Rend. Sem. Mat. Univ. Pol. Torino
Vol. 52, 3 (1994)
WALT1

E. Gallo

CONTROL AND SOLUTION OF "ALGEBRAIC PROBLEMS"

Abstract. The control provokes the adaptation of the models activated by the pupil during the problem solving process in order to create an ad hoc model for the solution: we are studying the control movement (descending or ascending) during the solution of algebraic exercises.

There are semantic controls linked to the objectives and meaning and syntactic controls linked to formal procedures and rules.

1. Models and control

The term *control* has acquired particular importance in our research: in fact, in problem solving the pupil recalls, uses and constructs models which suit his purpose and the action which links the subjective pole (what the pupil knows) to the objective pole (what the pupil produces) is precisely control.

As we see it, control is therefore the action which provokes the adaptation of the *models* activated by the pupil during the problem-solving process in order to create a model which suits the situation, namely an ad hoc model for the problem which needs to be solved.

The reference framework for the analysis of control takes into account the degree of structuring and the degree of applicability of the models at hand, thus enabling us to present the following four cases:

1. the model is formed and appropriate to the problem;
2. the model is formed but not appropriate to the problem;
3. the model is not formed but appropriate to the problem;
4. the model is not formed and not appropriate to the problem.

In case 1 (in which the model is already ad hoc) and case 4 (in which the model cannot become ad hoc) control is not decisive; it is in cases 2 and 3 that the pupil produces, or does not produce, an ad hoc model for the problem precisely due to the action of control.

It is therefore important to ask ourselves whether control is activated,

- when it becomes evident;
- what is it revealed by;
- what is it caused by;
- what it produces.

On the basis of our research on control we can now affirm that:

It appears	at times when	the problem is assimilated
	at times when	a strategy is elaborated
	at times when	a choice is acted upon
	at times when	the chosen answer is verified
	at times of	comparison
	at times of	doubt
	at times of	impasse
	....	....

It is revealed by	pauses
	questions
	gestures
	re-reading the text
	tests
	....
It is caused by	the interaction between information/models/products
	conflicts between different models
	the rigidity of standard and stereotype models
	the non-appropriateness of current models
	the incomplete formation of the current model
	knowledge-related obstacles
	....
It produces	the choice of starting model
	the evolution of the model: adjustment/formation
	changes in the model
	changes in strategy
	....

When control is activated, it is manifested by *descendant and ascendant dynamics*: descending from the mental model to the actions which produce the temporary solution; ascending from the solution produced to a new mental model.

Let us return to cases 2 and 3 which were described above and examine their control dynamics. In case 2 the model was formed in earlier situations, for which it was relevant, but it is not appropriate to the present situation and, in its present form, will not lead to the solution of the problem; the control is initially of the descendant type, from the pupil's representations to his actions, but what is produced by the pupil must provide information which may lead to modifications at the representative level, thus using an ascendant-type control. In case 3 the model is not formed and its subsequent construction takes place using an ascendant-type control whose purpose is to integrate the elements necessary to the solution of the problem by exploring the plane of actions; the model formed in this way then exerts a descendant-type control on actions.

The descendant/ascendant movement of control *produces a series of problem-interpretative cycles*, linked to one another to form a chain of re-interpretations of the problem-solving situation: it is during the transition phases from one interpretation to another that light is thrown on the variables involved in the situation.

2. Control in psychology

The term control, which we have used in the analysis of the pupil's actions in problem-solving situations, has been taken from psychological research and transferred into the context of research into mathematics teaching *by interpreting the psychologist's subjects as pupils*: the problem-solving process is in fact seen here as being centred on the subject/pupil in relation to whom it is possible to identify questions and results in psychology to be transferred to didactics so as to interpret the data observed and to construct a general explanation.

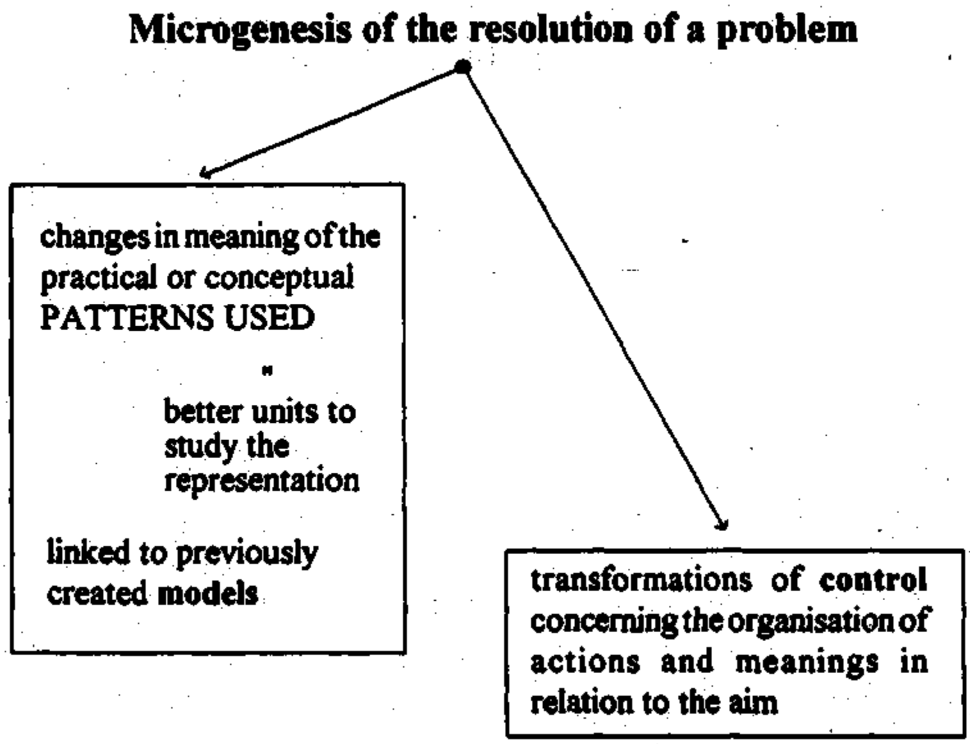
In particular, we referred to the theoretical framework on which Saada-Robert based his interpretations of individual problem-solving protocols, in which the cognitive activity in problem-solving situations is described as a microgenesis of functional units under construction which uses partially adapted models and leads to the formation of a representative ad hoc model for the problem set. During the microgenesis which constitutes the solution of a problem, representation fulfils the dual function of significantly linking appropriate knowledge and the data of the situation (translation function) and organising knowledge according to the conditions of the situation itself (control function).

In Saada-Robert's own words:

"*La microgénèse de la résolution d'un problème sera envisagée ici sous un double aspect: celui des changements de significations concernant les schèmes utilisés, pratiques ou conceptuel (liés à des modèles formés antérieurement) et concernant les objets (réels ou de pensée y compris leurs relations) et celui des transformations de contrôle, concernant l'organisation des actions et des significations en fonction du but*".

In the passage quoted the term "schème" indicates, in line with the meaning given to it by Vergnaud, "la meilleure unité pour étudier la représentation", defined by Vergnaud himself as the "conceptualisation du réel à travers les connaissances pour agir efficacement".

We have translated the passage from Saada-Robert as follows:



With reference to transformations of control, Saada-Robert again emphasizes that: *"il s'agit de transformation relatives au découpage opéré sur le problème (réduction heuristique). Elles visent la formation d'un bon "objet-à-penser", d'un micromonde de travail privilégié, en même temps que celle d'un prototype de résolution".*

Control therefore appears to be a set of mechanisms which organise the meanings constructed by the individual during the solution of a problem, namely during the functional path towards its solution.

3. Control in problem-solving

The term control is also used by Schoenfeld who attempts to understand the nature of mathematical thought through problem-solving.

For this purpose he focuses attention on resources, heuristics, control and belief systems which he presents in the following table.

Knowledge and Behaviour Necessary for an Adequate Characterization of Mathematical Problem-Solving Performance

---

Resources: Mathematical knowledge possessed by the individual that can be brought to bear on the problem at hand

- Intuitions and informal knowledge regarding the domain
- Facts
- Algorithmic procedures
- "Routine" nonalgorithmic procedures
- Understandings (propositional knowledge) about the agreed-upon rules for working in the domain

Heuristics: Strategies and techniques for making progress on unfamiliar or nonstandard problems; rules of thumb for effective problem solving, including:

- Drawing figures; introducing suitable notation
- Exploiting related problems
- Reformulating problems; working backwards
- Testing and verification procedures

Control: Global decisions regarding the selection and implementation of resources and strategies

- Planning
- Monitoring and assessment
- Decision-making
- Conscious metacognitive acts

Belief Systems: One's "mathematical world view", the set of (not necessarily conscious) determinants of an individual's behaviour

- About self
- About the environment
- About the topic
- About mathematics

Schoenfeld writes:

"The idea is that mathematical performance depends not only on what one knows but on how one uses that knowledge, and with what efficiency".

He adds that an important factor is:

"the set of understandings about mathematics that establish the psychological context within which individuals do mathematics".

Schoenfeld pursues his research in line with this theoretical framework and analyses protocols by breaking them down into episodes relating to the four centres of interest identified above.

His aim is to describe the behaviour of a good problem solver; in other words, to construct a general model for problem solving, or a model of cognitive function during this activity.

This context differs from that present in our research in which the terms control and model assume other meanings: our observations of pupils cannot be interpreted within such a systematic framework.

Nevertheless, *the problem-solving terminology might be used to interpret our material.*

4. Algebraic manipulation

Having interpreted algebra as a literal language, we now aim to focus attention on the construction and structuring stage of this language: in particular, our field of research is the *structural aspect of literal language* which gives meaning to formal transformations.

For this purpose we will examine pupils who are thought to have overcome the letter introduction phase and work using *literal calculation.*

This brings us to a very tricky didactic question which is worth examining; it is disguised by the term *algebraic manipulation.*

I quote from Thorpe:

"Whether it is the manipulation of algebraic expressions or the manipulation of algebraic equations, there is room for discussion and research on how proficient pupils in today's technological world need to be in manipulative skills. Just as the widespread availability of calculators has removed much of the rationale for insisting on a high level of competence in the algorithms of arithmetic, the growing availability of microcomputers and symbol manipulation software is about to remove much of the rationale for insisting that algebra pupils attain a high level of competence in symbol manipulation".

The pupil thus needs to understand which structure attributes meaning to what he does when he manipulates algebraic expressions which only mean something in this context.

And Thorpe cites Pollak:

"They will also need to understand something about why algebraic manipulation works, the logic behind it. In the past, such recognition skills and conceptual understanding have been learned as a by-product of manipulative drill, if learned at all. The challenge is now to teach skills and understanding even better while using the power of machines to avoid large time allotments to tedious drill" (Conference Board of the Mathematical Sciences, 1983).

Manipulation is therefore what is required to pass from what one has to what one wants to have; it is what stems from semantic comprehension; it is what is aimed at general algebraic processes.

The word manipulation is therefore freed from the negative connotation with which it is sometimes marked and raised to the level of useful, necessary, significant knowledge: *when manipulating algebraic symbols one must know where they come from, what they mean, how and why they can be modified.* The aim of algebraic teaching therefore shifts from acquiring skills in algebraic manipulation towards acquiring understanding at a conceptual level.

5. Literal calculation as problem-solving

The question at the basis of our research on algebraic manipulation is therefore: *what is the control movement shown during the solution of an algebraic problem?*

The control which interests us at this stage of the research is that exercised by the pupil without any outside influence; in other words, it is the control which leads the pupil to form his own solution to the problem, which may or may not be the true one; it is the pupil's inner control which is brought into play between his models and his actions of symbolic writing.

The exercises considered here are normal algebraic exercises rather than algebraic procedures used to solve problems of a different nature.

The pupils, aged between 14 and 16, are observed during the manipulation of algebraic expressions, thinking of

literal calculation as the problem to be solved.

In order to create an *experimental device* to suit our research in which

* control is necessary;

** the means of control are evident,

we chose to make the pupils work on *algebra exercises*

- *with a non-immediate objective*, that is to say exercises in which the rules to be applied must be found by re-elaborating data (direct exercises),
- *involving the completion* of an incomplete algebraic equality so that the equality becomes true (inverse exercises).

We set the following exercises since their solution would oblige pupils to control themselves

FACTORIZE

1a) $a^2 - b^2 + a^3 - a^2b =$

$\quad\downarrow \quad \downarrow \quad \downarrow \quad \downarrow\downarrow$

1b) $p^2 \quad q^2 \quad p^3 \quad p^2\, q$

2) $y^3 - x^2 + y^2 - xy^2 =$

3) $4x^2 - 9y^2 - 12x + 9 =$

4) $4a^2 - 4a^4 + b^4 - 4ab^2 =$

COMPLETE, SO THAT THE EQUALITY BECOMES TRUE

5a) $(\ldots\ldots) \cdot 5x = 30x^2 + 5ax$

$\quad\downarrow \quad \downarrow \quad \downarrow\downarrow$

5b) $\quad y \quad y^2 \quad b\, y$

6a) $(\ldots\ldots) \cdot 5x = 3x^2 + 10a^2x$

6b)

7a) $4a \cdot (\ldots\ldots) = \frac{1}{3}a^2x + 10ax^2$

7b)

8) $4a \cdot (\ldots\ldots) = \frac{1}{3}a^2x + 10x^2$

9) $(\ldots\ldots) \cdot 3x = 9x^2 - 3yz^2x$

10) $(\ldots\ldots) \cdot 3x = 5x^2 + 12a^2x$

11) $7b \cdot (\ldots\ldots) = \frac{1}{3}b^2x - 10ab$

12) $7b \cdot (\ldots\ldots) = \frac{1}{3}b^2x - 10a$

13) $\frac{5a}{a^2-1} + \frac{\cdots\cdots}{a+1} = \frac{8a-3}{a^2-1}$

14) $\frac{1}{a-x} + \frac{\cdots\cdots}{a+x} = \frac{3a-x}{a^2-x^2}$

15) $\frac{5b^2x^2}{\cdots\cdots} \cdot \frac{2a^2-2}{b} = \frac{x}{a-1}$

16) $\frac{3x^2-3}{a} \cdot \frac{2ay^2}{\cdots\cdots} = \frac{y}{x+1}$

17) $\frac{6-6x}{2x+2} = \frac{\cdots\cdots}{x^2-1}$

18) $\frac{10b-10a}{2a+2b} = \frac{\cdots\cdots}{a^2-b^2}$

The pupil receives the text of the exercise with instructions *to read and solve the exercise, thinking aloud*: it is explained to the pupil that *he must say what he is doing and what he is thinking while he is doing it*, justifying what he does and thinks if and when he feels this is necessary, not with the idea of having to explain to others.

The words must mediate his thoughts.

In this way we found ourselves using the observation method known as *thinking aloud* which was first developed in the 30s but only recently gained support in concomitance with the construction, by cognitive psychology, of a more powerful theoretical reference framework (based on the ideas of information and memory) and in parallel with the possibility of using more reliable recording instruments (e.g. video recorders).

6. A problem-solving process

The *materials* used to analyse the problem-solving process are, at this stage,

a) the pupil's written solution;

b) the video recording of the entire process.

We include a sample transcription of the materials produced in a three-column protocol in which the first column contains all that was said by the solver; the second contains everything that the solver wrote, placed in chronological sequence with what was said, and indicates errors and impasses; the third column contains the interpretation of the problem-solving process and identifies the phases of control.

The following protocol refers to one of the problems to be completed.

	$\frac{5a}{a^2-1}+\frac{\quad}{a+1}=\frac{8a-3}{a^2-1}$	COMPLETE, SO THAT THE EQUALITY BECOMES TRUE
I must complete this equality by writing something as the numerator so that the sum becomes true.		– the pupil selects his objective • descending control semantic control
Let me start with the denominator.		– selects his strategy • descending control
The denominator equals one of the sum's denominators		• ascending control symbolic/perceptive control
but I don't know where to put $(a+1)$		– is in doubt as to what to do • ascending control
The least common multiple...		– recalls a procedure • descending control (involving resources)
Now I understand: I can factorize $(a-1)(a+1)$	$\frac{5a}{(a-1)(a+1)}+\frac{\quad}{a+1}=\frac{8a-3}{a^2-1}$	
The denominator is O.K.		
I must take all the factors with the highest exponent		– recalls a rule • descending control (involving resources)

They're equal; let me take the first		
I've got to find something that, added to the first... no		– comes back to the objective • descending control semantic control
using the l.c.m. I must .... no		– comes back to l.c.m. • descending control
I'm writing it down again, as if it were $(a^2-1):(a^2-1)=1$ $1 \cdot 5a = 5a$ I've got it;		– applies a rule • descending control (involving resources)
$(a^2-1):(a+1)=a^2-1$; no $(a^2-1):(a+1)=a-1$	ERROR	• ascending control • descending control syntactic control
I've got to find a number that, when multiplied by $(a+1)$, is right when added to $5a$;	ERROR	
$8a$; so I've got to find $3a$		• descending control syntactic control
perhaps it's ... +	BLOCK produced by the addition that guides the solution	• ascending control (from the objective)
This gives me $(a-1)$		• descending control syntactic control
That -3 must be in the second addendum;		• ascending control textual control
which means it's +3	$\frac{5a + \qquad +3}{a^2-1} = \frac{8a-3}{a^2-1}$	
no... it must end up as -3.		
Let me write it down again; perhaps it'll come to mind. I must find a number that added...		– comes back to the objective to overcome the obstacle: the meaning remains the same

I can take this away from that	$\frac{8a-3}{a^2-1} - \frac{5a}{a^2-1}$	– changes strategy
This is wrong; it can't be done; it's wrong because it won't give 1....; done in this way it's wrong	BLOCK created by the comparison of the denominators a^2-1 and $a+1$ which differ from each other.	• ascending control symbolic/perceptive control
If I'm left with $(a-1)$ $(a+1)$..... it'll give $3a$		– uses the previous strategy again
Perhaps I've got it now	$\frac{5a}{(a-1)(a+1)} + \frac{\quad}{a+1}$	– returns to the addition
	$\frac{5a}{(a-1)(a+1)} + \frac{\quad}{a+1} = \frac{8a-3}{a^2-1}$ $\frac{5a}{(a-1)(a+1)} + \frac{3}{a+1} = \frac{8a-3}{a^2-1}$	– rewrites the text and completes the solution correctly by writing 3 in the empty space
	$\frac{5a+3a-3}{a^2-1} = \frac{8a-3}{a^2-1}$	– checks the solution • descending control syntactic control

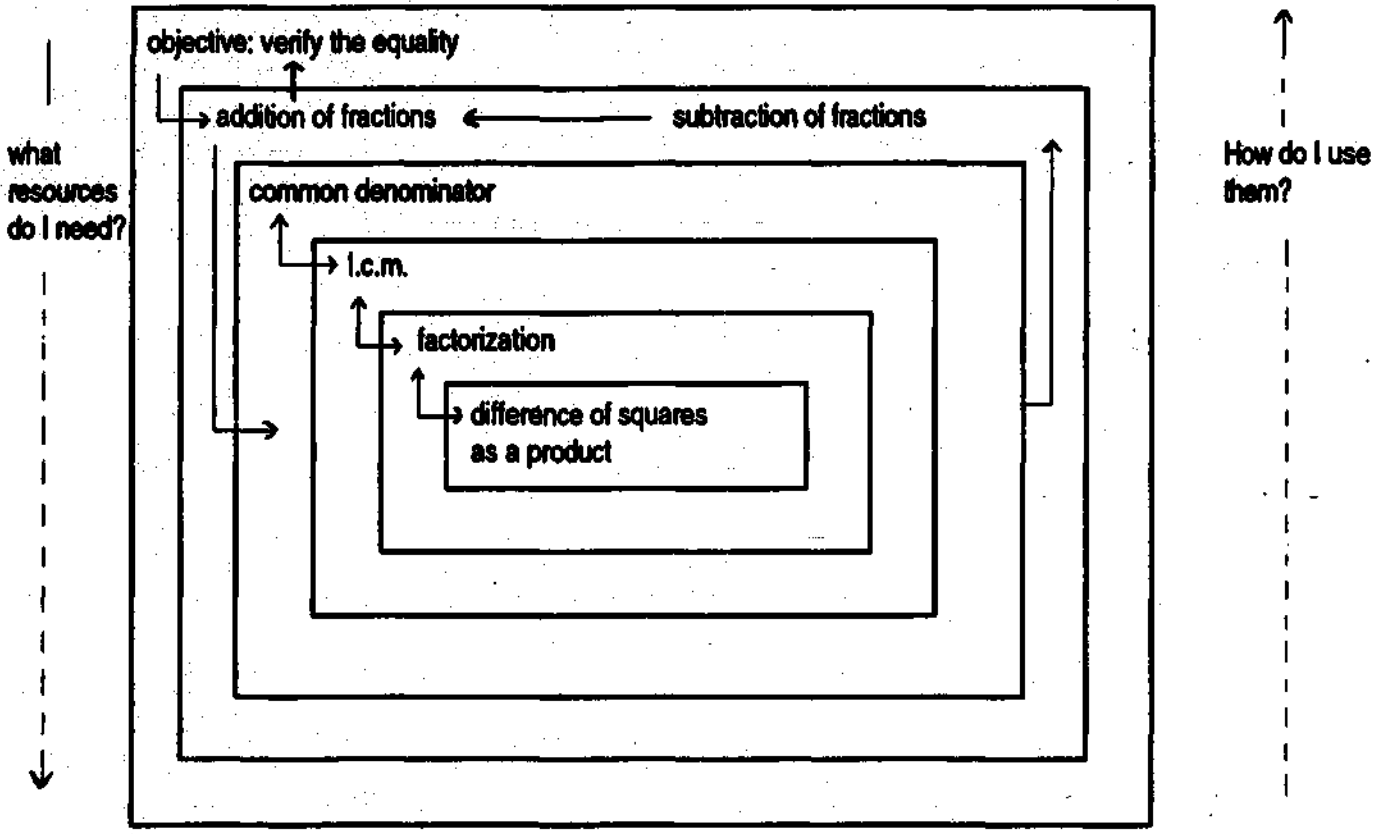
Protocols constructed in this way allow a sequential interpretation of the problem-solving process: they entail the "real time" involvement of strategies – resources – controls which influence one another throughout the entire process.

7. Interpretation of problem-solving process

From the synchronous interpretation of strategies – resources, – controls provided by the protocols it is possible to pass to overall interpretations relating to one of the above aspects; these are constructed by working back over the entire problem-solving process in question.

For example, a possible interpretation concerning strategies, taken from the protocol reported above, is represented in the following diagram (see next page).

It can be seen that, *from the point of view of pupils' strategies, the problem-solving process may be likened to a series of open cycles*, each of which starts with a representation of the problem (which prompts the pupil to proceed at the level of actions) and leads to a new representation (matured from what it has produced). Each cycle corresponds to a temporary representation of the problem, one within which the pupil is working at the time. From this point of view *the solution of a problem appears to be a series of reinterpretations*

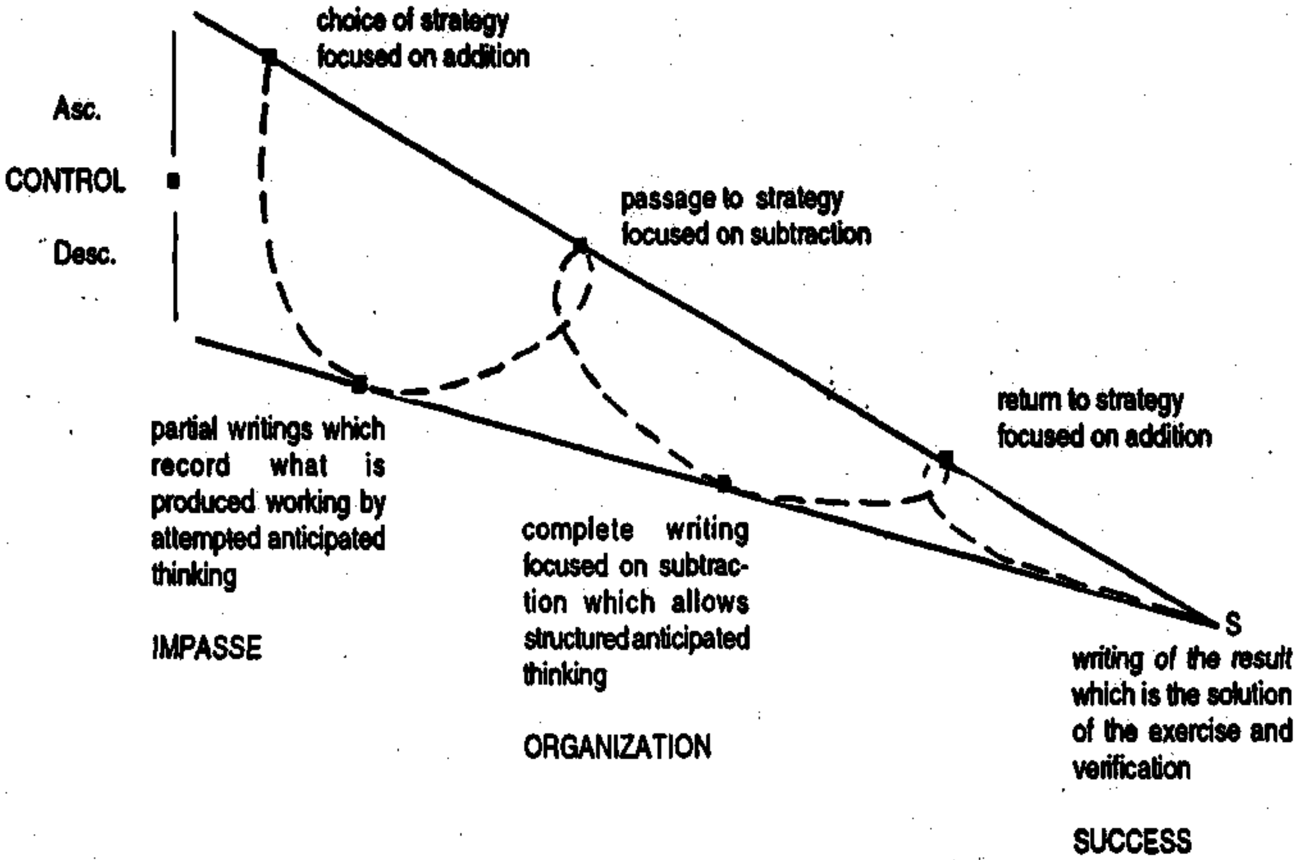


which result in the adjustment of the starting model until an adequate model is produced: control is the action which links the two levels of organisation of thought and the products of actions with an ascending and descending movement.

If we turn our attention, on the other hand, to the resources required for the task (both in terms of knowledge, but also knowing how to use knowledge), a possible interpretation of the protocol illustrated earlier is given in the following diagram (see next page), which translates the successive steps of the problem-solving process (taken more from the spoken than the written record).

It can be seen that *the organisation of resources is hierarchical and corresponds to a breakdown of the general objective into sub-objectives*: control acts as a descendant-type control at the moment of the choice of resources and their instrumental use for the attainment of the objective, and as an ascendant-type control during the stage when the resources are adapted to the context of the problem to be solved.

The analysis of the transformations of control in the above protocol confirms our overall affirmation: *control is manifested* in the choice and elaboration of strategies, during the recall and use of theoretical knowledge and rules, the comparison with objectives and verification, doubt and impasse at the level of semantic and syntactic anticipated thinking and action, the correcting of errors made; *control is revealed* by pauses, re- reading the text, attempted answers, changes in writing; *control is caused* by the rigidity of the model



focused on addition which allows only one procedure to be elaborated by trial and error, by the interference between different interpretations of a text, by the understanding of formal texts; *control causes* the choice of the starting strategy and its internal sub-objectives, the passage to the model focused on subtraction with the subsequent recovery of the strategy focused on addition, the correcting of errors at a syntactic level and of the "impasse" created by the + sign relating to the addition which "guides" the solution under the influence of the objective of the exercise.

There are *semantic controls* linked to the objectives and meanings, and *syntactic controls* linked to formal procedures and rules.

In conclusion, it may be said that the problem-solving process is a complex system whose analysis requires the use of concepts and languages taken from more than one interpretative sector: our analysis of the protocols is based on the individual's cognitive reinterpretation of the situation which can be described using the key words of problem-solving.

REFERENCES

[1] ACKERMANN-VALLADAO E., AUDÈTAT J., GIDDEY C., LOCK N., PIGUET-CHEVALLEY D., REITH E., SAADA-ROBERT M., *Formation et actualisation des modèles du sujet en situation de résolution de problème*, Archives de Psychologie **51** (1983).

[2] BELL A., ONSLOWO B., PRATT K., PURDY D., SWAN MB., *Diagnostic teaching: teaching for long term learning*, ESRC project, Shell Centre, Nottingham (1985).

[3] BELL A., *Putting purpose into school algebra*, Algebra Working Group, ICME 7-Quebec (1992).

[4] BOOTH L.R., *A question of structure*, Research Issues in the Learning and Teaching of Algebra (Eds. S.Wagner and C.Kieran), NCTM (1989).

[5] BRUN J., CONNE F., *Analyses didactiques de protocoles d'observation du déroulement de situations*, Education et recherche **12**, 3 (1990).

[6] CHEVALLARD J., JOHSUA M.A., MERCIER A., *Problèmes d'ingenierie didactique dans l'enseignement de debuts de l'algèbre*, III^e Ecole d'Eté de Didactique des Mathématiques, Orléans (1984).

[7] CHEVALLARD J., *Enseignement de l'algèbre et transposition didactique*, Seminari dell'IRRSAE-Piemonte, Torino (1986).

[8] FREUDENTHAL H., *Didactical phenomenology of mathematical structures*, Dordrecht, Reidel (1983).

[9] GAGNÉ E.D., *Psicologia cognitiva e apprendimento scolastico*, Torino, SEI (1989).

[10] GALLO E., GRANGE T., *Trasposizioni di livello nella rappresentazione mentale del linguaggio formale: studio dell'uso dell'incognita per la soluzione di problemi*, Ricerche di Didattica della Matematica CNR, Quaderno n. 5, Torino (1988).

[11] GALLO E., TESTA C., *A study of the conflicts in the production and in the comprehension of Mathematics texts*, Topic Group 8 "Language and Mathematics", ICME 6 - Budapest (1988).

[12] GALLO E., AMORETTI C., TESTA C., *Utilisation de modèles géométriques en situation de résolution de problèmes: contrôle descendant et ascendant*, Proceedings CIEAEM 41, Bruxelles (1989).

[13] GALLO E., BATTÙ M., TESTA C., *The control in problem resolution (poster)*, Proceedings PME XV, Assisi (1991).

[14] GALLO E., TESTA C., *Modèles, stratégies, types de contrôle dans la résolution d'un problème graphique de géométrie*, TDTM, n. 8, Tessalonica (1991).

[15] GALLO E., *Elaboration of models for problem resolution in interaction with 14/15 year-old pupils*, Proceedings II° Italian-German Bilateral Symposium, Osnabrück (1992).

[16] GALLO E., *Le contrôle dans la résolution de problèmes: une situation de classe*, Proceedings ICSIMT-CIEAEM 44, Chicago (1992).

[17] KIERAN C., *The early learning of algebra: a structural perspective*, Research Issues in the Learning and Teaching of Algebra (Eds. S.Wagner and C.Kieran), NCTM (1989).

[18] KIERAN C., *Cognitive processes involved in learning school algebra*, Mathematics and Cognition (Eds. P.Nesher and J.Kilpatrick), ICMI Study Series (1990).

[19] SAADA-ROBERT M., *La microgénèse de la représentation d'un problème*, Psychologie Française **34**, 2/3 (1989).

[20] SCHOENFELD A.H., *Mathematical problem solving*, San Diego, California, Academic Press (1985).

[21] SFARD A., *On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin*, Educational Studies in Mathematics **22** (1991).

[22] SFARD A., LINCHEVSKI L., *Equations and inequalities: processes without objects?*, Proceedings PME XVI, New Hampshire USA (1992).

[23] SFARD A., *The development of algebra. Confronting historical and psychological perspectives*, Algebra Working Group, ICME 7 - Quebec (1992).

[24] THORPE J.A., *Algebra: what should we teach and how should we teach it?*, Research Issues in the Learning and Teaching of Algebra (Eds. S.Wagner and C.Kieran), NCTM (1989).

[25] USISKIN Z., *Conceptions of school algebra and uses of variables*, The ideas of Algebra, K-12, 1988 Yearbook NCTM (1989).

[26] WAGNER S., KIERAN C., *Research issues in the learning and teaching of Algebra*, Research Agenda for Math. Education, vol. 4, Lawrence Erlbaum, NCTM (1989).

Elisa GALLO
Dipartimento di Matematica, Università di Torino
Via Carlo Alberto 10, 10123 Torino, Italy.

Lavoro pervenuto in redazione il 15.12.1993.