Finite-difference Eulerian hydrodynamics: second-order convective fluxes on a radially moving spherical mesh

Abstract. This paper describes the transport algorithm that has been implemented in the construction of a new multi-dimensional, finite-difference, explicit, radiation hydrodynamics code in spherical Eulerian coordinates. The temporal discretization is optionally first- or second-order accurate while the spatial differences are obtained using a volume-centred technique which results in a conservative, second-order scheme. The code has been primarily designed for studying protostellar collapse and fragmentation. For advection on a fixed mesh, the spatial second-order accuracy of the spherically symmetric code version is demonstrated through convergence testing. The use of a moving Eulerian grid, however, results in systematic errors in the advection of mass and momentum if special corrections are not included in the difference representation of the hydrodynamical fluxes. The pressureless collapse of a homogeneous sphere is used to derive and test these corrections for both the temporally first- and second-order schemes. Although the method specializes to the case of a radially moving spherical grid, it is in general applicable to any other Eulerian coordinate system. The new flux representations make the full transport algorithm invariant under grid motion, thereby preserving the second-order accuracy of the volume-centred scheme.

1. Introduction

Finite-difference (FD) methods for solving the Eulerian equations of hydrodynamics have been long applied in numerical astrophysics. In the context of the theory of star formation, a quite relevant problem that has most contributed to the expansion and testing of these methods concern the dynamical contraction and fragmentation of a self-gravitating
molecular gas cloud.

A protostellar gas cloud can be regarded as a continuous medium because the mean free path of the gas particles is much shorter than the typical distances over which significant physical changes occur in the system and so its time evolution can be completely described in terms of macroscopic quantities obeying the Eulerian equations of hydrodynamics. A self-consistent approach to the protostellar collapse problem then involves the solution of these equations in a full three-dimensional (3D) space. In multi-dimensions, however, the fluid-dynamic equations are analytically intractable even with restrictive assumptions and simplified physics, and so modelling of the flow will unavoidably require their numerical solution. Out of the variety of numerical methods that could be employed to obtain accurate solutions, one of the most attractive and widely tested is the method of finite differences. This method is essentially a discretization technique characterized by a high degree of simplicity, robustness and modularity. One further advantage of FD schemes is that they can be easily modified and extended to include new algorithms.

A general drawback of Eulerian FD collapse calculations is the progressive lack of resolution whenever small-scale condensations of size comparable to the local mesh-spacing form in the inner regions of the flow. This deficiency is usually mitigated by solving the governing differential equations in a moving orthogonal coordinate system that approximates the actual dynamic collapse of the flow. The definition of such coordinate systems leads to the notion of grid re-zoning (LeBlanc & Wilson 1970). In a particular convenient formulation of a moving Eulerian grid, originally due to Black & Bodenheimer (1975) and thereafter implemented in many other FD protostellar hydro-codes (e.g. Boss 1980; Tohline 1980; Norman & Winkler 1986; Sigalotti 1990; Stone & Norman 1992), only the convective part of the differential equations of hydrodynamics needs to be modified to account for the effects of fluid advection in a moving mesh and volume-zone changes due to grid expansion or contraction. In this formulation both effects are quantified simultaneously in the transport algorithm. The use of standard, second-order accurate FD replacements for the convective terms, however, results in systematic numerical errors in a moving grid. These errors amplify gradually with time resulting in a net loss of accuracy and are essentially independent of the method chosen to perform fluid advection.

The level of discussion of this paper involves a self-contained description of the transport algorithm that has been implemented in the construction of a new 3D FD radiation hydrodynamics code (Sigalotti, in preparation). Although the algorithm has been originally designed for studying protostellar collapse and fragmentation, it could be applied to a large variety of astrophysical systems. In its simplest form, the code solves the equation of continuity and motion for a compressible fluid, coupled with the Poisson equation, on a radially moving spherical grid. The time integration of the fluid-dynamic equations is
made explicitly and is optionally first- or second-order accurate. The spatial discretization is performed on a non-staggered mesh using a volume-centred technique (Mönchmeyer & Müller 1989) which results in a conservative, second-order preserving FD scheme in curvilinear Eulerian coordinates. Consistent advection (Norman, Wilson & Barton 1980) is fully implemented in the definition of the advective fluxes and a tensor artificial viscosity (Tscharnuter & Winkler 1979) is incorporated to stabilize the FD scheme against shock formation. The evolution of the fluid variables is calculated by means of a multi-step solution procedure.

New FD replacements have been used in the temporally first- and second-order versions of the code to correctly design the hydrodynamical fluxes on a moving mesh. Such replacements include appropriate temporal corrections for the advective quantities which make the transport scheme invariant under grid motion, thereby yielding numerical solutions which are free of systematic errors resulting from the straightforward application of standard FD replacements. The homologous collapse of a pressureless sphere (Hunter 1962) has been used as a fundamental test calculation to derive these corrections. Although the discussion specializes to the case of a radially moving spherical grid, the method is applicable in general to other coordinate systems as well as to any other problem involving the solution of the hydrodynamic equations on a moving FD orthogonal mesh. In Section 2 we write down the basic equations and outline the general method used to solve them. Sections 3 and 4 describe, respectively, the transport algorithm and the corrections made to the FD advective fluxes on a radially moving spherical grid for both the temporally first- and second-order transport schemes. The results obtained from a few important collapse test problems are discussed in Section 5 while Section 6 summarizes the relevant conclusions.

2. Basic Equations and Methodology

The fundamental equations that govern the gravitational collapse of a compressible, non-heat-conducting, inviscid, rotating gas configuration, written in conservative integral form in a moving coordinate system, are: the equation of continuity

\[ \frac{d}{dt} \int_V \rho dV + \int_S \rho \mathbf{U} \cdot \mathbf{n} dS = 0, \]

the vector equation of motion

\[ \frac{d}{dt} \int_V \rho \mathbf{V} dV + \int_S \rho \mathbf{V} \cdot \mathbf{n} dS + \int_V \nabla \rho dV + \int_V \rho \nabla \Phi dV = 0, \]

and the Poisson equation

\[ \nabla^2 \Phi = 4\pi G \rho, \]

where \( \mathbf{U} = \mathbf{v} - \mathbf{v}_g \) denotes the fluid velocity relative to the moving coordinate system.
The time derivatives and spatial integrations, in equations (1) and (2), operate on a time-dependent volume element $dV$ and surface element $dS$ with unit vector $n$, moving with velocity $v_g$ with respect to a fixed Eulerian observer. The quantities $\rho$, $v$, $p$ and $\Phi$ are functions of space and time, and denote, respectively, the mass-density, the Eulerian fluid velocity field, the gas pressure and the gravitational potential.

The use of the integral formulation is especially convenient because the fluid-dynamic equations when written in integral form still preserve their Eulerian structure in a moving coordinate system. The convective terms reduce to a flux-integrated form, thereby ensuring global conservation of the advected quantities. In addition, the integral equations do not demand continuity of the flow variables through the advecting fluid interfaces which are represented numerically as true discontinuities.

Equations (1)-(3) are sufficient to describe the gravitational collapse of isothermally (or adiabatically) evolving protostars if a pressure relation of the form $p \propto \rho^\gamma$, where $\gamma$ is the adiabatic index, is used to close the system. For axisymmetric (2D) and non-axisymmetric (3D) flows in spherical polar coordinates, equation (2) splits up into three scalar equations for the $r$-, $\theta$-, and $\phi$-momentum. These equations together with eqs. (1) and (3) form a coupled system of five non-linear equations which can be solved for given initial and boundary conditions once the grid velocity $v_g$ is specified.

An explicit multi-step solution method (e.g. Norman & Winkler 1986) is used to integrate equations (1) and (2). In this approach, the time integration is performed in a sequence of independent parts which can be grouped into two main steps, called the source and the transport step. The source step is designed by solving FD representations of the vector equation

$$\frac{d}{dt} \int_V \rho v dV = - \int_V \nabla p dV - \int_V \rho \nabla \Phi dV - \int_S Q \cdot n dS,$$

where the third term on the right-hand side, is purposely included to numerically stabilize the difference scheme in zones of shock formation. This term gives the contribution of artificial viscous stresses, $Q$, to the fluid acceleration. Equation (4) is actually solved under the assumption that the mass-density does not vary with time due to the effects of fluid acceleration consistently with the statement of equation (1). Furthermore, the source-step calculation is made independent of the effects of grid motion, and hence the time-derivatives and spatial integrations in equation (4) are not operating on moving volume and surface fluid elements.

The effects of cell-volume changes due to grid motion are evaluated simultaneously with fluid advection in the transport step. In this step, fluid variables are finally advanced
in time by solving difference representations of the conservative continuity-like equations

\[
\frac{d}{dt} \int_V \rho dV = - \int_S \rho \mathbf{U} \cdot n \, dS, \tag{5}
\]

\[
\frac{d}{dt} \int_V \rho \mathbf{v} dV = - \int_S \rho \mathbf{v} \mathbf{U} \cdot n \, dS, \tag{6}
\]

where \( dV \) and \( dS \) are now time-dependent as required on a moving coordinate system. The surface integral terms on the right-hand side of eqs. (5) and (6) represent, respectively, the time rate of change of mass and momentum due to the mechanical transport of fluid elements and are evaluated numerically using the updated velocity, \( \mathbf{v} = (v_r, v_\theta, v_\phi) \), resulting from the source step. In this way, the effects of the instantaneous fluid acceleration are properly included in the calculation of the advective fluxes. The FD scheme used for the numerical solution of these equations defines the transport algorithm which is the central discussion of this paper.

3. Transport Equations in Difference Form

On a non-staggered mesh, scalar and vector quantities are defined in the same type of zone and so any FD approximation of eqs. (5) and (6) will lead to identical difference equations. Using the generalized quantity \( \chi \), where \( \chi \) may be \( \rho \), \( v_r \), \( v_\theta \) or \( v_\phi \), all the transport equations can be written in the general scalar form

\[
\frac{d}{dt} \int_V \rho \chi dV = - \int_S \rho \chi \mathbf{U} \cdot n \, dS, \tag{7}
\]

where \( \mathbf{U} = (v_r - v_g, v_\theta, v_\phi) \) for a radially moving spherical grid.

Standard FD replacements which are spatially second-order accurate in Cartesian coordinates can still be used to approximate the model equation (7) in curvilinear coordinates within the same order of accuracy provided that all variables are assigned at volume rather than at geometrical cell centres. The fundamental reason for this distinction is that in a non-Cartesian mesh the value of any scalar quantity at the volume centre of a cell will represent a mean value for that cell to second-order accuracy (Mönchmeyer & Müller 1989). This leads to the notion of volume-centred discretization in general curvilinear coordinates.

We shall now specialize in the spatial discretization of equation (7) on a non-staggered spherical grid. The grid is defined by first specifying the coordinates of geometrical cell centres \( (r_i, \theta_j, \phi_k) \). Cell interface locations in each allowed direction are then calculated as simple averages: \( r_{i\pm1/2} = \frac{1}{2}(r_i + r_{i\pm1}) \), \( \theta_{j\pm1/2} = \frac{1}{2}(\theta_j + \theta_{j\pm1}) \) and \( \phi_{k\pm1/2} = \frac{1}{2}(\phi_k + \phi_{k\pm1}) \), so that the volume, \( V_{ijk} \), of a cell is given by

\[
V_{ijk} = \frac{1}{3} \left( r_{i+1/2}^3 - r_{i-1/2}^3 \right) \left( \cos \theta_{j-1/2} - \cos \theta_{j+1/2} \right) \left( \phi_{k+1/2} - \phi_{k-1/2} \right). \tag{8}
\]
For simplicity, the $\theta$- and $\phi$-meshes are here assumed to be perfectly equidistant ($\theta_{j+1} = \theta_j + \Delta \theta; \phi_{k+1} = \phi_k + \Delta \phi$) while the radial grid can become non-uniformly spaced ($\Delta r_{i+1} = r_{i+1} - r_i \neq \Delta r_i$) due to the assumption of grid motion along the $r$-direction. Following Mönchmeyer & Müller (1989), we can show that to second-order accuracy in space the coordinates of volume centres $(\hat{r}_i, \hat{\theta}_j, \hat{\phi}_k)$ obey the simple relations

\[ \hat{r}_i = \frac{3}{4} \left( \frac{r_{i+1/2}^4 - r_{i-1/2}^4}{r_{i+1/2}^3 - r_{i-1/2}^3} \right) \left[ 1 + O\{(\Delta \theta)^2\} + O\{(\Delta \phi)^2\} \right], \]

\[ \sin \hat{\theta}_j = \sin \theta_j + O\{(\Delta \theta)^2\}, \]

\[ \hat{\phi}_k = \phi_k, \]

where the definition of $\hat{\theta}_j$ is subject to the assumption that $\Delta \theta_{j+1} = \Delta \theta_{j} = \Delta \theta$. For non-equidistant $\theta$-meshes, $\hat{\theta}_j$ will coincide to $\theta_j$ only to first-order accuracy, and hence corrections must be performed in a non-uniform $\theta$-grid to achieve second-order accuracy. Only the $\phi$-coordinate behaves as a Cartesian one independently of the cell spacings. Furthermore, the radial distance between the volume and the geometrical centre, $\hat{r}_i - r_i$, is largest near the origin of the coordinate system and becomes progressively shorter at larger radii. For reasonable small angular spacings, the volume centre (9), represents a second-order approximation to the exact location of the centre of mass of a cell for an arbitrary density distribution. Thus the volume integral of a function $F$ over a grid cell can be calculated to spatial second-order accuracy by simply multiplying the local mean value of $F$, say $F_{ijk}$, defined at $(\hat{r}_i, \hat{\theta}_j, \hat{\phi}_k)$ with the cell volume

\[ \left\{ \int_V F dV \right\}_{ijk} = F_{ijk} V_{ijk} + O\{(\Delta r_i)^2(\Delta \theta)^2(\Delta \phi)^2\}. \]

Second-order FD representations for the surface integral in equation (7), are obtained in a similar fashion by multiplying the local mean surface fluxes with the surface area provided that the fluxes are evaluated at the area centres (Mönchmeyer & Müller 1989). Denoting the cell surfaces perpendicular to $r$, $\theta$- and $\phi$-direction by $S^r$, $S^\theta$ and $S^\phi$, respectively, the coordinates of area centres for a 3D zone in a spherical coordinate grid with uniform $\theta$- and $\phi$-spacing, are

\[ r_{S^r} = r_{i+1/2} \left[ 1 + O\{(\Delta \theta)^2\} + O\{(\Delta \phi)^2\} \right]; \theta_{S^r} = \hat{\theta}_j; \phi_{S^r} = \hat{\phi}_k, \]

\[ r_{S^\theta} = r_i^4 \left[ 1 + O\{(\Delta \phi)^2\} \right]; \tan \theta_{S^\theta} = \tan \theta_{j+1/2} \left[ 1 + O\{(\Delta \phi)^2\} \right]; \]

\[ \phi_{S^\theta} = \hat{\phi}_k, \]

\[ r_{S^\phi} = r_i^4 \left[ 1 + O\{(\Delta \theta)^2\} \right]; \sin \theta_{S^\phi} = \frac{(\cos \theta_{j-1/2} - \cos \theta_{j+1/2})}{\Delta \theta} \left[ 1 + O\{(\Delta \theta)^2\} \right]^{-1}; \phi_{S^\phi} = \phi_{k\pm1/2}, \]
where

$$ r_i^A = \frac{2}{3} \left( \frac{r_{i+1/2}^2 - r_{i+1/2}^2}{r_{i+1/2}^2 - r_{i-1/2}^2} \right). $$

With these definitions, the mean value of a variable at the area centre of cell surfaces $S^r$ and $S^\theta$ has to be obtained from its values at neighbouring volume centres by means of 2D interpolations while the calculation of area-centred values at cell surfaces $S^\phi$ requires the use of a 3D interpolation scheme. For instance, the mean value of a quantity $F$ at the area centre of cell surfaces $S^r$, denoted by $F_{i+1/2,j,k}$, can be obtained to second-order accuracy using the average relation

$$ F_{i+1/2,j,k} = \frac{1}{2} \left( F_{ijk} + F_{i+1,j,k} \right), $$

where the $F_{ijk}$ are calculated from their corresponding values at volume centres, $F_{ijk}$, through 1D linear interpolations of the form

$$ F_{ijk} = F_{ijk} - \left( \frac{\partial F}{\partial r} \right)_{ijk} (\hat{r}_i - r_i) + O((\hat{r}_i - r_i)^2), $$

where the radial derivative of $F$ is evaluated at volume centres by means of a three-point FD replacement modified by a non-linear filter to enforce monotonicity. The action of the non-linear filter is to allow the FD approximation of the derivative to make the transition from second to first-order accuracy whenever strong local variations of $F$ occur. The spatial correction implied by the interpolation scheme (14) will not affect the second-order accuracy of the surface flux $F_{i+1/2,j,k}$ because the radial off-set $\hat{r}_i - r_i$ is always a very small fraction of the local grid spacing. Second-order accurate mean values of $F$ at area centres of cell surfaces $S^\theta$ and $S^\phi$ are obtained in a similar fashion.

With the above construction for the mean surface values and using relation (10), the model equation (7) can be written in explicit FD form using the temporally and spatially second-order representation

$$ \frac{\rho x V_{i,j,k}^{n+1} - \rho x V_{i,j,k}^n}{\Delta t} = - \left( [S^r F_{i}^\prime x_{i+1/2,j,k} - [S^r F_{x}^\prime x_{i-1/2,j,k})^n + 1/2 \right. $$

$$ - \left. (S^\theta F_{x}^\prime x_{i,j+1/2,k} - [S^\theta F_{x}^\prime x_{i,j-1/2,k})^{n+1/2} \right) $$

$$ - \left. (S^\phi F_{x}^\prime x_{i,j,k+1/2} - [S^\phi F_{x}^\prime x_{i,j,k-1/2})^{n+1/2} \right), $$

where the superscript $n+1$ applies to update cell values (at the end of the time interval $\Delta t = t^{n+1} - t^n$) and $n+1/2$ denotes time-centred quantities (at time $t^{n+1/2} = t^n + \frac{1}{2}\Delta t$). The notion of consistent advection (Norman, Wilson & Barton 1980) is fully implemented in the definition of the hydrodynamical fluxes which are calculated by multiplying the mass flux $F$ with the mean surface value of $x$ associated with the advected fluid element. Note
that setting \( \chi = 1 \) in equation (15) we automatically recover the FD form of the continuity equation (5). The mass fluxes which enter in equation (15) are defined according to

\[
F_{i\pm 1/2,j,k} = \langle p \rangle_{i\pm 1/2,j} \left( v_{ri\pm 1/2,j,k} - v_{ri\pm 1/2} \right),
\]

\[
F_{i,j\pm 1/2,k} = \langle p \rangle_{i,j\pm 1/2,k} v_{r\phi_{i,j\pm 1/2,k}},
\]

\[
F_{ij,k\pm 1/2} = \langle p \rangle_{ij,k\pm 1/2} v_{ij,k\pm 1/2},
\]

where angled brackets are used to mean that cell surface values of the mass-density are determined by means of some stable advection method contrarily to the flux velocities which are calculated by linearly interpolating to the cell faces. Spatial second-order accuracy results if all interface quantities in (16) are evaluated at area centres of cell surfaces. With these minor spatial corrections, equation (15) represents a true volume-centred approximation to the conservative integral equation (7). Global conservation of the advected quantities is always ensured because summing up equation (15) over any collection of cells in the system, the contribution of the fluxes evaluated at common zone interfaces cancels pairwise so that the time rate of change of any numerically advected quantity will only depend on the flux across the grid boundaries.

4. Advective Fluxes on a Moving Mesh

The use of a moving grid imposes further modifications to the FD form of the advective fluxes to preserve the second-order accuracy of the volume-centred scheme. We shall refer to these modifications as the temporal flux corrections. The analytical solution of the collapse of a pressureless sphere is used to derive and test these corrections for the case of a radially moving spherical grid. The corrections are independent of the specific method chosen to perform fluid advection and they can be equally applied to any other orthogonal coordinate system.

4.1. The Pressureless Collapse Test Problem

The homologous contraction of a pressureless sphere provides a useful test case to design the temporal corrections that should be implemented for the advective fluxes in order to minimize the growth of systematic errors induced by grid motion. Hunter (1962) demonstrated analytically that a uniform-density \((p_0)\), unperturbed, spherically symmetric gas cloud with zero pressure, initially at rest \((v_r = 0)\), can condense to higher densities in such a way that the density remains uniform in space at any instant. Fixing the origin of time as the instant at which the collapse started \((t_0 = 0)\), Hunter's solution for \( t > t_0 \) reads

\[
\rho(r,t) = \beta_1(t),
\]

\[
v_r(r,t) = r\beta_2(t),
\]
for \( r < R(t) \), where \( R(t) \) is the time-varying radius of the homologously contracting sphere, and \( \beta_1 \) and \( \beta_2 \) are known functions of time. Any deviation of the solution from the uniform-density plateau induced by numerical errors then behaves as a perturbation. In a pressure-free object, density perturbations will grow unimpeded during collapse and so any small error caused by an improper numerical treatment can be easily detectable.

For this spherically symmetric test problem, the last two terms on the right-hand side of equation (15) vanish as a consequence of having \( v_\theta = v_\phi = 0 \) at any position for \( t \geq t_0 \). Thus only the radial fluxes will contribute to the time rate of change of mass and momentum. Assuming that the cell values of \( \rho \) and \( v_r \) at arbitrary time \( t = t^n \) are equal to their corresponding analytical values

\[
\rho_{i,j,k}^n = \beta_1(t^n), \\
v_{i,j,k}^n = \beta_2(t^n) r_i^n,
\]

the vector equation (4) can be written in FD form and solved to give

\[
v_{i,j,k}^{a,n} = \left[ \beta_2(t^n) + \Delta t \beta_3(t^n) \right] r_i^n \equiv \psi(t^n) r_i^n,
\]

for the accelerated \( r \)-velocities, where \( r^n \) denotes the radial coordinate of volume centres at time \( t = t^n \) and the function \( \beta_3 \) defines the time-dependent part of the gravitational acceleration forces, \( \partial^2 \Phi(r,t) / \partial r^2 = r \beta_3(t) \). The artificial viscous tensor is designed to vanish during perfect homologous collapse and so the last term appearing on the right-hand side of equation (4) will not contribute to the fluid acceleration. Correcting the accelerated velocities (18) for the effects of volume centring according to (14) and then interpolating to the cell faces in \( r \)-direction using (13), we find for the advective velocities the relation

\[
v_{r,i+1/2,j,k}^{a,n} = \psi(t^n) r_{i+1/2,j,k}^{n}.
\]

Thus linearly interpolating the volume-centred values of \( v_{r,i,j,k}^{a,n} \) to define flux velocities will not alter the linear behaviour of \( v_r \) expected for homologous collapse. The cell-surface velocities (19) will then be used in the transport step to construct mass and momentum fluxes in the radial direction.

### 4.2. Temporally First-Order Accurate Transport

A temporally first-order approximation to equation (7) is readily obtained from equation (15) by setting \( n + 1/2 \rightarrow n \) for the flux terms

\[
\frac{(\rho x V)^{i,j+1}_{k}}{\Delta t} - (\rho x V)^{i,j}_{k} = -\left( S_{i+1/2,j,k}^r \mathcal{F}_{i+1/2,j,k} x_{i+1/2,j,k} - S_{i-1/2,j,k}^r \mathcal{F}_{i-1/2,j,k} x_{i-1/2,j,k} \right)^n,
\]

where the mass fluxes, \( \mathcal{F} \), are as defined by (16) and \( x \) may be 1 or \( v_r \). The angular fluxes vanish during homologous collapse and so they have been omitted in writing equation
Using the pressureless collapse solutions (17) and (19), equation (20) can be solved explicitly to give

$$\rho_{ij,k}^{n+1} = \beta_1(t^n) \left[ 1 + 3\Delta t \psi(t^n) \frac{V_{ij,k}^n}{V_{ij,k}^{n+1}} \right],$$

(21)

$$\left( \rho u_r \right)_{ij,k}^{n+1} = \beta_1(t^n) \psi(t^n) \left[ 1 + 4\Delta t \psi(t^n) \frac{r_i^n V_{ij,k}^n}{r_i^{n+1} V_{ij,k}^{n+1}} \right] \hat{r}_i^{n+1}. $$

For a fixed grid ($V_{n+1} = V_n; \hat{r}_{n+1} = \hat{r}_n$), the above solutions reduce to those obtained by Mönchmeyer & Müller (1989) using a similar analysis on a staggered spherical mesh. For a radially contracting grid, however, $V_n > V_{n+1}$ for cells near the centre ($r = 0$) while $V_n < V_{n+1}$ in the outermost zones. Thus even for sufficiently small values of $\Delta t$, the factor $V_n/V_{n+1}$ will induce a numerical error in the mass advection algorithm which then amplifies, causing the solution to deviate from the analytical plateau after several timesteps. The error is somewhat smaller for advection of radial momentum due to the presence of the multiplying factor $r_i^n/r_{i+1/2}^{n+1}$ which is always less than unity. Therefore, any anomalous increase of the advective radial velocities in the central zones due to volume changes will be partially counterbalanced. Results (21) then show that mass and momentum are advected through the moving grid with progressively increasing velocities towards the centre and decreasing velocities towards the outer regions, eventually destroying the accuracy of the numerical solutions.

The analytical profiles can be, however, reproduced numerically by applying corrections to the fluxes in equation (20) in order to account for the effects of cell-interface motion during the time interval $\Delta t$

$$S_{i+1/2,j}^{n+1} = S_{i+1/2,j}^n + \Delta t \left( \frac{\partial S^r}{\partial t} \right)_{i+1/2,j}^n,$$

where $\partial S^r / \partial t < 0$ for radial grid contraction. Similarly, time extrapolated values of the advective velocities can be obtained using

$$u_i^{(n+1)} = u_i^n + \Delta t v_i^{n+1} \left( \frac{\partial u_r}{\partial r} \right)^{n+1}_{i+1/2,j},$$

(22)

where the superscript $(n+1)$ denotes interface values corrected for the effects of cell surface motion and is not related to the new hydrodynamical time level $n+1$. To temporal first-order accuracy, the velocity of the cell surfaces is simply defined by

$$v_{i+1/2}^n = \frac{r_i^{n+1} - r_i^{n+1}}{\Delta t},$$

(23)

and is assumed to not vary during the time interval $\Delta t$. Furthermore, the radial derivatives of the fluid velocity in (22) are evaluated at the area centres of cell surfaces $S^r$ by means...
of central FD replacements. With these prescriptions, the advective fluxes in equation (20) can then be corrected according to

$$ S_{t±1/2, jk}^r \cdot F_{t±1/2, jk}^{n,n} \cdot \chi_{t±1/2, jk}^n \rightarrow \langle \rho \rangle_{t±1/2, jk}^{n,n} \left[ \frac{S_{t±1/2, jk}^r \cdot v_{t±1/2, jk}^{(n+1)} \cdot \chi_{t±1/2, jk}^{(n+1)}}{\partial t} \right], $$

(24)

where the values of $\chi^{(n+1)}$ are obtained using corrections identical to (22). The origin of this formulation for the fluxes on a moving mesh can be better understood by a direct inspection of the surface integral term in equation (7) which can be written as the difference of two surface integrals

$$ \int \rho_\chi U \cdot n dS \rightarrow \int_\Sigma \rho_\chi v_r dS_r - \int_\Sigma \rho_\chi v_g dS_r. $$

Expanding the first term in Taylor series about the time $t = t^n$ and retaining only first-order terms in time, we then recover the flux corrected form (24). Note that for homologous collapse, the mass-density cannot vary due to grid contraction and so no corrections are needed for the cell-surface values of $\rho$, say $\langle \rho \rangle$. In a few test runs carried with the 1D version of the temporally first-order code, the inclusion of temporal corrections for the advected mass-density in the sense of equation (22) did not appear to alter the final solution for the isothermal collapse of a pressure-supported configuration. Thus only the advective fluid velocities need be corrected for advection on a moving grid. In the absence of such corrections they will represent the main source of error in the transport of mass and momentum as is just inferred from (21).

Substitution of relations (19) and (23) into equation (22) yields

$$ v^{(n+1)}_{t±1/2, jk} = \psi(t^n) r^{n+1}_{t±1/2, jk}, $$

(25)

which is consistent with definition (19). Using the temporally corrected fluxes (24) together with relations (17), (19) and (25), the FD equation (20) can now be solved to give the solutions

$$ \rho^{n+1}_{i±1/2,jk} = \beta_1(t^n)[1 + 3\Delta t\psi(t^n)], $$

$$ (\rho v_r)^{n+1}_{i±1/2,jk} = \beta_1(t^n)\psi(t^n)[1 + 4\Delta t\psi(t^n)] r^{n+1}_{i±1/2,jk}, $$

(26)

which are independent of the temporal variations of cell centres and volumes. These solutions reproduce the analytical profiles in a radially moving grid within first-order accuracy in time and second-order accuracy in space. In deriving the numerical solutions (21) and (26), it has been assumed that any high-order advection method will yield $\langle \rho \rangle_{t±1/2, jk} = \beta_1(t^n)$ which is true for a uniform-density distribution.
4.3. Temporally Second-Order Accurate Transport

In the general case of both spatially and temporally varying velocity fields, time-centred mass fluxes are commonly defined as (cf. Boss & Myhill 1992)

\[ \mathcal{F}_{i \pm 1/2, j k}^{n+1/2} = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \langle \rho(t) \rangle_{i \pm 1/2, j k} v_{r \pm 1/2, j k}(t) dt, \]

where for simplicity we have considered only radial fluxes and neglected momentarily grid motion. Similar definitions to (27) will also hold for the angular fluxes. The time integral can be approximated by expanding \( \langle \rho(t) \rangle \) and \( v_r(t) \) in Taylor series about the time \( t = t^n \) as separate functions. Integrating and neglecting all terms containing time-derivatives of order higher than the first, the result is

\[
\mathcal{F}_{i \pm 1/2, j k}^{n+1/2} = (\rho)_{i \pm 1/2, j k}^n v_{r \pm 1/2, j k}^n + \frac{\Delta t}{2} \left[ (\rho)_{i \pm 1/2, j k}^n \left( \frac{\partial v_r}{\partial t} \right)_{i \pm 1/2, j k}^n \right]
+ \frac{\Delta t}{3} \left[ (\rho)_{i \pm 1/2, j k}^n \left( \frac{\partial^2 \rho}{\partial t^2} \right)_{i \pm 1/2, j k}^n \right]
+ O((\Delta t)^2).
\]

Using forward replacements for the time-derivatives, e.g. \( (\partial X/\partial t)^n = (X^{n+1} - X^n)/\Delta t + O(\Delta t) \), equation (28) reduces to the flux form

\[ \mathcal{F}_{i \pm 1/2, j k}^{n+1/2} = \frac{1}{3} \left( (\rho)_{i \pm 1/2, j k}^{n+1} v_{r \pm 1/2, j k}^{n+1} + (\rho)_{i \pm 1/2, j k}^n v_{r \pm 1/2, j k}^n \right)
+ \frac{1}{6} \left( (\rho)_{i \pm 1/2, j k}^{n+1} v_{r \pm 1/2, j k}^{n+1} + (\rho)_{i \pm 1/2, j k}^n v_{r \pm 1/2, j k}^n \right), \]

where the previous knowledge of quantities at the time level \( n + 1 \) demands the use of a predictor-corrector method for the transport step. Note that for velocity fields which vary only spatially \( (v_r(r, t) = v_r(r)) \), equation (28) reduces to the “velocity-corrected” scheme proposed by Finn & Hawley (1989). The additional terms appearing in the flux definition (28) then represent the corrections needed to make the scheme second-order accurate in the presence of temporally varying velocity fields.

An improved formulation for the mass-fluxes is here introduced using the slightly different definition

\[ \mathcal{F}_{i \pm 1/2, j k}^{n+1/2} = \frac{1}{\Delta t^2} \left( \int_{t^n}^{t^{n+1}} \langle \rho(t) \rangle_{i \pm 1/2, j k} dt \right) \left( \int_{t^n}^{t^{n+1}} v_{r \pm 1/2, j k}(t) dt \right). \]

After expanding the integrands in Taylor series about \( t = t^n \) and discarding all terms in
the resulting expansion with second and higher order time-derivatives, we obtain

\[
F_{n+1/2}^{i\pm 1/2,jk} = \left(\langle \rho \rangle_{i\pm 1/2,jk} \right)^{n} + \frac{\Delta t}{2} \left[ \langle \rho \rangle_{i\pm 1/2,jk} \left( \frac{\partial \rho}{\partial t} \right)_{i\pm 1/2,jk} + \frac{\Delta t^2}{4} \left( \frac{\partial \langle \rho \rangle}{\partial t} \right)_{i\pm 1/2,jk} \left( \frac{\partial \rho}{\partial t} \right)_{i\pm 1/2,jk} \right]
\]

\[
+ \left( v_{i\pm 1/2,jk}^{n} \right)^{2} \langle \rho \rangle_{i\pm 1/2,jk} \left( v_{i\pm 1/2,jk}^{n} \right)^{2} \left( \frac{\partial \rho}{\partial t} \right)_{i\pm 1/2,jk} \left( \frac{\partial v_{i\pm 1/2,jk}^{n}}{\partial t} \right)_{i\pm 1/2,jk} + O\{ (\Delta t)^2 \},
\]

which differs from (28) only in the constant value of the coefficient dividing the second-order term on the right-hand side. Again using a forward replacement for the time-derivatives, equation (31) becomes

\[
F_{n+1/2}^{i\pm 1/2,jk} = \frac{1}{2} \left( \langle \rho \rangle_{i+1/2,jk}^{n+1} + \langle \rho \rangle_{i\pm 1/2,jk}^{n} \right) \left( v_{i\pm 1/2,jk}^{n+1} + v_{i\pm 1/2,jk}^{n} \right),
\]

which can be written in the equivalent form

\[
F_{n+1/2}^{i\pm 1/2,jk} = \frac{1}{4} \left( \langle \rho \rangle_{i+1/2,jk}^{n+1} v_{i\pm 1/2,jk}^{n+1} + \langle \rho \rangle_{i\pm 1/2,jk}^{n} v_{i\pm 1/2,jk}^{n} \right)
\]

\[
+ \frac{1}{4} \left( \langle \rho \rangle_{i+1/2,jk}^{n+1} v_{i\pm 1/2,jk}^{n+1} + \langle \rho \rangle_{i\pm 1/2,jk}^{n} v_{i\pm 1/2,jk}^{n} \right),
\]

as an average of second-order straightforward (first component) and ZIP-like (second component) fluxes. The terminology “straightforward” and “ZIP-like” refers here only to the temporal discretization and not to the spatial discretization form of the convective fluxes for which it is currently used in the literature (e.g. see Zalesak 1981). By way of comparison, the second-order fluxes (29) weight the straightforward component more strongly than the stabilizing ZIP-like one. The reason for preferring (33) to (29), however, is not due to stability concerns but simply to the fact that the former gives a superior response for the pressureless collapse test as long as a moving grid is incorporated.

The generalization of equation (30) for advection on a radially moving grid, is obtained by defining the mass-fluxes according to

\[
F_{n+1/2}^{i\pm 1/2,jk} = \frac{1}{\Delta t^2} \left( \int_{t_n}^{t_{n+1}} \langle \rho(t) \rangle_{i\pm 1/2,jk} dt \right)
\]

\[
\times \left( \int_{t_n}^{t_{n+1}} \left( v_{i\pm 1/2,jk}(t) - v_{i\pm 1/2}(t) \right) dt \right),
\]

which to second-order accuracy in time takes a form identical to (31) with \( v_r \) replaced by \( v_r - v_g \), and using forward replacements for the time-derivatives reduces to

\[
F_{n+1/2}^{i\pm 1/2,jk} = \frac{1}{2} \left( \langle \rho \rangle_{i+1/2,jk}^{n+1} + \langle \rho \rangle_{i\pm 1/2,jk}^{n} \right)
\]

\[
\times \left[ \frac{1}{2} \left( v_{i\pm 1/2,jk}^{n+1} + v_{i\pm 1/2,jk}^{n} \right) - v_{i\pm 1/2}^{n+1/2} \right],
\]
where the time-centred grid velocity (at cell surfaces $S^r$) can be calculated directly from the grid-motion algorithm. The use of equation (35) requires a predictor-corrector approach. This demands solving the entire set of transport equations twice per timestep resulting in an excessive computational cost. A particularly convenient way of preserving temporal second-order accuracy for the transport step calculation without predicting and correcting, is by approximating the time-derivatives in equation (31) by means of backward replacements, e.g. $(\partial X/\partial t)^n = (X^n - X^{n-1})/\Delta t + O(\Delta t)$, so that equation (35) becomes

$$ F_{i\pm 1/2,jk}^{n+1/2} = \frac{1}{2} \left( 3(f^n_{i\pm 1/2,jk} - f^{n-1}_{i\pm 1/2,jk}) \right) $$

which is consistent with extrapolating flux quantities at the old time levels $n$ and $n-1$ to find time-centred values. For uniform temporal increments of the fluid variables, where $X^{n+1} - X^n = \Delta X$ is a constant in time, eqs. (35) and (36) are quantitatively equivalent. For non-uniform $\Delta X$, the second-order truncation error carried by (36) is a factor of 5 larger than the corresponding one resulting from (35). In a true collapse calculation, however, the difference between successive values of $\Delta X$ is usually so small that (35) and (36) produce essentially the same results.

Substituting the mass-fluxes as given by (35) into equation (15) and then solving for the mass and radial momentum, we obtain the temporally second-order accurate solutions

$$ \rho_{ijk}^{n+1} = \beta_1(t^n) \left[ \frac{1 + 3 \Delta t \psi(t^{n+1/2})}{1 - \frac{3}{2} \Delta t \psi(t^{n+1/2})} \right] $$

for the homologous contraction of a pressureless sphere. For $\Delta t \to 0$, the above expressions can be expanded in Taylor series to give

$$ \rho_{ijk}^{n+1} = \beta_1(t^n) \left\{ 1 + 3 \Delta t \psi(t^{n+1/2}) + \frac{9}{2} \Delta t^2 \psi^2(t^{n+1/2}) + \frac{27}{4} \Delta t^3 \psi^3(t^{n+1/2}) + \ldots \right\} \rho_i^{n+1} $$

Expanding $\psi(t^{n+1/2})$ about $t = t^n$, substituting the result into the above expressions and
retaining only first-order terms in time, we then find that

\[ P^n_{ij}^{t+1} = \beta_1(t^n) \left[ 1 + 3 \Delta t \psi(t^n) + O((\Delta t)^2, (\Delta t)^3, \ldots) \right], \]

\[(\rho v_r^n)_{ij}^{t+1} = \beta_1(t^n) \psi(t^n) \left[ 1 + 4 \Delta t \psi(t^n) + \frac{1}{2} \frac{\Delta t}{\psi(t^n)} \left( \frac{\partial \psi}{\partial t} \right)^n + O((\Delta t)^2, (\Delta t)^3, \ldots) \right] s_{ij}^{t+1}, \]

where the first two terms in both expansions are just the temporally first-order solutions (26). Therefore, the collapse of a pressureless sphere is a temporal first-order problem that could be better solved using a temporally first-order code.

Temporal corrections have been also implemented to construct the \( \theta \)- and \( \phi \)-advective fluxes. In a radially moving grid, the radial coordinates of area centres of cell interfaces \( S^ \theta \) and \( S^ \phi \) [see relations (11) and (12)] will be time-varying, and hence the surface values of the velocity components in the angular directions must be also corrected to preserve spatial second-order accuracy for non-axisymmetric (3D) flows.

5. Test Results

In this section we describe the results obtained for a number of spherically symmetric collapse tests using both the temporally first- and second-order versions of the code. The test cases were first chosen to demonstrate through convergence testing the second-order of accuracy of the volume-centred FD scheme (Section 3) and then to assess how the solutions are actually improved on a moving grid by applying the temporal flux corrections described in Section 4. The transport of the mass-density [i.e. the calculation of \( \langle \rho \rangle \) in the flux definitions (24) and (35)] has been performed using a generalization to spherical coordinates of one of the following three “upwind” methods: (i) the first-order accurate donor-cell method (DC); (ii) the second-order accurate van Leer (1977) monotonic interpolation scheme (VL); and (iii) the third-order accurate piecewise parabolic advection method (PPA) of Collela & Woodward (1984). The special modifications that have been made to each of these methods for advection on a volume-centred spherical mesh will be described elsewhere (Sigalotti, in preparation).

5.1. Convergence Testing

The order of the truncation error associated with the FD representation of a partial differential equation can be determined through convergence testing as demonstrated by Finn & Hawley (1989). In the absence of analytical solutions, the truncation error can be estimated empirically by simply comparing the numerical results at several different resolutions. For instance, in the limit as the grid-spacing and the time interval tend to zero, the FD equation (15) becomes equivalent to the differential form of the integral equation (7). This property is called consistency and is clearly necessary if the FD solution is to converge.
to the solution of the differential equation being approximated. However, this condition is not sufficient to ensure that the solution of the FD equation will approach the solution of the exact differential equation (cf. Noye 1984). Therefore, we must always trace the functional dependence of the truncation error on resolution to see how rapidly the solution of the FD equation approaches the solution of the differential equation as the grid-spacing gets smaller and smaller at a fixed point in the computational domain. The asymptotic behaviour of this functional dependence at sufficiently high resolution will define the rate of convergence of the FD approximation which then provides a direct measure of its order of accuracy.

The spherically symmetric (1D) collapse of an initially uniform-density \( \rho_0 = 1.0 \times 10^{-18} \text{ g cm}^{-3} \), isothermal \( T = 10K \) protostellar cloud of mass \( M = 1M_\odot \) and radius \( R_0 = 7.80 \times 10^{16} \text{ cm} \), is used here as a test calculation to determine the rate of convergence of both the temporally first- and second-order versions of the code. This test problem involves the solution of eqs. (1)-(3) closed by a pressure relation of the form \( p \propto \rho \), for both spatially and temporally varying density and velocity fields. A constant-volume boundary condition \( \left[ R(t) - R_0 \right] \) is applied at the external spherical surface of the grid. The radial mesh is defined to be uniformly spaced and is kept fixed during the evolution so that the temporally corrected fluxes (24) reduce to the standard form given by (16) with \( v_r - v_g \rightarrow v_r \). Convergence was measured at \( 1.0t_{ff} \left[ t_{ff} = \sqrt{3\pi/32G\rho_0} \right] \), where \( \rho_0 \) refers to the initial density in the collapse. At this time, the central density has increased by a factor of \( 2.0 \times 10^6 \) and the flow is highly supersonic throughout most of the cloud. The error in the numerical solution is quantified by comparing the results at resolutions \( n_r \) and \( 2n_r \) by means of the \( L_2 \) error norm

\[
L_2(n_r) = \frac{1}{\sqrt{N}} \left\{ \sum_{i=1}^{N} \left[ \frac{\tilde{\rho}_{n_r}(r_i) - \tilde{\rho}_{2n_r}(r_i)}{\tilde{\rho}_{2n_r}(r_i)} \right]^2 \right\}^{1/2}
\]

where the summation includes \( N \) fixed radial grid-points and the \( \tilde{\rho}_{n_r}(r_i) \) are third-order accurate values of the mass-density at geometrical cell centres determined from the solution at volume centres by means of a quadratic interpolation scheme. The value of \( N \) is specified depending on the advection method used to compute the mass fluxes. Due to the constant-volume boundary condition, the second-order VL fluxes revert to first-order accuracy at the outermost radial cell-interfaces while for the PPA scheme, the mass fluxes at the two radial cell surfaces near the cloud boundary are calculated using the DC and the VL method. In general, the PPA method uses a five point molecule centred on the zone being updated and so lower order fluxes must also be implemented near the cloud centre \( r = 0 \). Furthermore, since the lowest resolution run has \( n_r = 10 \) grid-points, convergence was measured setting \( N = 10 \) for DC, \( N = 8 \) for VL and \( N = 5 \) for PPA in equation (39). Figure 1a shows the \( L_2 \) error norm as a function of the spatial resolution for the temporally first-order
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code using DC (Curve 1) and VL (Curve 2) fluxes and for the temporally second-order code using VL (Curve 3) and PPA (Curve 4) fluxes. Clearly the temporally second-order scheme gives the most accurate results with the PPA method exhibiting the smallest error at every resolution. As long as the resolution is increased, the $L_2$ error norm should decrease as $\Delta r^{-q}$, where $q$ is the rate of convergence. Thus calculating the slope of the log-log graph of error versus resolution, we trace the behaviour of $q$ for any of the four code versions. The result is shown in Figure 1b. The asymptotic rates are, respectively, $0.894$ (Curve 1) and $2.170$ (Curve 2) for the temporally first-order DC and VL code while for the temporally second-order code $q \rightarrow 2.056$ (Curve 3) using VL fluxes and $q \rightarrow 2.045$ (Curve 4) using PPA fluxes. Although the PPA method is formally third-order accurate, the whole scheme converges to spatial second-order accuracy as expected for the volume-centred discretization. However, by comparing curves 3 and 4 we see that the PPA solution converges to second order relatively faster than does the VL solution at high resolution.

5.2. Relaxation Problem

A useful 1D test involves relaxing an initially uniform-density ($\rho_0$) gas sphere of radius $R_0$ with a linear velocity field $v_r = v_0r$. If $v_0$ is assumed to be constant in space and time, then the equation of continuity (1) can be solved analytically to yield $\rho(t) = \rho_0 e^{-3v_0 t}$ in spherical geometry. Therefore, as the gas flows outwards the density should decrease exponentially with time while remaining uniform in space at any instant. For this calculation, the initial model is chosen to be a sphere of mass $M = 4.189g$ and radius $R_0 \approx 1.0cm$ with $\rho_0 = 1.0gcm^{-3}$ and $v_0 = 1.0cm^{-1}$, subject to an outflow boundary condition. Figure 2 shows the numerical solution obtained using the temporally first-order code with VL fluxes and $n_r = 50$ fixed grid-pints, after four seconds had elapsed. At this time, the numerical profile is exactly uniform and reproduces the analytical solution ($\rho_{analytic} = 6.144 \times 10^{-6}gcm^{-3}$) with a relative error of $7.45\%$ when the timestep is defined to be a fraction $f = 0.50$ of the Courant time $\Delta t_{CFL}$. By way of comparison, Stone & Norman (1992) obtained an error of $8.50\%$ for the same test using their spherical FD code version. When the calculation is repeated with a smaller fraction ($f = 0.05$) of the Courant time, the error in the numerical amplitude drops to $0.18\%$ (Fig. 2). Thus using small timesteps, the magnitude of the temporal discretization error becomes smaller resulting in improved numerical solutions.

5.3. Pressureless Collapse Test

The results derived in Section 4 on the optimal FD representation of the advective fluxes on a moving mesh are now assessed by calculating numerically the homologous contraction of a pressureless sphere. The initial conditions for this problem are chosen to be a static, uniform-density ($\rho_0 = 1.0 \times 10^{-18}gcm^{-3}$) sphere of mass $M = 10M_\odot$ filling
the computational volume. An inflow boundary condition is applied at the surface of the spherical grid so that mass is allowed to enter the grid as the collapse proceeds. This would be equivalent to follow the collapse of an infinitely extended cloud. Figures 3, 4 and 5 show the results as they are obtained using the temporally first-order code with VL fluxes and \( n_r = 50 \) grid-points for three different choices of the numerical techniques. In each case a Courant factor of 0.10 is used and the evolution is halted at 0.999\( t_{\text{ff}} \) when the density had increased by a factor of \( \sim 2.0 \times 10^5 \) times over its initial value. Identical results are also obtained using the same code with DC and PPA fluxes as is actually expected for advection of a uniform-density distribution.

Figure 3 shows the resulting numerical profiles using a fixed radial grid. At 0.999\( t_{\text{ff}} \), the density plateau is preserved to the eleventh digit out of fourteen carried in double precision accuracy (64 bits) and the radial velocity field remains perfectly linear. By 0.708\( t_{\text{ff}} \), the analytical solution is reproduced with a relative error of \( 8.73 \times 10^{-4} \)\% for the density and \( 1.59 \times 10^{-3} \)\% for the velocity. These errors progressively increase to less than 0.32\% by 0.999\( t_{\text{ff}} \) when the density has increased by five orders of magnitude. While the volume-centred scheme ensures the preservation of the density plateau on a fixed grid, the same is not necessarily true on a moving mesh if special corrections are not included in the FD representation of the advective fluxes. The effects of modifying the fluxes in a straightforward manner for use on a radially moving mesh are illustrated in Figure 4. A numerical solution which is now dependent on the temporal variations of cell centres and volumes results as predicted by (21). This dependence introduces a systematic error in the mass and momentum advection towards the cloud centre which then amplifies with time causing the solution to significantly deviate from the analytical profiles. These features are evident in Figure 4 at 0.999\( t_{\text{ff}} \) when the central density had erroneously risen above the plateau of the rest of the cloud and the velocity is only approximately linear. Beyond this time, the error propagates outwards involving the intermediate cloud regions and grows rapidly due to the shorter free-fall time for collapse of the central zones. This steadily growing excess of density must be regarded as a serious numerical problem for practical applications requiring the use of a moving mesh. For instance, in 2D simulations of rotating protostellar collapse, the transport of angular momentum which is also described by a continuity-like equation, would probably exhibit a similar wrong behaviour. Thus the effects due to centrifugal forces would be calculated incorrectly and angular momentum would be no longer conserved.

As demonstrated in Section 4, this class of errors can be essentially eliminated in a temporally first-order scheme by correcting the advective fluxes according to (24). Similar results can be obtained with a temporally second-order scheme if the mass fluxes are computed according to (31). The use of these temporal corrections yields solutions which
are completely independent of geometrical mesh properties in a moving grid [see relations (26) and (37)]. Figure 5 shows the results obtained using the fluxes as given by (24). No systematic errors due to grid motion are now apparent in the numerical solution at 0.999$t_{ff}$. By this time, the grid had become considerably stretched about the centre that an equidistant mesh with roughly $n_r = 500$ points would be necessary to achieve a comparable central resolution. On comparing with Figure 3, the density plateau is now preserved to the fourth digit throughout the cloud and the radial velocity still exhibits the correct linear behaviour at 0.999$t_{ff}$. However, the relative deviations of the numerical density and velocity profiles from the analytical solution are, respectively, $2.46 \times 10^{-2}\%$ and $1.40 \times 10^{-2}\%$. These errors are considerably smaller than those resulting from the fixed-grid calculation. This difference can be explained in terms of the Courant timestep restriction for explicit hydrodynamics. In a radially moving grid, the timestep governing the evolution will be generally determined by the central zones where the grid-spacings are smaller. This results in timesteps of size smaller than those needed on a fixed equidistant mesh to maintain stability and consequently the numerical solution will have smaller temporal discretization errors associated with it.

5.4. Protostellar Collapse Tests

A further important 1D test case concerns the non-homologous collapse of an isothermal protostellar cloud. The initial model for this calculation is chosen to be a static, uniform-density ($\rho_0 = 1.4 \times 10^{-19}$gcm$^{-3}$), $1M_\odot$ cloud composed of molecular hydrogen at constant temperature ($T = 10K$). A constant-volume boundary condition is applied at the edge of the computational volume and a radially moving grid with $n_r = 50$ points is used to represent the fluid. Figure 6 shows the resulting density profiles at different times in the collapse through seven orders of magnitude increase in central density, using the temporally first-order code with VL fluxes (Fig. 6a) and the temporally second-order code with VL (Fig. 6b) and PPA (Fig. 6c) fluxes. The profiles look extremely similar in spite of differences in the numerical methods, with the gas in the cloud envelope approximately matching the slope $r^{-2}$ (Larson 1969). The temporally second-order calculations are expected to achieve higher densities more rapidly than the first-order one. This is in fact consistent with the models of Figures 6b and c which collapsed to a central density $\rho_c = 1.0 \times 10^{-12}$gcm$^{-3}$ slightly before $(1.2430t_{ff})$ than did the model of Figure 6a $(1.2435t_{ff})$.

So far all test calculations have considered simplified thermodynamics. A more realistic approach to the protostellar collapse problem requires the inclusion of the effects of radiation transfer. We test a version of the 1D code which includes such effects in the Eddington approximation. The details of the combined hydrodynamics and radiation transfer methods will be described elsewhere (Sigalotti, in preparation). The test involves the collapse of an initially uniform-density ($\rho_0 = 3.1 \times 10^{-19}$gcm$^{-3}$), uniform-temperature
(\(T_0 = 10K\)), \(1M_\odot\) cloud of molecular hydrogen through the isothermal (optically thin) and non-isothermal (optically thick) regime. Figure 7 shows the numerical density profiles at different times in the collapse as obtained using the temporally first-order radiation code with \(n_r = 50\) moving points. The evolution is halted at \(1.055t_{ff}\) after the formation of the first quasi-equilibrium core. At this time, the central density had increased by a factor of \(10^9\) over its initial value and the maximum core temperature is \(\sim 478K\). The gas in the infalling cloud envelope is nearly isothermal (with temperature \(\leq 30K\)) and closely approaches the \(r^{-2}\) power law. The total energy has been continuously monitored during the evolution and by \(1.055t_{ff}\) it is conserved to better than 3\%. Figure 7 also depicts the resulting profiles for the same collapse model using the isothermal approximation (solid curves). For densities \(\rho < 10^{-12}gcm^{-3}\), the isothermal and non-isothermal profiles compare quite favourably at the same collapse times, a confirmation that both code versions are numerically consistent.

6. Conclusions

In this paper we have described the transport algorithm that has been implemented in the construction of a new multi-dimensional FD radiation hydrodynamics code. The spatial differences are obtained using a volume-centred discretization technique which results in a conservative, second-order accurate scheme in spherical Eulerian coordinates. The temporal discretization is made explicitly and is optionally first- or second-order accurate. Although the code has been originally written for studying protostellar collapse and fragmentation, it could be in general applied to a large variety of astrophysical flows.

Whereas the spatial second-order accuracy of the volume-centred scheme is preserved for advection on a fixed Eulerian mesh as demonstrated through convergence testing, the same is no longer true for advection on a moving orthogonal mesh. Systematic errors appear in the transport of the fluid variables which then amplify with time resulting in a net loss of accuracy. An analysis involving the pressureless collapse of a homogeneous sphere on a radially moving spherical grid, has shown that these errors arise due to the intrinsic dependence of the numerical solution on the temporal variations of cell centres and volumes whenever standard FD fluxes are straightforwardly applied. This causes the mass and the momentum to be transported inwards with steadily increasing velocities. New FD flux representations have been derived for the temporally first- and second-order schemes to minimize these effects on a moving mesh. These forms include appropriate temporal corrections for the advective quantities which make the full transport algorithm invariant under grid motion, resulting in a high degree of accuracy on the pressureless collapse test case. Results have been also presented for the isothermal and non-isothermal collapse of a molecular cloud that validate this conclusion. Although the flux corrections specialize to the case of a radially moving spherical grid, they are in general applicable to other coordinate systems as well as to any other problem involving the FD solution of the hydrodynamic
equations on a moving orthogonal mesh. The formal description of the full code and the results from axisymmetric (2D) and non-axisymmetric (3D) tests will be presented in a forthcoming paper (Sigalotti, in preparation).

Acknowledgments

The calculations of this paper were performed on the CONVEX C210 computer of the International Centre for Theoretical Physics (ICTP). This work was supported in part by the Ministero Italiano dell'Università e della Ricerca Scientifica e Tecnologica.

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Fig. 1. (a) $L_2$ error norm and (b) rate of convergence as a function of resolution for the temporally first-order 1D code using DC (curve 1) and VL (curve 2) fluxes and for the temporally second-order 1D code using VL (curve 3) and PPA (curve 4) fluxes.
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Fig. 1b.
Fig. 2. Density profile (dots) for the radial advection of a uniform-density sphere (relaxation test problem) from $\rho_0 = 1.0 \text{g cm}^{-3}$, using the temporally first-order VL code with $n_r = 50$ fixed radial points. The solid line corresponds to the analytical solution at $t = 4$ seconds when the density is $6.144 \times 10^{-6} \text{g cm}^{-3}$. 
Fig. 3. (a) Density and (b) radial velocity profiles (dots) compared with the analytical solution (solid line) for the pressureless collapse of a homogeneous sphere from $\rho_0 = 1.0 \times 10^{-15} \text{g cm}^{-3}$, using the temporally first-order VL code with $n_r = 50$ fixed radial points. The sequence of times, starting from the lowest curves on each plot, are: $t/t_{ff} = 0.708, 0.961, 0.994$ and 0.999.
Fig. 3b.
Fig. 4. (a) Density and (b) radial velocity profiles (dots) compared with the analytical solution (solid line) for the same pressureless collapse model calculation of Fig. 3 but now using a moving $n_r = 50$ grid without flux corrections.
Fig. 4b.
Fig. 5. (a) Density and (b) radial velocity profiles (dots) compared with the analytical solution (solid line) for the same model calculation of Fig. 4 but now correcting the radial fluxes for treatment on a moving mesh.
Fig. 5b.
Fig. 6: Density profiles for the isothermal collapse of a molecular cloud initially at rest with $\rho_0 = 1.4 \times 10^{-16} \text{g cm}^{-3}$ through seven orders of magnitude increase in central density, using (a) the temporally first-order 1D code with VL fluxes and the temporally second-order code with (b) VL and (c) PPA fluxes. The solid line on each plot corresponds to the slope $\rho \propto r^{-2}$. 
Fig. 6b.
Fig. 6c.
Fig. 7. Density profiles for the isothermal (solid line) and non-isothermal (crosses) collapse of a molecular cloud initially at rest with $\rho_0 = 3.1 \times 10^{-19} \text{g cm}^{-3}$, using the temporally first-order code with VL fluxes. The calculation is halted after the formation of the first quasi-equilibrium core (last profile for the non-isothermal run) at $1.055t_{ff}$ when the central density has increased by nine orders of magnitude. For comparison the $r^{-2}$ slope is also shown.

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*Lavoro pervenuto in redazione il 8.7.1993.*