NUMERICAL SOLUTION OF A BOUNDARY-LAYER PROBLEM
WITH A CHEBYSHEV PSEUDOSPECTRAL MULTI-DOMAIN METHOD

Abstract. A multi-domain pseudospectral method, which is based on Chebyshev polynomials expansions, is presented to solve an initial-boundary value problem in incompressible MHD, the tearing instability, in which a boundary-layer is spontaneously generated inside the spatial domain. The method is based on a property of Chebyshev pseudospectral expansions which accurately describe functions having strong gradients localized near one of the Chebyshev domain boundaries. A comparison with the results of a single-domain pseudospectral method is performed, showing that, in the considered case, the multi-domain technique furnishes a higher accuracy keeping the truncation error to a lower level. Because of the steeper Chebyshev spectra lower aliasing errors are obtained during the nonlinear stage of the instability.

1. Introduction

Pseudospectral methods represent a powerful tool in the numerical solution of nonlinear partial differential equations, and in particular, the fluid dynamics or the magnetohydrodynamics (MHD) equations. When the solutions are sufficiently regular, such methods give an accuracy which is generally higher than the accuracy of finite-differences methods which uses the same number of points in the spatial grid. The accuracy of spectral methods is strictly related to the regularity properties of the solution: when the quantities are of $C^p$-class, spectral methods give origin to $(p+1)$-order schemes (see, e.g., [1]). As a consequence, a discontinuous solution is very poorly represented by a spectral method. This happens, for instance, in the inviscid fluid dynamics when a shock forms; the Gibbs phenomena generated by the discontinuity completely destroy the accuracy of the numerical solution. A way to overcome this problems is to use a multi-domain technique; if the quantities are piecewise $C^\infty$-class the spatial domain $D$ can be divided into the
union of $K$ subdomains $D^{[a]}$. We indicate by $\delta D^{[a,\beta]} = D^{[a]} \cap D^{[\beta]}$ the internal boundary between the $a$-th and the $\beta$-th subdomain. This partition must be such that any quantity remains of $C^\infty$-class in each subdomain $D^{[a]}$ during the time interval of interest. Then, a spectral method is used in each subdomain giving the appropriate boundary conditions at the internal boundaries. This kind of technique was employed by Bonazzola and Marck [2] to describe the shock propagation in fluid dynamics, by defining a moving internal boundary placed at the shock location, at any time step.

If diffusive phenomena are present, such as in the viscous fluid dynamics, the discontinuities associated with the shocks are smoothed out and are replaced by more or less narrow regions where the quantities of the problem vary with continuity from their values upstream to the values downstream. In such a case, in consequence of the regularity of the solution, a spectral method could be in principle used.

However, in many applications the Reynolds number is quite large; this implies that the shock width is small and large gradients develop at the shock location. In such a case, even though the quantities keep continuous, a large number of terms (harmonics) are necessary in their spectral representation in order to obtain a good accuracy in the numerical solution. If the number of harmonics is not sufficient to accurately describe fast variations in the solution, oscillations will appear at the shock location, similar to the Gibbs phenomena of the nonviscous case. However, in spite of this analogy between the inviscid and the weakly viscous fluid dynamics, in the latter case a multi-domain technique does not necessarily improve the accuracy of the numerical solution. Actually, when fast variation regions are present instead of real discontinuities, a high spatial resolution is required in these regions. Then, the advantage of using a multi-domain or a single-domain spectral technique depends only on the capability of this method to give such a high resolution in the "diffusivity" regions.

This can be actually achieved if a multi-domain technique is employed in association with a Chebyshev pseudospectral method. Solomonoff and Turkel [3] studied the properties of pseudospectral methods based on a Chebyshev expansion in cases when sharp variations are present in the solution. The standard pseudospectral Chebyshev expansion uses the following set of $(N+1)$ collocation points

$$x_j = \cos \left( \frac{\pi j}{N} \right) , \quad j = 0, \ldots, N$$

in the spatial domain $D = [-1, 1]$; however, sets of collocation points different from (1) can also be used. In particular, Solomonoff and Turkel [3] considered both the standard collocation points (1) and a set of uniformly spaced collocation points. In both cases these authors found that the Chebyshev collocation methods give lower errors when sharp gradients or discontinuous derivatives occur near the boundaries of the spatial domain $D$.
than when they are near the center. This implies that a multi-domain technique can be advantageous to describe continuous solutions with localized sharp gradients, provided that these fast variation regions are near a boundary (internal or not) of some subdomain $D^i$.

In this paper we present an application of a multi-domain pseudospectral Chebyshev technique. Our aim is to show the advantages of this method with respect to a single-domain technique in studying boundary layer problems in cases when the thickness of the boundary layer is small but finite. In particular, we will use this technique to study a problem of the incompressible MHD; namely, the nonlinear development of the tearing instability in a plane sheet pinch. The results are compared with those of a similar single-domain technique. It will be shown that the multi-domain method gives a good accuracy with a number of Chebyshev polynomials much lower than the single-domain method. This, in turn, allows to keep aliasing errors to a low level during the nonlinear stage of the instability.

2. The Numerical Method

The Chebyshev pseudospectral multi-domain technique has been applied to the solution of an initial-boundary value problem for the incompressible magnetohydrodynamic equations. We considered two-dimensional configurations, where any physical quantity depends only on the $X$ and $Y$ variables, and only the $X$ and $Y$ components of vector quantities are non-vanishing. The equations of the incompressible MHD can be written in the following dimensionless form:

\[
\frac{\partial Z_i^\sigma}{\partial t} + Z_k^\sigma \frac{\partial Z_i^\sigma}{\partial x_k} + \frac{\partial P}{\partial x_i} = \chi \frac{\partial^2 Z_i^\sigma}{\partial x_k \partial x_k}, \quad i = 1, 2, \quad \sigma = \pm
\]

\[
\frac{\partial^2 P}{\partial x_k \partial x_k} = -\frac{\partial Z_i^+ \partial Z_j^-}{\partial x_m \partial x_k}
\]

where $Z^\sigma$ and $P$ are the dimensionless Elsasser variables and total pressure, respectively

\[
Z_i^\sigma = \frac{v_i}{c_A} + \sigma \frac{B_i}{B_0}, \quad i = 1, 2, \quad \sigma = \pm; \quad P = \frac{p + \frac{B^2}{2\sigma}}{\rho_0 c_A^2}
\]

In these equations $v$ represents the velocity of the fluid, $B$ the magnetic field, $p$ the gas pressure, and $c_A = B_0/(4\pi\rho_0)^{1/2}$ is the Alfvén velocity, $\rho_0$ being the mass density which keeps constant and uniform, and $B_0$ a characteristic value for the magnetic field. The dimensionless space-time variables are $x_1 = x/X$, $x_2 = y/Y$, and $t = T(c_A/a)$, where $a$ represents some characteristic length of the problem. Lower indices identify the vector cartesian components (e.g. $Z_1 \equiv Z_x, Z_2 \equiv Z_y$) and summation over lower dummy indices is hereinafter understood. Finally, it has been assumed that $\nu = \eta c^2/(4\pi)$ (with $\nu$ the kinematic viscosity, $\eta$ the resistivity and $c$ the velocity of the light) and the dissipation
coefficient $\chi = \nu/(acA) = \eta c^2/(4\pi acA)$ has been defined. The space domain is given by
the rectangle

$$D = \{(x, y) : x \in [-l, l], y \in [0, \pi R l]\}$$

where $R$ is a parameter which determine the aspect ratio, and $l$ gives a measure of the
domain width in normalized units.

The initial condition will be chosen in such a way that a boundary-layer forms along
the line $x = 0$. The location of this boundary-layer does not change in time. Then, an
enhanced spatial resolution is required close to the line $x = 0$, where the largest gradients
develop. This is achieved by dividing the spatial domain $D$ in two subdomains, with
the internal boundary located at the line $x = 0$. Pseudospectral Chebyshev expansions are
carried out in both subdomains in the $x$-direction, i.e. parallel to such gradients. The results
of this method will be compared with those obtained using a single-domain method. In this
latter case, a single pseudospectral Chebyshev expansion is performed in the $x$-direction
and the location of the boundary layer corresponds to the center of the Chebyshev mesh.
It will be shown that the double-domain technique improves the accuracy of the numerical
solution, in accordance with the results of Solomonoff and Turkel [3].

Concerning the boundary conditions, periodicity conditions are imposed on the
boundaries $y = 0$ and $y = 2\pi R$, i.e. on those boundaries crossed by the boundary-layer.
The following free-slip conditions are imposed on the other pair of boundaries

$$\frac{\partial Z^\sigma_y}{\partial x} = 0 \quad, \quad \sigma = \pm, \text{ at } x = \pm l, \quad t \geq 0$$

(4)

$$\frac{\partial P}{\partial x} = 0 \quad, \quad \text{ at } x = -l, \quad t \geq 0$$

$$P = P_0 \quad, \quad \text{ at } x = l, \quad t \geq 0$$

t = 0 being the initial time, along with the divergenceless conditions

$$\frac{\partial Z^\sigma_x}{\partial x} = \frac{\partial Z^\sigma_y}{\partial y} \quad, \quad \sigma = \pm, \text{ at } x = \pm l, \quad t \geq 0.$$  

(5)

If the divergence of $Z^\sigma$ is vanishing at the initial time

$$\frac{\partial Z^\sigma_k}{\partial x_k} = 0 \quad, \quad \sigma = \pm \quad, \quad \text{ for } (x, y) \in D \text{ at } t = 0$$

(6)

the equations (2) and (3) with the boundary condition (5) ensure that the fields $Z^\sigma$ remain
divergenceless for any time $t \geq 0$.

The problem will be solved using both a double-domain and a single-domain
 technique. In the former case the domain $D$ is divided in two subdomains (left and
right) $D = D^{[L]} \cup D^{[R]}$, with $D^{[L]} = [-l, 0] \times [0, \pi R l]$; and $D^{[R]} = [0, l] \times [0, \pi R l]$;
hereinafter we will indicate quantities relative to these two subdomains by the upper indices.
$[L]$ and $[R]$, respectively. Extra boundary conditions must be given on the internal boundary $x = 0$. Even though strong gradients forms at this boundary, we assume that the solution is continuous, at least up to the second order space derivatives; this is actually verified for the particular problem (the tearing instability) which will be studied [4]. Then, we impose the following matching conditions:

\[(7a)\] \[Z_i^\sigma[L](0, y, t) = Z_i^\sigma[R](0, y, t), \quad y \in [0, \pi R], \quad t \geq 0, \quad i = 1, 2, \quad \sigma = \pm\]

\[(7b)\] \[\frac{\partial Z_i^\sigma[L]}{\partial x}(0, y, t) = \frac{\partial Z_i^\sigma[R]}{\partial x}(0, y, t), \quad y \in [0, \pi R], \quad t \geq 0, \quad i = 1, 2, \quad \sigma = \pm\]

\[(7c)\] \[P_i^L(0, y, t) = P_i^R(0, y, t), \quad y \in [0, \pi R], \quad t \geq 0\]

\[(7d)\] \[\frac{\partial P_i^L}{\partial x}(0, y, t) = \frac{\partial P_i^R}{\partial x} = (0, y, t), \quad y \in [0, \pi R], \quad t \geq 0\]

These conditions imply the continuity at the internal boundary $x = 0$ of the space derivatives of $Z^\sigma$ and $P$, up to the second order. In fact, let us consider a given quantity $f(x, y, t)$ and assume that $f(x, y, t)$ is piece-wise of $C^r$-class (with $r \geq 2$) with respect to the space variables, and of $C^1$-class with respect to $t$, in the two subdomains $D^L$ and $D^R$ separately. Then, the continuity of $f(x, y, t)$ at $x = 0$ for any time $t \geq 0$ (conditions (7a) and (7c)) implies both the continuity across the line $x = 0$ of $\partial^n f/\partial y^n$, for $n \leq r$, and of $\partial f/\partial t$. Moreover, from the equations (2) and (3) it follows that $\partial^2 f/\partial x^2$ is a continuous function of its first-order space derivatives, of $\partial f/\partial t$, and of $\partial^2 f/\partial y^2$; then, $\partial^2 f/\partial x^2$ is continuous at $x = 0$ and for any time $t \geq 0$, as well.

In the double-domain method, the equations (2) and (3) are numerically solved using a pseudospectral method in both $D^L$ and $D^R$; any quantity $f^{[\alpha]}(x, y, t)$ in the subdomain $D^{[\alpha]}$ at a given time $t$ is approximated as a linear combination of a finite number of functions

\[(8)\] \[f^{[\alpha]}(x, y, t) = \sum_{m=0}^{M-1} \sum_{n=0}^{N} a_{nm}^{[\alpha]}(t) \varphi_{nm}^{[\alpha]}(x, y)\]

where

\[(9)\] \[\varphi_{nm}^{[L]}(x, y) = T_n(2x/l + 1)e^{2i\pi y/(\pi l)}, \quad (x, y) \in D^L\]

\[\varphi_{nm}^{[R]}(x, y) = T_n(2x/l - 1)e^{2i\pi y/(\pi l)}, \quad (x, y) \in D^R\]

$T_n(\xi), \xi \in [-1, 1]$, is the Chebyshev polynomial of degree $n$, and $i$ is the imaginary unity.

The spatial grids of the collocation points used in the pseudospectral expansions are the following sets

\[(10)\] \[S^{[\alpha]} = \{(x_i^{[\alpha]}, y_j^{[\alpha]}), \quad i = 0, \ldots, N, \quad j = 0, \ldots, M - 1, \quad \alpha = L, R\}\]

where

\[x_i^{[L]} = l[-\cos(\pi i/N) - 1]/2, \quad x_i^{[R]} = l[-\cos(\pi i/N) + 1]/2\]

\[y_j^{[L]} = y_j^{[R]} = \pi Rj/M\]
These correspond to the standard collocation points used in the Fourier and Chebyshev pseudospectral expansions, respectively. We chose to use the standard collocation points because of the following reasons: Solomonoff and Turkel [3] considered the Chebyshev pseudospectral expansion of functions with large gradients near the boundaries, and compared the accuracy when standard collocation points, or uniformly spaced points are respectively used. They found that the former choice is preferable, since in such a case lower absolute errors are obtained. Moreover, in the calculation of the expansion coefficients with standard collocation points Fast Fourier Transform (FFT) algorithms can be employed; this transformation technique is faster than the matrix multiplication if the order $N$ of the Chebyshev transform is sufficiently high (typically $N \gg 100$) [5].

When the single-domain technique has been used, the pseudospectral expansion of a quantity $f(x,y,t)$ is performed using $2N + 1$ polynomials, so that the total number of polynomials in the two cases is almost the same and this fact makes clearer the comparison between the results obtained by the two different methods.

The time dependence in the equations (2) and (3) is treated by a second-order numerical scheme. In particular, the following semi-implicit [6,7] method is used in order to avoid the constraint of the Courant condition, which would become very severe in consequence of the high density of collocation points near the boundaries at $x = \pm 1$ and $x = 0$ [8]. Given the solution $(Z^{(n)}, P^{(n)})$ at the time $t_n = n\Delta t$ ($\Delta t$ being the time step amplitude), the solution $(Z^{(*)}, P^{(*)})$ at an intermediate time $t^* = t_n + \theta \Delta t$, $1/2 < \theta \leq 1$, is first calculated using the following equations

$$
\frac{Z_i^{(*)} - Z_i^{(n)}}{\theta \Delta t} + Z_k^{-\sigma(n)} \frac{\partial Z_i^{(n)}}{\partial x_k} + \frac{\partial P^{(n)}}{\partial x_i} = \frac{\chi}{2} \frac{\partial^2}{\partial x_k \partial x_k} [Z_i^{(*)} + Z_i^{(n)}], \quad i = 1, 2, \quad \sigma = \pm
$$

$$
\frac{\partial^2 P^{(*)}}{\partial x_k \partial x_k} = -\frac{\partial Z_k^{(*)}}{\partial x_m} \frac{\partial Z_m^{(*)}}{\partial x_k}
$$

and imposing the boundary conditions (4) and (5) and the matching conditions (7) to $Z_i^{(*)}$ and $P^{(*)}$. The upper index in parentheses identifies the time step. The solution at the time $t_{n+1}$ is calculated using the following equations

$$
\frac{Z_i^{(n+1)} - Z_i^{(n)}}{\Delta t} + Z_k^{-\sigma(*)} \frac{\partial Z_i^{(*)}}{\partial x_k} + \frac{\partial P^{(*)}}{\partial x_i} - \Delta t c_k^\sigma \frac{\partial^2 Z_i^{(n+1)}}{\partial^2 x_k} = -\Delta t c_k^\sigma \frac{\partial^2 Z_i^{(n)}}{\partial^2 x_k} + \frac{\chi}{2} \frac{\partial^2}{\partial x_k \partial x_k} [Z_i^{(n+1)} + Z_i^{(n)}], \quad i = 1, 2, \quad \sigma = \pm
$$

$$
\frac{\partial^2 P^{(n+1)}}{\partial x_k \partial x_k} = -\frac{\partial Z_k^{(n+1)}}{\partial x_m} \frac{\partial Z_m^{(n+1)}}{\partial x_k}
$$
and imposing the boundary conditions (4) and (5) and the matching conditions (7) to $Z_{i}^{(n+1)}$ and $P^{(n+1)}$. The whole scheme (11)-(12) depends on the numerical parameters $\theta, c_{1}$ and $c_{2}$, as well as on $\Delta t$. These parameters must be tuned in order to have a good accuracy and to avoid numerical instabilities. Both equations (11) and (12) are implicit, because it is necessary to invert some differential operator in order to calculate $(Z_{i}^{(n)}, P^{(n)})$ and $(Z_{i}^{(n+1)}, P^{(n+1)})$, respectively.

We showed that the solutions of the equations (2) and (3) with the matching conditions (7a)-(7d) are continuous at $x = 0$ up to the second order derivatives with respect to the space variables. It can be easily proved that the same property also holds for the solutions of the equations (11) and (12), which represent the time-discretized version of the equations (2) and (3).

The equations (11) and (12) are transformed in the spectral space. Let us consider first the double-domain case. In the following we will indicate by $\hat{f}^{[\alpha]}(x,y)$ the $m$-th coefficient of the Fourier expansion of a quantity $f^{[\alpha]}(x,y)$ at a given time. The equations (11) and (12) are written in the Fourier space, for the subdomains $D^{[L]}$ and $D^{[R]}$ separately. All these equations can be set in the following general form, along with their boundary conditions:

\begin{align}
(13a) \quad \left[ \frac{d^2}{dx^2} - \kappa^2(m) \right] \hat{f}^{[\alpha]}(x,m) &= \hat{G}^{[\alpha]}(x,m) \quad m = 0, \ldots, M - 1, \alpha = L, R \\
(13b) \quad \hat{e}^{[\alpha]}[\hat{f}^{[\alpha]}(x,m)] &= \hat{\beta}^{[\alpha]}(m)
\end{align}

and the matching conditions

\begin{align}
(14a) \quad \hat{f}^{[L]}(0,m) &= \hat{f}^{[R]}(0,m) \quad m = 0, \ldots, M - 1 \\
(14b) \quad \frac{d\hat{f}^{[L]}}{dx}(0,m) &= \frac{d\hat{f}^{[R]}}{dx}(0,m)
\end{align}

where $\hat{f}^{[\alpha]}(x,m)$ indicates the Fourier coefficients of $Z_{i}^{[\alpha](*)}, Z_{i}^{[\alpha](n+1)}, P^{[\alpha](*)},$ or $P^{[\alpha](n+1)}$; $\hat{G}^{[\alpha]}(x,m)$ is the R.H.S., which depends on quantities relative to previous time steps; $\hat{e}^{[L]}$ and $\hat{e}^{[R]}$ are the linear operators which give the boundary conditions (4) and (5) in the Fourier space at the boundaries $x = -l$ and $x = l$, respectively; $\hat{\beta}^{[\alpha]}(m)$ are the corresponding R.H.S.; $\kappa^2(m)$ are positive constants. Note that the solution of the problems (13)-(14) can be carried out independently for each Fourier harmonic, i.e. in the Fourier space we have to solve $M$ decoupled problems.

For a given Fourier harmonic, the equations (13a) can be considered as two distinct problems, each relative to one subdomain, which are coupled by the matching conditions (14a,b). The problem (13)-(14) is solved by a Chebyshev pseudospectral technique and the boundary conditions are imposed by means of the Tau method [9]. In order to speed up the numerical procedure we splitted the problem (13)-(14) in two decoupled problems, using the following technique. We write the solutions $\hat{f}^{[\alpha]}(x,m)$ of the problem (13)-(14)
in each subdomain as a linear combination of two functions:

\[
\hat{f}^{[\alpha]}(x, m) = \hat{v}^{[\alpha]}(x, m) + \lambda_m \hat{w}^{[\alpha]}(x, m), \quad m = 0, \ldots, M - 1, \quad \alpha = L, R
\]

where \( \lambda_m \) is an appropriate constant which is defined by the equation (19), and \( \hat{v}^{[\alpha]}(x, m) \) and \( \hat{w}^{[\alpha]}(x, m) \) are respectively solutions of the following problems:

\[
\begin{align*}
(16a) & \quad \left[ \frac{d^2}{dx^2} - \kappa^2(m) \right] \hat{v}^{[\alpha]}(x, m) = \hat{G}^{[\alpha]}(x, m) \\
(16b) & \quad \hat{\beta}^{[\alpha]}[\hat{v}^{[\alpha]}(x, m)] = \hat{\beta}^{[\alpha]}(m) \quad m = 0, \ldots, M - 1, \quad \alpha = L, R \\
(16c) & \quad \hat{v}^{[\alpha]}(0, m) = \hat{\beta}_{\text{guess}}(m) \\
\end{align*}
\]

and

\[
\begin{align*}
(17a) & \quad \left[ \frac{d^2}{dx^2} - \kappa^2(m) \right] \hat{w}^{[\alpha]}(x, m) = 0 \\
(17b) & \quad \hat{\beta}^{[\alpha]}[\hat{w}^{[\alpha]}(x, m)] = 0 \quad m = 0, \ldots, M - 1, \quad \alpha = L, R \\
(17c) & \quad \hat{w}^{[\alpha]}(0, m) = 1
\end{align*}
\]

where \( \hat{\beta}_{\text{guess}}(m) \) represents a guess value for the solution \( \hat{f}^{[\alpha]}(x, m) \) at the internal boundary \( x = 0 \).

It can be easily proved that the linear combination (15) gives the solution of the problem (13)-(14), which is unique. First, summing the equation (16a) to the equation (17a) multiplied by \( \lambda_m \) it is seen that the form (15) represents a solution of the equation (13a). The same linear combination of the equations (16b) and (17b) shows that the form (15) satisfies the boundary condition (13b). The matching condition (14a) is satisfied also by the form (15), since, from the equations (16c) and (17c), it is found

\[
\begin{align*}
\hat{v}^{[L]}(0, m) + \lambda_m \hat{w}^{[L]}(0, m) &= \hat{G}^{[R]}(0, m) + \lambda_m \hat{w}^{[L]}(0, m) \\
&= \hat{\beta}_{\text{guess}}(m) + \lambda_m, \quad m = 0, \ldots, M - 1 .
\end{align*}
\]

Finally, the remaining matching condition (14b) is satisfied by a proper choice of the parameter \( \lambda_m \); in particular

\[
\lambda_m = -\frac{\frac{d\hat{G}^{[R]}}{dx}(0, m) - \frac{d\hat{G}^{[L]}}{dx}(0, m)}{\frac{d\hat{\beta}^{[R]}}{dx}(0, m) - \frac{d\hat{\beta}^{[L]}}{dx}(0, m)}
\]

The correction solutions \( \hat{v}^{[\alpha]}(x, m) \) do not depend on the time step; so they are calculated only once, at the beginning of the time advancing procedure, by solving the problems (17).
The guess solutions \( \tilde{v}^{[\alpha]}(x, m) \) depend on the quantities \( \tilde{\beta}_{\text{guess}}(m) \). These are free parameters which can in principle be chosen in an arbitrary way, since for any choice of \( \tilde{\beta}_{\text{guess}}(m) = \tilde{v}^{[\alpha]}(0, m) \) the corresponding guess solutions \( \tilde{v}^{[\alpha]}(x, m) \) can be corrected by adding the appropriate correction term \( \lambda_m \tilde{v}^{[\alpha]}(x, m) \). However, in order to avoid too large roundoff errors in the linear combination (15), the guess solutions \( \tilde{v}^{[\alpha]}(x, m) \) should not be too far from the final solutions \( \tilde{f}^{[\alpha]}(x, m) \). The amount of the correction can be evaluated defining the following quantities

\[
\Delta \tilde{f}^{[\alpha]}(m) = \| \tilde{f}^{[\alpha]}(x, m) - \tilde{\beta}_{\text{guess}}(m) \|_{\infty} / \| \tilde{f}^{[\alpha]}(x, m) \|_{\infty}
\]

where, as usual, \( \| g(x) \|_{\infty} = \sup \{ g(x), x \in [-1, 1] \} \). In particular, we will require that

\[
\Delta \tilde{f}^{[\alpha]}(m) \ll 1 \quad m = 0, \ldots, M - 1 \quad \alpha = L, R.
\]

It can be easily shown that \( \| \tilde{\omega}^{[\alpha]}(x, m) \|_{\infty} = 1 \); then, from the equation (15) it follows that

\[
\Delta \tilde{f}^{[\alpha]}(m) = |\lambda_m| / \| \tilde{f}^{[\alpha]}(x, m) \|_{\infty}
\]

i.e. the correction is proportional to \( |\lambda_m| \). Evaluating the equation (15) at \( x = 0 \), and taking into account that \( \tilde{\omega}^{[\alpha]}(0, m) = 1 \), we find

\[
\lambda_m = \tilde{f}^{[\alpha]}(0, m) - \tilde{\beta}_{\text{guess}}(m).
\]

Thus, in order to have small values for the correction parameters \( \Delta \tilde{f}^{[\alpha]}(m) \) at any time step we choose the trial boundary condition \( \tilde{\beta}_{\text{guess}}(m) \) as

\[
\tilde{\beta}_{\text{guess}} = \tilde{f}^{[\alpha]}(n-1)(0, m)
\]

where the RHS of the equation (24) indicates the value of \( \tilde{f}^{[\alpha]}(0, m) \) at the previous time step. With this choice of the trial boundary condition the parameters \( \lambda_m \) are quantities of order \( \Delta t \):

\[
\lambda_m = \frac{\partial \tilde{f}^{[\alpha]}}{\partial t} (0, m) \mu \Delta t + O(\Delta t^2)
\]

where \( \mu = 0 \) when the solution is calculated at the intermediate time \( t^* \) or \( \mu = 1 \) when the solution is calculated at the time \( t_{n+1} \). The correction \( \Delta \tilde{f}^{[\alpha]} \) becomes, to the lowest order in \( \Delta t \)

\[
\Delta \tilde{f}^{[\alpha]} \approx \frac{\Delta t}{\tau_m}
\]

where

\[
\tau_m = \frac{\| \tilde{f}^{[\alpha]}(x, m) \|_{\infty}}{\left| \frac{\partial \tilde{f}^{[\alpha]}}{\partial t} (0, m) \right|}
\]
represents a characteristic evolution time of the solution $\tilde{f}^{[a]}$ at $x = 0$. In conclusion, using the boundary condition (24), condition (21) is satisfied for sufficiently small values of $\Delta t$, namely

$$\Delta t \ll \min\{\tau_m, m = 0, \ldots, M - 1\}.$$ 

In the case of the single-domain technique, the equations (13)-(14) are replaced by the following equations

\begin{align*}
(26a) \quad & \frac{d^2 \hat{f}(x, m)}{dx^2} - \kappa^2(m) \hat{f}(x, m) = \hat{G}(x, m) \\
(26b) \quad & \hat{\ell}[L]| \hat{f}(x, m)| = \hat{\beta}[L](m), \quad m = 0, \ldots, M - 1 \\
(26c) \quad & \hat{\ell}[R]| \hat{f}(x, m)| = \hat{\beta}[R](m)
\end{align*}

which are solved by a Chebyshev pseudospectral technique, with the standard spatial mesh given by the equation (13). The method is the same as that employed to solve the problem (16) or (17). Since in the single-domain case the order of the Chebyshev transforms is $2N$, the solution of the equations (26a,b,c) requires to solve a $(2N + 1) \times (2N + 1)$ linear system.

### 3. Numerical results

The tearing instability represents the physical problem which will be used as a test for the numerical technique described in the previous section. Since the first work by Furth et al. [4], the tearing instability has received a lot of attention in plasma physics research. It develops in a magnetofluid for configurations in which some component of the magnetic field vanishes on a given surface (neutral surface). A small but finite value of the resistivity $\eta$ is also required, in order to allow changes in the magnetic lines topology. In particular, we will consider the case of a plane sheet pinch, where the equilibrium configuration is given by a vanishing velocity $v_{eq} = 0$ and the following magnetic field

\begin{align*}
(27) \quad & B_{eq} = B_0 \left[ lgh \left( \frac{X}{a} \right) + b \left( \frac{X}{a} \right) \right] e_y
\end{align*}

where $B_0$ gives the magnitude of the magnetic field, $a$ is the shear length, $e_y$ is the unity vector along the $Y$-direction of a cartesian frame of reference and $b$ is a constant. For this equilibrium structure the neutral surface is represented by the plane $X = 0$. At this location a boundary layer develops in consequence of the tearing instability. In this layer the velocity and the magnetic field are continuous but they present sharp gradients in the direction of the $X$-axis, i.e. perpendicularly to the neutral surface. During the growth of the unstable modes, these gradients remain localized near the neutral surface of the equilibrium structure. The thickness of the boundary layer depends on the resistivity $\eta$, in that it is smaller for lower values of $\eta$. 

Because of these features, a multi-domain technique with Chebyshev expansions is well-suited to give an accurate numerical solution of this problem. In this section we present the results of some simulations of the tearing instability. In particular, the results obtained using the double-domain technique will be compared with those obtained using the single-domain technique. In order to excite the instability, at the initial time $t = 0$ a small amplitude divergenceless random perturbation is superposed on the equilibrium magnetic field $B_{eq}$ (equation (27)). The amplitude of the initial perturbation has been chosen so that the perturbation energy to the equilibrium structure energy ratio is a small quantity. In particular, indicating such a ratio by

$$
\varepsilon = \frac{1}{4E_0} \int_D \left[ Z_{pert}^2(x, y) + Z_{pert}^{-2}(x, y) \right] dx \, dy
$$

where

$$
E_0 = \frac{1}{4} \int_D \left[ Z_{pert}^2(x, y) + Z_{pert}^{-2}(x, y) \right] dx \, dy
$$

is the equilibrium structure initial energy, we chose the perturbation amplitude such that $\varepsilon = 10^{-7}$, i.e. much less than the equilibrium magnetic field amplitude.

The values of the parameters used in the runs are: $\chi^{-1} = 2 \times 10^3$, $R = 8/3$, $l = 10$. With this choice the Fourier harmonic $m = 1$ turns out to be comprised in the unstable wavelengths range, close to the most unstable wavelength [10-12]. Moreover, the spatial domain width ($2l$ in our units) is much larger than the width of the equilibrium magnetic field shear (equal to 1 in our units); then, the boundary condition at $x = \pm l$ do not sensibly affect the evolution of the instability.

In order to study the capability of the pseudospectral Chebyshev expansion to describe the formation of the boundary layer, different runs have been performed using a different number of Chebyshev polynomials. In particular, we considered the cases $N=32$, $N=64$, and $N=128$, corresponding respectively to $33+33$, $65+65$, and $129+129$ Chebyshev polynomials in the double-domain case (see equation (8)), and to $65$, $129$, and $257$ Chebyshev polynomials in the single domain case (see equation (11)). The couplings among Fourier harmonics are rather weak because of the low energy level in the perturbation. So the Fourier spectrum of the solution is very steep and a small number of Fourier harmonics ($M=32$) has been used. The values of the other numerical parameters are $\Delta t = 4$, $\theta = 0.505$, $c_1 = 0.025$, $c_2 = 1$. We verified that with these values the scheme is numerically stable, in accordance with the results by Harned and Schnack [7].

Starting from the initial condition, for increasing time the $m = 1$ harmonic evolves giving origin to the unstable eigenmode which corresponds to that wavelength and to the given boundary conditions. In figures 1a and 1b the normalized kinetic energy $E_{1kin}(t)$ of the $m = 1$ Fourier harmonic is plotted vs time, the normalized kinetic energy of the $m$-th
Fourier harmonic being defined as

\[(29) \quad E_m^{\text{kin}}(t) = \frac{1}{8 E_0} \int_{-l}^{l} \left| \hat{Z}^+(x, m, t) + \hat{Z}^-(x, m, t) \right|^2 dx. \]

The time evolution of the normalized kinetic energy $E_m^{\text{kin}}(t)$ is plotted both in the double-domain (figure 1a) and in the single-domain (figure 1b) case, using different numbers of Chebyshev polynomials in the spectral expansions of the quantities.

The main features of the time evolution can be summarized as follows: during about the first 600 time units the kinetic energy grows in time, but relevant oscillations are superposed on this trend. These oscillations are gradually damped. This stage corresponds to the formation of the unstable eigenmode, which is characterized by strong gradients in the $x$-direction, localized around the line $x = 0$. From the time $t \approx 600$ on, the kinetic energy of the $m = 1$ harmonic grows exponentially up to the time $t \approx 1000$. This stage corresponds to the exponential growth of the unstable eigenmode, predicted by the linear theory of the tearing instability [4].

In the double-domain case (figure 1a) the plots corresponding to $33+33, 65+65, 129+129$ Chebyshev polynomials are, in practice, superposed. Then, if the double-domain technique is used, the evolution of the kinetic energy in the perturbation is well described also when a relatively low number of Chebyshev polynomials is employed, namely a total number of 66 polynomial in the whole domain.

A different behavior is observed when the single-domain technique is used (figure 1b). In particular, the time evolution is still well described with 129 and 257 Chebyshev polynomials, but this is not the case when spectral expansions with 65 polynomials are used. Actually, in this latter case the exponential growth of the kinetic energy appears to be sensibly slower than in all the other cases. The value of the growth rate $\gamma$ has been determined fitting a linear function of the time on $\ln[E_1^{\text{kin}}(t)]$. We obtained $\gamma \approx 3.41 \times 10^{-3}$ in the single-domain case with 65 Chebyshev polynomials, while $\gamma \approx 4.23 \times 10^{-3}$ in all the other cases. This indicates that 65 polynomials in the single-domain case are not sufficient to accurately describe the time evolution of the energy in the perturbation.

The profiles of the real part of the $m = 1$ Fourier harmonic of the velocity and of the magnetic field, as functions of the $x$-variable, are shown in figures 2 and 3 at the time $t = 1000$, i.e. during the exponential growth stage. It can be seen that the solution has strong space variations around $x = 0$, as predicted by the linear theory [4,10,11]. The solutions obtained using the double-domain technique (figure 2) with $33+33, 65+65, 129+129$ Chebyshev polynomials are very close to one another, so to appear superposed. Only the $x$-component of the velocity perturbation (figure 2a) calculated with $33+33$ polynomials presents some difference near the boundaries with respect to the profiles calculated using a higher number of polynomials. On the contrary, the boundary layer
Higher Godunov Methods for Astrophysical Flows

is well described also using $33 + 33$ polynomials. Then, the resolution which is obtained using 66 polynomials with the double-domain technique is sufficient to describe the structure of the boundary layer associated with the tearing instability.

The results obtained using the single-domain technique are shown in the figures 3; it is seen that the profiles obtained using 129 or 257 Chebyshev polynomials are superposed. On the contrary, the solution is sensibly different when 65 polynomials are employed: the amplitude of the perturbation is lower, in accordance with the results shown in figure 1b. Moreover, relevant short-scale oscillations are present in the $y$-component velocity perturbation profile (figure 3b); the amplitude of such oscillations increases with the time. This phenomenon indicates that when 65 polynomials are used the Chebyshev expansion is unable to correctly describe the strong gradients which forms around $x = 0$.

In order to obtain further information on the accuracy of the solution we calculated the Chebyshev spectrum $z_n^p(x,t)$ of the excited unstable eigenmode $z^p(x,t)$. In the single-domain case, these two quantities are defined by

$$z^p(x,t) = \frac{\tilde{Z}^p(x,1,t)}{\left[ \int_{-1}^{1} |\tilde{Z}^p(x',1,t)|^2 dx' \right]^{1/2}}$$

and

$$z^p(x,t) = \sum_{n=0}^{N} z_{i,n}^p(t) T_n(x,l) \quad x \in [-l,l], \quad i = x, y, \quad \sigma = \pm.$$ 

In the double domain case we represent only the Chebyshev spectra relative to the left subdomain $D^L$, those relative to the right subdomain $D^R$ being very similar in consequence of the symmetry properties of the unstable eigenmode [4,11].

In figures 4a and 4b the spectrum $|z_{x,n}^p(t)|$ is plotted for $t = 1000$ and $t = 2000$, respectively, in the double-domain case, using different values of $N$. The same quantity is plotted in figures 5a and 5b, in the single-domain case. The time $t = 1000$ approximately corresponds to the end of the exponential growth stage, while $t = 2000$ is during the instability saturation (see figures 1). Comparing the results shown in figures 4 and 5 it is seen that the Chebyshev spectra $|z_{x,n}^p(t)|$ obtained using the double-domain technique are much steeper than those obtained by the single-domain technique. Then, using the double-domain technique a small number of Chebyshev polynomials is required to represent the solution with a boundary layer at $x = 0$. On the contrary, using the single-domain technique the contribution of high-order Chebyshev polynomials in the spectral expansion of the boundary layer solution is much more relevant.

In consequence of this, representing the solution with the same number of Chebyshev polynomials, much lower truncation errors are obtained using the double-domain then the
single-domain technique. An order of magnitude evaluation of the truncation error in the 
mode \( z_f^s(x, t) \) is given by

\[
\Delta_N z_f^s(t) = \sum_{n=N+1}^{128} |z_{f,n}^s(t)|
\]

which represents an estimation of an upper bound for the truncation error of the mode 
\( z_f^s(x, t) \) in the cases in which the double-domain technique has been used with 33 + 33 
or 65 + 65 polynomials. This quantity is plotted in figure 6 as a function of time, relative 
to the component \( z_f^s(x, t) \). For instance, at the time \( t = 1000 \) the truncation error of 
the mode \( z_f^s(x, t) \) is less than \( 3 \times 10^{-5} \) when 33 + 33 polynomials are used, while it is 
less than \( 2 \times 10^{-5} \) using 65 + 65 polynomials. These values are actually small because 
the truncation error is evaluated for the normalized solution (see equation (30)), i.e. for a 
quantity whose \( L^2 \)-norm is equal to 1. For increasing time the truncation error \( \Delta_N z_f^s(t) \) 
increases in consequence of nonlinear effects which transfer energy towards the higher-order 
coefficients of the spectral expansion.

A similar procedure could be used to estimate the truncation error for the single- 
domain runs. Indeed, from figures 5a and 5b it can be seen that, though at \( t = 1000 \) the relation (43) is still relatively well satisfied (at least as an order of magnitude estimation), 
this is no more true at \( t = 2000 \). In fact, figure 5b shows that the high-\( n \) part of the 
spectrum strongly changes when increasing the order of the Chebyshev transform. Then, 
in the single-domain runs we can assume as an upper limit for the truncation error the 
quantity

\[
\Delta_N z_f^s(t) = \sum_{n=N+1}^{256} |z_{f,n}^s(t)| , \quad \text{with } N = 64 \text{ or } N = 128
\]

but only during the first part of the runs, say for \( t \leq 1000 \). This quantity is also plotted 
on figure 6 (thick lines). Comparing with the results obtained with the double-domain 
technique, it can be seen that in this latter case the truncation error is orders of magnitude 
smaller than that obtained using the single-domain technique with the same number of 
Chebyshev polynomials.

This result is due to the fact that the strongest gradients in the solution are localized 
around the \( x = 0 \), which corresponds to the internal boundary of the double-domain case. 
Then, the accuracy of the Chebyshev expansion is much higher in this case, in accordance 
with the results of Solomonoff and Turkel [3].

The pseudospectral method used to calculate the nonlinear (products) terms in 
equations (11) leads to aliasing errors [1,13,14]. These errors are more relevant during 
the last part of the runs, when nonlinear effects play an important role in the instability 
saturation. Such errors are smaller if the spectra of the quantities which are multiplied are
steeper. This implies that lower aliasing errors should be obtained with the double-domain technique.

The effects of aliasing errors in the single-domain runs can be observed in figures 5a and 5b: indeed, as already observed, the high-n part of the spectrum changes when the order of the Chebyshev polynomials is increased. This effect becomes more relevant with increasing time; actually, the amplitude of the perturbation grows in time, so the nonlinearities in the evolution equations (11) become more important. The same effect is much less relevant in the double-domain case (see figures 4a and 4b), even for \( t = 2000 \). Then, the small truncation errors obtained with the double-domain technique allow to keep aliasing errors to a lower level; this is useful in particular during the nonlinear stage of the instability.

A more quantitative evaluation of the effects of the truncation errors can be obtained considering the divergence of the fields \( \mathbb{Z}^\sigma \). The time evolution equations (2)-(3) ensures that \( \partial \mathbb{Z}_i^\sigma / \partial x_i = 0 \) for \( t > 0 \), provided that: (i) the initial condition is divergenceless and (ii) \( \partial \mathbb{Z}_i^\sigma / \partial x_i \) is kept vanishing at the boundaries for any time \( t \geq 0 \). Let us discuss how the truncation errors affect the conservation of the divergence in the numerical scheme. The truncation in the representation of the fields \( \mathbb{Z}^\sigma \) does not introduce any error in the evaluation of linear operators, such as space derivatives. On the contrary, aliasing errors arise in the calculation of nonlinear terms. Consider the numerical scheme (11)-(12) for the truncated quantities, and taking the divergence of the equation (11a) and using the equation (12b) evaluated at the time \( t = t_n \), we have

\[
\left[ 1 - \frac{\partial \chi \Delta t}{2} \frac{\partial}{\partial x_k \partial x_k} \right] \left( \frac{\partial \mathbb{Z}_i^\sigma(n)}{\partial x_i} \right) = \left[ 1 - \frac{\partial \chi \Delta t}{2} \frac{\partial^2}{\partial x_k \partial x_k} \right] \left( \frac{\partial \mathbb{X}_i^\sigma(n)}{\partial x_i} \right) \\
- \theta \Delta t \left[ \mathbb{Z}_k^{-\sigma(n)} \frac{\partial}{\partial x_k} \left( \frac{\partial \mathbb{Z}_i^\sigma(n)}{\partial x_k} \right) \left( \frac{\partial \mathbb{X}_i^\sigma(n)}{\partial x_i} \right) \right] + \epsilon^\sigma
\]

(34)

where \( \epsilon^\sigma \) is the aliasing error which arises in the calculation of the nonlinear terms. Even assuming that the truncated field \( \mathbb{Z}^\sigma(n) \) at the time \( t = t_n \) is divergenceless: \( \partial \mathbb{Z}_i^\sigma(n) / \partial x_i = 0 \) the aliasing error \( \epsilon^\sigma \) acts as a source term for the field \( \mathbb{Z}^\sigma(x) \) at time \( t^* = t_n + \theta \Delta t \). A similar argument applied to the equations (12a) and (11b) shows that aliasing errors generate divergence in the truncated fields \( \mathbb{Z}^\sigma(n+1) \). The time evolution of the divergence of \( \mathbb{Z}^\sigma(n) \) then gives account of the effects of the cumulation of the aliasing errors in the pseudospectral numerical scheme. In figure 7 the quantities

\[
d^\sigma(t) = \max_{(x_i,y_i) \in D} \left| \frac{\partial \mathbb{Z}_i^\sigma(x_i,y_j,t)}{\partial x_k} \right|, \quad \sigma = \pm
\]

(35)

are plotted as functions of time, for the different runs performed using both the single-domain and the double-domain technique. The maximum is calculated over all the points
in the spatial grid. For any run it has been found that, at any time \( t \), \( d^+(t) \) is very close to \( d^-(t) \); so, these quantities appear superposed. From figure 7 it can be seen that, using the same number of Chebyshev polynomials, much lower values of the maximum divergence are obtained with the double-domain technique than with the single-domain one, in accordance with the much lower truncation errors obtained in the former case.

4. Conclusions

In this paper we have described the application of a pseudospectral Fourier-Chebyshev method to the solution of a problem in which a boundary layer forms, namely the development of the tearing instability in a plane sheet pinch. In the considered two-dimensional configuration the boundary layer is located along a straight line, where strong gradient in the direction perpendicular to such a line develops.

Our aim was to show that a better accuracy can be obtained using a double-domain technique rather than a standard single-domain technique. The idea is based on the results by Solomonoff and Turkel [3], who studied the properties of pseudospectral Chebyshev expansions in cases when sharp gradients are present. These authors found that lower errors result when these gradients are localized near the boundary of the Chebyshev domain than when they are near the center.

On the base of such results, we have splitted the computational domain in two subdomains, the internal boundary being along the line \( x = 0 \), where the boundary layer is located. In each subdomain a Chebyshev expansion has been performed with respect to the \( x \)-variable, i.e. in the direction of the strongest gradients, while a Fourier expansion has been used with respect to the \( y \)-variable. The calculation of numerical solution has been splitted in two by a decoupling technique, which allows to calculate the solution independently in each of the two subdomains. In this way two \((N+1) \times (N+1)\) linear systems are solved at each time step and for any Fourier harmonic, instead of one \(2(N+1) \times 2(N+1)\) linear system.

The results obtained by the double-domain technique have been compared with those obtained by a single-domain technique, in which a unique Chebyshev expansion is performed in the \( x \)-direction. In this latter case, the boundary layer is located at the center of the Chebyshev mesh. The \( m = 1 \) Fourier harmonic corresponds to the unstable eigenmode in which the boundary layer forms. It has been found that the Chebyshev spectrum of this harmonic at a given time is much steeper when the double-domain method is used than in the single-domain case. This implies that if the same number of Chebyshev polynomials is used with the two techniques, much lower truncation errors are obtained with the double-domain method. For instance, considering the quantity \( Z_x \), the truncation error obtained using \( 65 + 65 \) Chebyshev polynomials with the double-domain technique is smaller by a
factor \(\approx 106\) with respect to the truncation error obtained using 129 polynomials with the single-domain technique.

As a consequence, the double-domain technique allows to obtain an accurate solution with a relatively low number of Chebyshev polynomials: in the considered case we found that using 33 + 33 polynomials a good accuracy is obtained in the numerical solution. On the contrary, using 65 polynomials with the single-domain technique a much worse result is obtained: strong oscillations are present in the space profiles of the solution and the time evolution is sensibly changed; in particular the instability growth rate is lowered.

The pseudospectral method leads to aliasing errors in the evaluation of the nonlinear terms in the equations. Since steeper Chebyshev spectra are obtained with the double-domain technique, smaller aliasing errors are obtained using such a technique. As a consequence, the growth of the \(Z^a\) fields divergence in the numerical solution is much slower using the double-domain than the single-domain technique. This is relevant, in particular, during the nonlinear stage of the instability, when the nonlinearities of the equations play a more important role in the time evolution.

**Acknowledgments**

The author is grateful to A. Mangeney and V. Carbone for several helpful discussions on the subject of the paper. A particular thanks is due to P. Veltri for many suggestions and a critical reading of the manuscript. This work was partially supported by the "Ministero della Università e della Ricerca Scientifica e Tecnologica" (MURST) on National Project Funds and by the "Consiglio Nazionale delle Ricerche" (CNR) under Contract N.90.01253.CT02.

**REFERENCES**

Figure 1: The kinetic energy associated to the $m = 1$ Fourier harmonic, as a function of time. Using the double domain technique the plots corresponding to $33 + 33$, $65 + 65$, and $129 + 129$ Chebyshev polynomials are superposed (a). Using the single domain technique, this holds when $129$ or $257$ polynomials are used, while a different behavior is found with $65$ polynomials (b).
Figure 2: The profiles, as functions of $x$, of the real part of the $m = 1$ Fourier harmonic of the velocity $x$-component (a) and $y$-component (b) at the time $t = 1000$, obtained using the double-domain technique with $33 + 33$, $65 + 65$, and $129 + 129$ Chebyshev polynomials.
Figure 3: The same profiles as those of figure 2, obtained using the single-domain technique with 65, 129, and 257 Chebyshev polynomials.
Figure 4: The spectrum $|z_{x,a}(t)|$ of the normalized mode $z + x(x,t)$ plotted as a function of $n$ at the time $t = 1000$ (a) and $t = 2000$ (b), using $33+33$, $65+65$, and $129+129$ Chebyshev polynomials with the double-domain technique.
Figure 5: The same as in figure 4, using 65, 129, and 257 Chebyshev polynomials with the single-domain technique.
Figure 6: The truncation error of the normalized mode $x$-component is plotted as a function of time. The total number of the employed Chebyshev polynomials is indicated. Thick lines refer to single-domain runs while thin lines refer to double-domain runs.

Figure 7: The maximum divergences $d^+(t)$ and $d^-(t)$ are plotted as function of time. The total number of the employed Chebyshev polynomials is indicated. Thick lines refer to single-domain runs while thin lines refer to double-domain runs. For all the cases $d^+(t)$ and $d^-(t)$ are almost equal, so they appear superposed.

Francesco MALARA
Dipartimento di Fisica, Università della Calabria
87030 Cosenza, Italy.

Lavoro pervenuto in redazione il 10.11.1993.