h] = 0\).

Indeed, this is already known to hold for \(k = h\). Assume that it holds for \(h < k \leq s - 1\), and let us prove it for \(h < k = s\). Using (1.25) (for \(\Pi_{s-1}, \Pi_{s-1}\) instead of \(\Pi_0\)) we get
\[
[\Pi_s, \Pi_h] = [\Pi_{s-1}, \Pi_h]' = [\Pi_h, \Pi_{s-1}]' = [\Pi_{h+1}, \Pi_{s-1}] = 0,
\]
by the inductive hypothesis. ■

REMARKS. 1) Two Poisson bivectors \(\Pi_1, \Pi_2\) such that \([\Pi_1, \Pi_2] = 0\) are called compatible. Thus \(\{\Pi_k\}\) is a set of pairwise compatible Poisson bivectors. Moreover, using again (1.25) we also see that \([\Pi_k, \Pi_h]' = 0\). The set of Poisson-Nijenhuis structures \(\{(\Pi_k, A^p)\}\) is called the hierarchy of Poisson-Nijenhuis structures of \((M, \Pi, A)\). 2) If \(A\) is nondegenerate, using \(L_X(AA^{-1}) = 0\), we obtain
\[
N_{A^{-1}}(X, Y) = A^{-2}N_A(A^{-1}X, A^{-1}Y),
\]
\[
C_{(\Pi, A^{-1})}(\alpha, \beta) = -C_{(\Pi, A)}(\alpha \circ A^{-1}, \beta) \circ A^{-1},
\]
which shows that \(\Pi, A^{-1}\) is again a Poisson-Nijenhuis structure. Hence, in this case the hierarchy \((\Pi_k, A^p)\) is defined for any integers \(k, p\).
Moreover, the hierarchy of Poisson-Nijenhuis structures can also be much enlarged by the following procedure. If $\mathcal{N}_A = 0$, (1.19) leads to

$$L_{P(A)(X)} A = P(A) \circ L_X A$$

for any polynomial $P(A)$ with constant coefficients and, therefore, for any convergent power series $P(A)$. Then, the computation which proved (1.20) now proves that any such polynomial (series) is again a Nijenhuis tensor. (By convergence, we understand both the existence and the differentiability of the sum.) Now, from the $\mathbb{R}$-linearity of $C_{(\Pi,A)}$ with respect to $A$ and $\Pi$, and using (1.22), we easily deduce that, if $(\Pi, A)$ is a Poisson-Nijenhuis structure, so is any pair $(Q, P_1(A))$, where $#Q = P_2(A) \circ #_\Pi$ and $P_1(A); P_2(A)$ are either polynomials or convergent power series with constant coefficients. Furthermore, any pair of such Poisson structures are compatible.

In particular, if the binomial series

$$A^t = 1 + t(A - I) + \frac{1}{2}t(t - 1)(A - I)^2 + \frac{1}{6}t(t - 1)(t - 2)(A - I)^3 + \ldots$$

is convergent, $(\Pi, A^t)$ is a Poisson-Nijenhuis structure, and $A^t #_\Pi$ defines a deformation of the Poisson structure $\Pi$. Easy evaluations show that existence and differentiability of $A^t$ are ensured if each point of $M$ has a compact coordinate neighbourhood where

$$||A - I|| < 1, \quad ||\partial A|| < 1,$$

for all the derivatives $\partial$ of all orders with respect to the corresponding local coordinates.

3) If $\{\Pi, ([, ]^A)\}$ is a generalized Poisson-Nijenhuis structure, Theorem 1.3 with $p = 1$ remains true i.e., $(\Pi_k, A)$ are again generalized Poisson-Nijenhuis structures, and $[\Pi_k; \Pi_h] = 0$. Indeed, the condition $i(#_{\Pi} \alpha) \Theta = 0$ included in the definition of a generalized Poisson-Nijenhuis structure implies $i(#_{\Pi} \alpha) \Theta = i(#_{\Pi} (\alpha \circ A)) \Theta = 0$, and also (because of (1.6)), $i(#_{\Pi} \alpha) \mathcal{N}_A = 0$, and the proof given for Theorem 1.3. still holds.

Another interesting result is given by

**THEOREM 1.4.** [MM] Let $\phi$ be a closed 2-form of a Poisson manifold $(M, \Pi)$, and put $A = #_\Pi \circ #_\phi^{-1}$. then $(M, \Pi, A)$ is a Poisson-Nijenhuis manifold iff $\forall \alpha \in \wedge^1 M$ one has $i(#_{\Pi} \alpha) d\phi = 0$, where $\tilde{\phi}(X, Y) = \phi(AX, Y)$.

**Proof.** Above we denoted $\#_\phi^{-1} X = i(X) \phi$, $X \in \Gamma(TM)$ (while, of course, $\#_\phi$ itself may not exist). Thereby

$$AX = #_\Pi(i(X)\phi), \alpha(AX) = \Pi(i(X)\phi, \alpha),$$

and we may use this to compute $\alpha(\mathcal{N}_A(X, Y))$. The result will be

$$\alpha(\mathcal{N}_A(X, Y)) = d\phi(Y, AX, #_\Pi \alpha) - d\phi(X, AY, #_\Pi \alpha)$$

$$+ d\tilde{\phi}(X, Y, #_\Pi \alpha) - \sigma_{#\Pi}(i(X)\phi, i(Y)\phi, \alpha).$$
To check (1.29), we just express its right-hand side by using the classical formula of the exterior differential and the similar formula of $\sigma_{\Pi}$, which is also an exterior differential for the suitable Lie algebroid. A lot of cancellations will occur, based on the definition of $A$ which implies the skew-symmetry of $\bar{\phi}$, (1.28), and the relation $\Pi(i(X)\phi, \alpha) = \phi(\Pi_{\alpha} X)$, and, also, based on (1.3).

Since $[\Pi, \Pi] = 0$, (1.24) shows that the last term of (1.29) vanishes. Accordingly, since $\phi$ is closed, $i(\Pi_{\alpha} \phi)d\phi = 0 \forall \alpha \in \Lambda^1 M$ is equivalent to $\mathcal{N}_A = 0$.

Furthermore, in this situation we can also check that condition (1.11) is satisfied. Indeed, with (1.28) we have, $\forall f \in C^\infty(M), \forall Y \in \Gamma(TM)$,

\[(1.30) \quad (df \circ A)(Y) = \phi(X_f, Y),\]

and the equality (1.11) evaluated on $Y$ becomes

\[(1.31) \quad d\phi([X_f, AY]) = d\phi([X_f, Y]) = d\phi(X_g, [X_f, Y]).\]

By (1.30), $d\phi([X_f, AY]) = \phi(X_g, [X_f, Y])$. Then

\[d\phi([X_f, AY]) = d\phi([\Pi_{\alpha} df, \Pi_{\beta} i(Y)\phi]) = d\phi(\Pi_{\alpha} df, \Pi_{\beta} i(Y)\phi) =
-\{df, i(Y)\phi\}_{\Pi}(X_g) \begin{array}{c}
(1.2) \\
\end{array} -L_{X_f}(i(Y)\phi)(X_g) + X_g(df(AY))

+ X_g(\Pi(df, i(Y)\phi) = -L_{X_f}(i(Y)\phi)(X_g),

where the last equality follows from (1.28). If these results are inserted in (1.31), and Lie and exterior derivatives are evaluated, we get the equivalent form

\[(1.32) \quad (L_Y \phi)(X_f, X_g) = (di(Y)\phi)(X_f, X_g).\]

Since $d\phi = 0$, this is true, and (1.11) holds. ■

Remark. Theorem 1.4 holds under the weaker hypothesis $i(\Pi_{\alpha} \phi)i(\Pi_{\beta} \phi) d\phi = 0$ ($\alpha, \beta \in \Lambda^1 M$), which is what one really uses in the proof, since the arguments $AX, AY$ in (1.29) and $X_f, X_g$ in (1.32) belong to $\text{im}\#_{\Pi}$. Notice also that (1.32) is implied by $i(\Pi_{\alpha} \phi) = 0, \forall \alpha$.

Corollary 1.5. Let $(M, \omega)$ be a symplectic manifold, and $\Pi$ its Poisson structure (i.e., $\#_{\Pi} \circ \#_{\omega}^{-1} = -\text{Id}$). Then, if $\Pi'$ is any Poisson structure compatible with $\Pi$ (i.e., $[\Pi, \Pi'] = 0$), and if $A = \#_{\Pi'} \circ \#_{\omega}^{-1}, (\Pi, A)$ is a Poisson-Nijenhuis structure.

Proof. Define $\bar{\omega}$ by $\bar{\omega}(X, Y) = \omega(AX, Y)$. If $\#$ is extended to differential forms by acting with it on the arguments, then $\forall \alpha, \beta \in \Lambda^1 M$, we get

\[(\Pi_{\alpha} \phi)(\alpha, \beta) \overset{\text{def}}{=} \bar{\omega}(\Pi_{\alpha} \phi, \Pi_{\beta} \phi) = \omega(AX, Y).

= -i(\Pi_{\beta} \phi) = \beta(\Pi_{\alpha} \phi) = \Pi'(\alpha, \beta).
Hence,
\[ [\Pi, \Pi'] = -\sigma_{\Pi} \Pi' = -\sigma_{\Pi}(\#_{\Pi}(d\check{\sigma})) = -\#_{\Pi}(d\check{\sigma}), \]
where the last equality follows from the expressions of \( \sigma \) and \( d \) in view of (1.3) (e.g., [V]). Thus, \([\Pi, \Pi'] = 0\) yields \( d\check{\sigma} = 0 \), and Theorem 1.4 tells us that \((\Pi', A)\) is a Poisson-Nijenhuis structure.

In particular, we see that \( A \) is a Nijenhuis tensor, and we shall check that (1.1) holds for the pair \((\Pi, A)\). Indeed, it is easy to see that we also have \( A = \#_{\Pi} \circ \#_{\check{\sigma}}^{-1}, \) whence, as for (1.30), we get \( df \circ A = i(X_f)\check{\sigma}, X_f = \#_{\Pi} df \). Now, the same computation as for (1.32) shows that (1.11) for \((\Pi, A)\) is equivalent to
\[ (L_f \check{\sigma})(X_f, X_g) = d(i(Y)\check{\sigma})(X_f, X_g), \]
where \( f, g \in C^\infty(M), Y \) is a vector field on \( M \). Since \( d\check{\sigma} = 0 \), this relation is true. ■

If \((\Pi, A)\) is a Poisson-Nijenhuis structure such that \( \Pi \) comes from a symplectic structure \( \check{\sigma} \), we shall say that \((\omega, A)\) is a symplectic-Nijenhuis structure.

2. Affine Poisson structures of cotangent bundles

In this section, we study a natural class of Poisson structures of a cotangent bundle \( T^*M \), and the particular case where such a structure is compatible with the canonical symplectic structure of \( T^*M \) and, thereby, when added to the latter, it provides \( T^*M \) with a symplectic-Nijenhuis structure and a hierarchy of Poisson-Nijenhuis structures.

First, we recall that the canonical symplectic form of \( T^*M \) is \( \omega_0 = d\lambda \), where \( \lambda \) is the Liouville 1-form. We shall use local coordinates \( x^i \) on \( M \) and corresponding covector coordinates \( \xi_i \) for \( \xi \in T^*M \). Then we have
\[ \lambda = \xi_i dx^i, \quad \omega_0 = d\xi_i \wedge dx^i, \]
where we use the Einstein summation convention.

The function space \( C^\infty(T^*M) \) has some interesting subspaces. Namely, we have (fiberwise) homogeneous \( k \)-polynomials
\[ \psi = t^{i_1 \cdots i_k}(x)\xi_{i_1} \cdots \xi_{i_k}, \]
where \( t \) is a symmetric tensor field on \( M \) (the form of (2.2) does not change by a change of the local coordinates \( x^i \)), and it is meaningful to define the space \( \mathcal{P}_k(T^*M) \) of nonhomogeneous \( k \)-polynomials (and the polynomial algebra \( \mathcal{P}(T^*M) = \bigcup_{k \geq 0} \mathcal{P}_k(T^*M) \)), and, in particular, the space \( \mathcal{A}(T^*M) = \mathcal{P}_1(T^*M) \) of the affine functions.

The latter are of the form
\[ a(x, \xi) = f(x) + m(X), \]
where \( f \in C^\infty(M), X \) is a vector field on \( M \), and \( m(X) \) is the momentum of \( X = X^i(\partial / \partial x^i) \), defined by
\[ m(X)(\xi) = \xi(X_u) = \xi_i X^i(u) \quad (u \in M, \xi \in T^*M). \]
The $\omega_0$-Hamiltonian vector field on $m(X)$ is given by

$$X^0_{m(X)} \overset{\text{def}}{=} \dot{X} = X^i \frac{\partial}{\partial x^i} - \xi^i \frac{\partial X^h}{\partial x^i} \frac{\partial}{\partial \xi_i},$$

(we have $i(X)\omega_0 = -d(m(X))$, and this vector field of $T^*M$ is also known as the complete lift of $X$ to $T^*M$ [Y], i.e., the flow of $\dot{X}$ is the natural lift of the flow of $X$ to $T^*M$). Notice that $\dot{X}$ is also characterized by

$$X(f \circ \pi) = Xf, \quad X(m(Y)) = m([X,Y]),$$

where $\pi : T^*M \rightarrow M$ is the natural projection, and $f \in C^\infty(M)$.

Hereafter, we write $f$ both for $f \in C^\infty(M)$, and for $f \circ \pi$. The Hamiltonian field of $f$ on $T^*M$ is

$$X^f = - \frac{\partial f}{\partial x^i} \frac{\partial}{\partial \xi_i},$$

and, up to the sign, this is just the vertical lift of $-df$ ($\forall x = \alpha_k dx^k$, the vertical lift is defined as $\alpha_k(\partial/\partial x^i)$ [YII]).

Using (2.7), we also get

$$fX = f\dot{X} - m(X)X^f.$$  

**DEFINITION 2.1.** A Poisson structure $\Pi$ on $T^*M$ is affine if the Poisson bracket of any two affine functions is again an affine function.

**REMARK.** An affine structure in the sense of Definition 2.1. is also affine in the sense of [W] and [DLSW] with respect to the natural affinoid structure of $T^*M$. This follows just as for $\mathbb{R}^{2n}$ in Example 4.4. of [W] but, now, the $\omega_{ij}$ of [W] are affine in our sense, and the brackets $\{\phi_i, \phi_j\}$ of [W] will be linear combinations of $\phi_k$.

The affine Poisson structures of $T^*M$ are determined by

**THEOREM 2.2.** Each affine Poisson structure $\Pi$ of $T^*M$ is bijectively associated with a pair $((\omega, \lambda)^t, A; \Phi)$ consisting of a Lie algebroid structure of $TM$, and a $d'$-closed 2-form $\Phi$ of $M$.

**Proof.** Any Poisson structure $\Pi$ of $T^*M$ is completely determined by the brackets $\{f,g\}$, $\{m(X),f\}$, and $\{m(X),m(Y)\}$ ($f,g \in C^\infty(M), X,Y \in \Gamma(TM)$) since these include the brackets of the local coordinates $x^i, \xi_i = m(\partial/\partial x^i)$. If $\Pi$ is affine, all these brackets are in $\mathcal{A}(T^*M)$. Thus,

$$\{m(X),m(fY)\} = f\{m(X),m(Y)\} + \{m(X),f\}m(Y) \in \mathcal{A}(T^*M),$$

$\forall f,X,Y$, whence, necessarily, $\{m(X),f\} \in C^\infty(M)$. If this happens, and if we look at

$$\{f,m(gX)\} = \{f,g\}m(X) + g\{f,m(X)\}$$
we see that we must have
\[ (2.9) \quad \{ f, g \} = 0. \]

Now, \( \{ m(X), . \} \) is a derivation of \( C^\infty(M) \). Hence, \( \forall X \in \Gamma(TM) \), there is a vector field \( AX \in \Gamma(TM) \) such that
\[ (2.10) \quad \{ m(X), f \} = (AX)f. \]
Moreover, if we change \( X \) by \( gX \) in (2.10), and we use again the Leibniz rule for the Poisson bracket, and (2.9), we see that \( A \) is an endomorphism of the tangent bundle \( TM \).

Furthermore, for given vector fields \( X, Y \) on \( M \), we must have some expression of the form
\[ (2.11) \quad \{ m(X), m(Y) \} = \Phi(X, Y) + m(\Psi(X, Y)), \]
where \( \Phi(X, Y) \in C^\infty(M) \) and \( \Psi(X, Y) \in \Gamma(TM) \) are skew-symmetric, and, then, a replacement of \( Y \) by \( fY (f \in C^\infty(M)) \) leads to
\[ (2.12) \quad \Phi(X, fY) = f\Phi(X, Y), \quad \Psi(X, fY) = f\Psi(X, Y) + [(AX)f]Y. \]

Therefore, \( \Phi \) is a 2-form on \( M \), and we can define a bracket of two vector fields of \( M \) by
\[ (2.13) \quad [X, Y]' = \Psi(X, Y). \]

Then, the Jacobi identity
\[ \sum_{\text{Cycl}(X,Y,Z)} \{ m(X), \{ m(Y), m(Z) \} \} = 0 \]
is equivalent to the pair of conditions
\[ (2.14) \quad \sum_{\text{Cycl}(X,Y,Z)} [X, [Y, Z]']' = 0; \]
\[ \sum_{\text{Cycl}(X,Y,Z)} \{ (AX)\Phi(Y, Z) - \Phi([X, Y]', Z) \} = 0, \]
where the first is the Jacobi identity for the new bracket, and the second means \( d'\Phi = 0 \), as required by the theorem.

Finally, the Jacobi identity
\[ \{ f, \{ m(X), m(Y) \} \} + \{ m(X), \{ m(Y), f \} \} + \{ m(Y), \{ f, m(X) \} \} = 0 \]
yields
\[ (2.15) \quad A[X, Y]' = [AX, AY], \]
which, together with (2.12) and (2.14) shows that \( ([ , ]', A) \) is a Lie algebroid structure on \( TM \).
Conversely, if \(((\cdot, \cdot), A); \Phi)(d^\Phi = 0)\) are given, the associated affine Poisson structure \(\Pi\) of \(T^*M\) will be determined by asking that the formulas (2.9), (2.10) and (2.11) hold good. This defines brackets of the functions \(f, m(X)\) which satisfy the Leibniz rule and the Jacobi identity for the functions mentioned.

In particular, we get some brackets for the local coordinates:
\[
\{x^i, x^j\} = 0, \{\xi_i, x^j\} = -x^j, \{\xi_i, \xi_j\} = \{m(\partial/\partial x^i), x^j\},
\]
and the Leibniz rule allows us to check that these are components of a well defined bivector field \(\Pi\) on \(T^*M\). (They have a tensorial behaviour when the local coordinates of \(M\) are changed). Then, the Jacobi identity is equivalent to \([\Pi, \Pi] = 0\), and we are done.

**Remarks.**

1) Affine Poisson structures may be defined similarly on the total space of any vector bundle \(\pi: E \to M\). Then, it follows as in Theorem 2.2. that such a structure is associated with a pair \(((\cdot, \cdot), A); \Phi)\) where \(((\cdot, \cdot), A)\) is a structure of a Lie algebroid on the dual bundle \(E^*\), and \(\Phi\) is a \(dE^*\)-closed cross-section of \(\wedge^2 E\). The linear case \(\Phi = 0\) was noticed in [DS].

2) For an affine Poisson structure \(\Pi\) on \(T^*M\), the Leibniz rule shows that, if \(\phi \in \mathcal{P}_k, \psi \in \mathcal{P}_h\), then \(\{\phi, \psi\}_\Pi \in \mathcal{P}_{k+h-1}\).

3) The basic Hamiltonian vector fields of the affine Poisson structure \(\Pi\) of \(T^*M\) are given by
\[
(2.16) \quad X^\Pi_f g = 0, \quad X^\Pi_f (m(X)) = -(AX)f
\]
(i.e., \(X^\Pi_f\) is the vertical lift of \(-df \circ A\)), and
\[
(2.17) \quad X^\Pi_{m(Y)} f = (AY)f, \quad X^\Pi_{m(Y)} (m(Z)) = \Phi(Y, Z) + m(\Psi(Y, Z))
\]
\((f, g \in C^\infty(M); Y, Z \in \Gamma(TM))\). In particular, using (2.6) we see that
\[
V_Y \overset{\text{def}}{=} X^\Pi_{m(Y)} - AY
\]
is a vertical vector field.

4) We can also give equivalent expressions of the brackets (2.10), (2.11) which show straightforwardly that the results depend only on the differentials of the involved functions. First, we have
\[
(2.10') \quad \{m(X), f\} = [d(m(X))](\text{vertical lift of } df \circ A).
\]
Furthermore, let us express the Lie bracket (2.13) as (1.4), which is equivalent to
\[
\Psi(X, Y) = (AX)(Y) + (L_X A)(Y) + \Theta(X, Y),
\]
where \(\Theta\) is a tensor, and introduce the following objects which are easily seen to be well defined on \(T^*M\):
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vertical lift of \( \Phi = \frac{1}{2} \Phi_{ij} dx^i \wedge dx^j \) def \( \frac{1}{2} \Phi_{ij} \frac{\partial}{\partial \xi_i} \wedge \frac{\partial}{\partial \xi_j} \),

vertical lift of \( \Theta = \frac{1}{2} \Theta^t_{ij} \frac{\partial}{\partial x_t} \otimes dx^i \wedge dx^j \) def \( \frac{1}{2} \Theta^t_{ij} \xi_k \frac{\partial}{\partial \xi_i} \wedge \frac{\partial}{\partial \xi_j} \),

vertical lift of \( B = B^t_{ij} \frac{\partial}{\partial x_t} \otimes dx^j \) def \( B^t_{ij} \xi_k \frac{\partial}{\partial \xi_i} \),

where \( B \) is any \((1,1)\)-tensor field on \( M \). Then, using also (2.6), we see that (2.11) is equivalent to

\[
(2.11') \quad \{m(X), m(Y)\} = \frac{1}{2} i(\text{vertical lift of } \Phi)(dm(X) \wedge dm(Y)) + dm(Y)(AX + (\text{vertical lift of } LXA)) + \frac{1}{2} i(\text{vertical lift of } \Theta)(dm(X) \wedge dm(Y)).
\]

This formula shows that \( \{m(X), m(Y)\} \) depends only on the differential \( dm(Y) \), and the same holds for \( dm(X) \) because of the skew-symmetry.

\textbf{Theorem 2.3.} An affine Poisson structure \( \Pi \) of \( T^*M \) is symplectic iff its corresponding anchor map \( A \) of (2.10) is nondegenerate, and the affine symplectic structures are exactly those associated with the symplectic forms

\[
(2.18) \quad \omega_\Pi = \pi^* \nu + d\mu,
\]

where \( \nu \) is a closed 2-form on \( M \), and \( \mu \) is the 1-form defined on \( T^*M \) by

\[
(2.19) \quad \mu_\xi(\mathcal{X}) = \xi^t A^{-1} \pi_\ast \mathcal{X} \quad (\xi \in T^*M, \mathcal{X} \in T_\xi T^*M).
\]

\textbf{Proof.} For a given \( \Pi \) with a nondegenerate \( A, \Psi \) of (2.11) is determined by \( A \) since (2.13) and (2.15) yield

\[
(2.20) \quad \Psi(X, Y) = A^{-1} [AX, AY].
\]

On the other hand, on \( M \) we may define the 2-form

\[
(2.21) \quad \nu(X, Y) = -\Phi(A^{-1}X, A^{-1}Y),
\]

and, because of (2.15), \( d'\Phi = 0 \) is equivalent to \( d\nu = 0 \). Conversely, \( A \) and the closed 2-form \( \nu \) define \( \Phi \) and \( \Psi \) by (2.20), (2.21) and we are provided, thereby, with an affine Poisson structure \( \Pi \).
Now, the local components of $\Pi$ are given by

\begin{align}
(2.22) \quad \Pi(dx^i, dx^j) &= \{x^i, x^j\} = 0, \\
(2.23) \quad \Pi(dx^i, d\xi^j) &= -\Pi(d\xi_j, dx^i) = \left\{ x^i, m \left( \frac{\partial}{\partial x^j} \right) \right\} = -A^i_j, \\
(2.24) \quad \Pi(d\xi_i, d\xi_j) &= \left\{ m \left( \frac{\partial}{\partial x^i} \right), m \left( \frac{\partial}{\partial x^j} \right) \right\} \\
&= -\nu \left( A \frac{\partial}{\partial x^i}, A \frac{\partial}{\partial x^j} \right) + m \left( A^{-1} \left[ A_i \frac{\partial}{\partial x^i}, A_j \frac{\partial}{\partial x^j} \right] \right) \\
&= -A^h_i A^j_h \nu_{hk} + \xi_k \alpha^k_i \left( A^u_i A^h_j - A^u_j A^h_i \right) \\
&= -A^h_i A^j_h \nu_{hk} - \left( A^u_i A^h_j - A^u_j A^h_i \right) \alpha^k_{h,i} \xi_k,
\end{align}

where $A(\partial/\partial x^i) = A^h_i (\partial/\partial x^j)$, the commas denote partial derivatives, $\alpha^k_i A^i_k = \delta^h_i$, and the last equality follows by using the derivative of this condition i.e.,

$$\alpha^k_{h,i} A^i_k = -\alpha^k_{h,i} A^i_k.$$ 

It follows that $det \Pi = det^2 A$, which justifies the first assertion of the theorem.

Then, the symplectic form $\omega_{\Pi}$ is determined by the relation

\begin{align}
(2.25) \quad \omega_{\Pi}(X, Y) &= \Pi(\alpha, \beta),
\end{align}

where $\alpha, \beta, x, y \in \Lambda^1 T^* M$, and $\alpha = \#_\Pi \alpha, Y = \#_\Pi \beta$. With the calculated components of $\Pi$, we get

\begin{align}
(2.26) \quad \#_\Pi(dx^p) &= -A^i_p \frac{\partial}{\partial \xi^j}, \\
(2.27) \quad \#_\Pi(d\xi_p) &= A^i_p \frac{\partial}{\partial x^i} + A^u_p A^h_i \nu_{uv} \frac{\partial}{\partial \xi^i} \\
&- \left( A^u_i A^h_j - A^u_j A^h_i \right) \alpha^k_{h,i} \xi_k \frac{\partial}{\partial \xi^i},
\end{align}

whence

\begin{align}
(2.28) \quad \frac{\partial}{\partial \xi^j} &= -\alpha^j_p \#_\Pi(dx^p), \\
(2.29) \quad \frac{\partial}{\partial x^i} &= -\alpha^j_p \#_\Pi(d\xi^p) + \nu_{s\gamma} + (\alpha^k_{s,j} - \alpha^k_{s,i}) \xi_k \#_\Pi(dx^s).
\end{align}

With (2.25), (2.28), (2.29) and (2.23), (2.24) we can compute the local components

$$\omega_{\Pi} \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right), \quad \omega_{\Pi} \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial \xi^j} \right), \quad \omega_{\Pi} \left( \frac{\partial}{\partial \xi^i}, \frac{\partial}{\partial \xi^j} \right),$$

$$\omega_{\Pi} \left( \frac{\partial}{\partial \xi^i}, \frac{\partial}{\partial x^j} \right).$$
and it turns out that the result is precisely the 2-form (2.18). The local expression of \( \mu \) is
\[
\mu = \xi_k \alpha^k_i dx^i .
\]

**Remark.** At the opposite end, we can consider affine Poisson structures with \( A = 0 \). It is easy to see that these consist of differentiable (with respect to \( M \)) families of affine Poisson structures of the fibers of \( T^*M \).

**Proposition 2.4.** For every pair consisting of a closed 2-form \( \nu \) of \( M \), and a Nijenhuis tensor \( A \) of \( M \) one can define an associated affine Poisson structure of \( TM \) by taking
\[
(2.30) \quad \Phi(X,Y) = \nu(AX,AY); \Psi(X,Y) = [AX,Y] + [X,AY] - A[X,Y].
\]

**Proof.** It is known from Section 1 that \( \Phi(X,Y) = [X,Y]_A, A \) is a Lie algebroid structure on \( T^*M \), and \( d\nu = 0 \) implies \( d\Phi = 0 \).

Now, we give a result which relates the present section to section 1.

**Theorem 2.5.** The affine Poisson structure \( \Pi \) defined by the formulas (2.9), (2.10), (2.11) is compatible with the Poisson structure of the canonical symplectic form \( \omega_0 \) iff its associated algebroid \( (\Psi(X,Y), A) \) is a Nijenhuis algebroid, and its 2-form \( \Psi \) is closed.

**Proof.** If \( \Pi_0 \) and \( \Pi \) are Poisson structures of \( M \), then \( [\Pi_0, \Pi] = 0 \) iff \( [\Pi_0 + \Pi, \Pi_0 + \Pi] = 0 \), i.e., \( \Pi_0 + \Pi \) is again a Poisson structure. If \( \Pi_0 \) belongs to \( \omega_0 \) (i.e., \( \# \Pi_0 = \#^{-1}_0 = - \text{Id} \)), and if we denote by an index + the bracket of the sum, then (2.6), (2.7), (2.9), (2.10), (2.11) yield
\[
(2.31) \quad \{f, g\}_+ = 0, \{m(X), f\}_+ = (A + \text{Id})(X) f, \quad \{m(X), m(Y)\}_+ = \Phi(X,Y) + m([X,Y] + \Psi(X,Y)).
\]
If this is a Poisson structure, it is affine and this happens iff \( [X,Y] + \Psi(X,Y) \) is a Lie algebroid bracket of anchor \( A + \text{Id} \), and \( \Phi \) is closed for this bracket. In view of the fact that \( \Phi, \Psi \) satisfy the conditions of Theorem 2.2, the previous conditions become
\[
(2.32) \quad \Psi(X,Y) = [AX,Y] + [X,AY] - A[X,Y],
\]
\[
(2.33) \quad d\Phi = 0 ,
\]
\[
(2.34) \quad \sum_{\text{cur}(X,Y,Z)} \{\Psi(X,[Y,Z]) + [X,\Psi(Y,Z)]\} = 0 .
\]
If we apply \( A \) to (2.32), we get \( \mathcal{N}_A = 0 \), and we see that \( \Pi \) is associated with a Nijenhuis algebroid. It follows also easily that (2.34) is identically satisfied, modulo (2.32).
COROLLARY 2.6. Let $A$ be a Nijenhuis tensor, and $\nu$ a closed 2-form of $M$ such that its associated 2-form $\Phi(X, Y) = \nu(AX, AY)$ is also closed. Let $\Pi$ be the affine Poisson structure associated by Proposition 2.4. with $((\cdot, \cdot)_A, A, \Phi)$ on $T^*M$. Then $\Pi$ is compatible with the canonical structure $\Pi_0$, and $T^*M$ has a corresponding hierarchy of Poisson-Nijenhuis structures, which starts with $(\Pi_0, A = \#_{\Pi} \circ \#_{\omega}^{-1})$. In particular, the cotangent bundle $T^*M$ of a Kaehler manifold $M$ has a natural pair of compatible symplectic structures.

Proof. The conditions of Theorem 2.5. are satisfied. The existence of the Poisson-Nijenhuis hierarchy follows from Corollary 1.5. If $M$ is Kaehler with the complex structure $J$, the metric $g$, and the Kaehler form $\Omega$, we may use the first part of the present corollary for $A = J$, $\nu = \Omega$, and get the second symplectic structure of $T^*M$ required. In this case, the hierarchy contains no other structures because of $J^2 = -Id$.

Corollary 1.5. can be applied to $(\Pi_0, \Pi)$ for any structure $\Pi$ as in Theorem 2.5., and we have a corresponding Poisson-Nijenhuis structure $(\Pi_0, A)$ on $T^*M$, where the Nijenhuis tensor is $A = \#_{\Pi} \circ \#_{\omega}^{-1}$.

PROPOSITION 2.7. For any $\Pi_0$-compatible affine Poisson structure $\Pi$ of $T^*M$, the associated Nijenhuis tensor $A$ is $\pi$-related to $-A$, and the vertical subspaces of $TT^*M$ are $A$-invariant.

Proof. $\pi : T^*M \to M$ is the natural projection, and the required $\pi$-relatedness means

$$\pi_*(A\mathcal{X}) = -A(\pi_*\mathcal{X}) \quad (\mathcal{X} \in T_\xi T^*M)$$

It suffices to check (2.35) for the generators of $T_\xi T^*M$ i.e., for $\mathcal{X} = \#_{\Pi_0} df (f \in C^\infty(M))$ and $\mathcal{X} = \#_{\Pi_0} dm(X) (X \in \Gamma(TM))$. In these cases, $A\mathcal{X} = -X_f^\Pi$ and $A\mathcal{X} = -X_m^\Pi(X)$, respectively, and (2.35) follows from (2.5), (2.7), (2.16) and (2.17).

Furthermore, as usual, by vertical we mean tangent to the fibers of $T^*M$, and (2.7) shows that the vertical subspace $\mathcal{V}(T^*M)$ is spanned by $\#_{\Pi_0} df (f \in C^\infty(M))$. Since $A\#_{\Pi_0} df = -X_f^\Pi$ which is vertical by (2.16), we see that $\mathcal{V}$ is $A$-invariant.

We end by a result on infinitesimal automorphisms of the symplectic-Nijenhuis structures of $T^*M$ given by Theorem 2.5. These are interesting since they are the Hamiltonian dynamical systems whose integrability could be studied by the method of [MM] (when applicable).

PROPOSITION 2.8. Let $\Pi$ be a $\Pi_0$-compatible affine Poisson structure of $T^*M$. Then, the following vector fields are infinitesimal automorphisms of the associated symplectic-Nijenhuis structure: i) the vertical lifts of the closed 1-forms $\alpha$ of $M$ such that $d(\alpha \circ A) = 0$; ii) the complete lifts $\bar{X}$ of the vector fields $X$ of $M$ such that $L_X A = 0$, $L_X \Phi = 0$; iii) the $\Pi_0$-Hamiltonian fields $X^0_{m^\alpha(X)}$, where $X \in \Gamma(TM)$, and $i(X)\Phi = 0$, $AX = \lambda X$ with $\lambda = \text{const.}$, $L_X A = 0$. 

Proof. Of course, in this formulation, $A$ is a Nijenhuis tensor on $M$, and $\Phi$ is a closed 2-form such that $\Pi$ is associated with $([, A, A], \Phi)$ in the sense of Theorems 2.2, 2.5.

Clearly, a vector field $\mathcal{X}$ of $T^*M$ is an infinitesimal automorphism iff $L_{\mathcal{X}}\omega_0 = 0$ and $L_{\mathcal{X}}\Pi = 0$. Since any $\omega_0$-Hamiltonian vector field $\mathcal{X}$ satisfies $L_{\mathcal{X}}\omega_0 = 0$, (2.5) and (2.7) show that all the vector fields of i), ii) and iii) satisfy this condition. Therefore, we must only check $L_{\mathcal{X}}\Pi = 0$.

We have the general formula

$$(L_{\mathcal{X}}\Pi)(d\phi, d\psi) = \mathcal{X}\{\phi, \psi\}_\Pi - \{\mathcal{X}\phi, \psi\}_\Pi - \{\phi, \mathcal{X}\psi\}_\Pi,$$

where $\phi, \psi \in C^\infty(T^*M)$. Hence, $L_{\mathcal{X}}\Pi = 0$ if (2.36) vanishes for $\phi, \psi$ either in $C^\infty(M)$ or equal to momenta of vector fields of $M$. That this happens for the vector fields i), ii), iii) of the proposition can be checked by lengthy but technical computations. (Notice that, since $d\Phi = 0$, we also have $L_{\mathcal{X}}\Phi = 0$ in case iii).

REFERENCES


[MM] MAGRI F., MOROSI C., A geometrical characterization of integrable Hamiltonian systems through the theory of Poisson-Nijenhuis manifolds, Quaderno S. 19, Univ. of Milan, 1984.


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