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F. Tricerri

LOCALLY HOMOGENEOUS RIEMANNIAN MANIFOLDS

Abstract. A Riemannian manifold is locally homogeneous if the pseudogroup of the local isometries acts transitively on it. A complete locally homogeneous Riemannian manifold is locally isometric to a globally homogeneous Riemannian space. This is not longer true if we drops the completeness assumption. The aim of the present paper is to discuss this phenomenon in some detail.

1. Introduction

A Riemannian manifold (M,g) is *locally homogeneous* if the pseudogroup of the local isometries acts transitively on it.

If (M,g) is in addition *complete*, then its universal Riemannian covering is globally homogeneous. Therefore, (M,g) is locally isometric to a Riemannian homogeneous space G/H, endowed with a G-invariant metric.

This is no longer true if we drop the completeness assumption. In fact, in such a case, there exist locally homogeneous Riemannian manifolds which are not locally isometric to any Riemannian homogeneous spaces (see [3] [5] and [9]).

The aim of the present paper is to discuss this phenomenon in some detail.

Our approach is developping as follows.

We start by recalling that (M,g) is locally homogeneous if and only if there exists a metric linear connection ∇ with parallel torsion and curvature tensor fields (we say that ∇ is *invariant* by *parallelism* or that it is an *Ambrose-Singer connection*).

To each connection of this kind it is possible to attach an algebraic object, the so-called *infinitesimal model*. Conversely, to each infinitesimal model corresponds a uniquely defined (up to local isometries) locally homogeneous Riemannian manifold (see [5] and section 2).

The *Nomizu construction* associates to each infinitesimal model (and therefore to each Ambrose-Singer connection) a Lie algebra $\mathfrak{g}$ together with a reductive decomposition $\mathfrak{g} = V \oplus \mathfrak{h}$ ($\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ and $[\mathfrak{h}, V] \subseteq V$). The infinitesimal model is *regular* if the connected subgroup H, with Lie algebra $\mathfrak{h}$, of the simply connected Lie group G, whose Lie algebra is $\mathfrak{g}$, is *closed* in G. In such a case, the locally homogeneous Riemannian manifold (M, g) is locally isometric to the homogeneous space G/H endowed with a suitable G-invariant Riemannian metric (section 2).

The converse also holds and it will be proved in section 5 (see Theorem 5.2, which is a slightly improvement of a result of A. Spiro. Its proof is inspired by [10]). The key point in proving these results is the existence of the *canonical AS-connection* (see[4] and section 4). This connection is a purely Riemannian invariant. In particular, the Lie algebra $\mathfrak{g}_0$ associated with it via the Nomizu construction is isomorphic to the Lie algebra of the *Killing generators* introduced by K. Nomizu in [7]. Recall that in the case of a simply connected homogeneous space this algebra is isomorphic to the Lie algebra of the isometry group (see [7] and n.4).

It follows from Theorem 5.2 that, if (M, g) is locally isometric to a globally homogeneous Riemannian space, then *all* the infinitesimal models associated with it are *regular*. Moreover, it is also possible to prove that the *transvection algebra* of each infinitesimal model is "regular" (see Theorem 6.1). This gives at once the existence of simple examples of locally homogeneous Riemannian manifolds which are not isometric to any globally homogeneous Riemannian space (see section 6).

As remarked by O. Kowalski (see the introduction and Remark 4.3 of [5]), these examples show that there exist manifolds endowed with a linear connection with parallel torsion and curvature which are not affinely diffeomorphic to any reductive homogeneous space (in contrast with the claim of [6] p.60).

2. General facts

The basic facts about locally homogeneous Riemannian manifolds is contained in the following theorem.

THEOREM 2.1. *A Riemannian manifold (M, g) is locally homogeneous if and only if there exists a metric connection ∇ such that*

$$\nabla T' = \nabla R' = 0\,, \tag{2.1}$$

where T' and R' are the torsion and the curvature of the connection ∇.

Proof. If such a connection ∇ exists, then the parallel transport (w.r. to ∇) along a curve connecting two points p and q can be extended to a local isometry f sending p to q (see *f.* ex. [11]). Conversely, the *canonical connection* constructed in [4] (see also section 4) always satisfies (2.1).

Each connection satisfying (2.1) is called an *Ambrose-Singer connection* or briefly an *AS-connection.*

Each AS-connection has a natural associated algebraic object, namely its *infinitesimal model.*

An *infinitesimal model* on Euclidean vector space V endowed with the inner product $<>$ is a pair (T, K) of tensors on V,

$$T : V \longrightarrow \mathrm{End}(V),\ X \longrightarrow T_X\,,$$
$$K : V \times V \longrightarrow \mathrm{End}(V),\ (X, Y) \longrightarrow K_{XY}$$

such that

(2.2) $T_X Y = -T_Y X,\ K_{XY} = -K_{YX}\,.$

(2.3) $\langle K_{XY}Z, W\rangle + \langle K_{XY}W, Z\rangle = 0\,.$

(2.4) $K_{XY} \cdot T = K_{XY} \cdot K = 0\,.$

(2.5) $K_{XY}Z + K_{YZ}X + K_{ZX}Y + T_{T_X Y}Z + T_{T_Y Z}X + T_{T_Z X}Y = 0\,.$

(2.6) $K_{T_X Y}Z + K_{T_Y Z}X + K_{T_Z X}Y = 0\,.$

In (2.4) K_{XY} is acting as derivation on the tensors.

Two infinitesimal models (T, K) on V, and (T', K') on V', are *isomorphic* if there exists an isometry $F : V \longrightarrow V'$ such that

$$T'_{FX}FY = FT_X Y$$

and

$$K'_{FXFY}FZ = FK_{XY}Z\ .$$

The infinitesimal model associated with an AS-connection ∇ is defined by taking $V = T_pM$ and putting $<>= g_p$, $T = T'_p$, $K = R'_p$. In fact (2.2) and (2.3) are trivially satisfied. The Ricci identities give (2.4). Moreover (2.5) and (2.6) follow from the Bianchi identities.

Of course, if we choose another point p of M, we get simply an infinitesimal model which is isomorphic to the previous one.

Conversely we have

THEOREM 2.2. ([5]) *Let (T,K) be an infinitesimal model on V. Then, there exists a locally homogeneous Riemannian manifold (M,g) and an AS-connection ∇ on it, whose infinitesimal model is isomorphic to the given one. Moreover, (M,g) is uniquely defined up to local isometries.*

Two locally homogeneous Riemannian manifolds (M,g) and (M',g') with isomorphic infinitesimal models are locally isometric. Indeed, any isometry F between $V = T_pM$ and $V' = T_{p'}M'$, preserving (T,K) and (T',K'), can be extended to a local affine diffeomorphism f between the corresponding AS-connections ∇ and ∇' on (M,g) and (M',g'). This diffeomorfism turns out to be an isometry, because its differential at p coincides with the isometry F (see *f.* ex. [11]).

We refer to [10] for a proof of Theorem 2.2 by using the theory of *transformation pseudogroups.*

The *Nomizu construction* associates a Lie algebra $\mathfrak{g}$ with each infinitesimal model (T,K) on V in the following way. Let $\mathfrak{so}(v)$ be the Lie algebra of the skewsymmetric endomorphisms of V. Let $\mathfrak{h}$ be the subalgebra of $\mathfrak{so}(v)$ defined by

$$\mathfrak{h} = \{A \in \mathfrak{so}(v)/A\cdot T = A\cdot K = 0\}\ . \tag{2.7}$$

Note that K_{XY} is an element of $\mathfrak{h}$ for all X,Y. Then, the Lie algebra $\mathfrak{g}$ is the direct sum of V and $\mathfrak{h}$ endowed with the following brackets

$$[X,Y] = -T_XY + K_{XY}\ , \tag{2.8}$$

$$[A,X] = A(X)\ , \tag{2.9}$$

$$[A,B] = AB - BA\ , \tag{2.10}$$

where X, Y are elements of V and A, B belong to $\mathfrak{h}$.

Let G be the connected and simply connected Lie group whose Lie algebra is $\mathfrak{g}$ and H its connected subgroup corresponding to $\mathfrak{h}$. If H is *closed* in G, then the infinitesimal model is called *regular*.

In such a case, the locally homogeneous Riemannian manifold (M, g) associated with (T, K) in the sense of Theorem 2.2 is *locally isometric* to the homogeneous space G/H endowed with the G-invariant metric induced by the inner product of V. In fact, $\mathfrak{g} = V \oplus \mathfrak{h}$ is a *reductive decomposition* of the Lie algebra of G. Then, the "canonical connection of second kind" associated with this decomposition is an AS-connection, whose torsion and curvature at the origin O coincide with T and K (see [2] vol. II). Therefore, (M, g) is locally isometric to G/H.

The converse holds, but it is a more delicate question and it will be proved only in section 5.

3. The AS-connections

Some technical facts concerning AS-connections, their infinitesimal models and the Lie algebras obtained by the Nomizu construction will be useful in the sequel. We collect and prove them in this section.

Let ∇ be an AS-connection on (M, g). Denote by D the Levi Civita connection of the metric g and put

$$S' = \nabla - D \,. \tag{3.1}$$

Then the torsion of ∇ is given by

$$T'_X Y = S'_X Y - S'_Y X \,, \tag{3.2}$$

and

$$2g(S'_X Y, Z) = g(T'_X Y, Z) - g(T'_Y Z, X) + g(T'_Z X, Y) \,, \tag{3.3}$$

for any vector fields X, Y, Z on M. Since $\nabla g = 0$, from (3.2) and (3.3) we get at once that $\nabla T' = 0$ if and only if $\nabla S' = 0$. On the other hand, the curvatures of ∇ and D are related by

$$R_{XY} = R'_{XY} - [S'_X, S'_Y] + S'_{T'_X Y} \,. \tag{3.4}$$

Hence, from $\nabla R' = 0$, we get

$$\nabla R = 0 . \tag{3.5}$$

More in general, we have

PROPOSITION 3.1. *Let ∇ be an AS-connection on (M,g). Then,*

$$\nabla(D^m R) = 0$$

for all $m \geq 0$.

Proof. As remarked above, the proposition is true if $m = 0$. We prove that it is true for all m by induction on m. Let $\nabla(D^r R)$ be equal to zero for all $r \leq m-1$. Then , we have

$$\begin{aligned}
&(\nabla(D^m R))(X, Y_1, \dots, Y_{m+4}) = -X\Big((S'_{Y_1} \cdot D^{m-1}R)\Big)(Y_2, \dots, Y_{m+4}) \\
&+ (S'_{\nabla_{X_1} Y_1} \cdot D^{m-1}R)(Y_2, \dots, Y_{m+4}) + (S'_{Y_1} \cdot D^{m-1}R)(\nabla_X Y_2, \dots, Y_{m+4}) \\
&+ \dots + (S'_{Y_1} \cdot D^{m-1}R)(Y_2, \dots, Y_{m+4}) = -\nabla_X\Big((S'_{Y_1} \cdot D^{m-1}R)\Big)(Y_2, \dots, Y_{m+4}) \\
&+ (S'_{\nabla_{X_1} Y_1} \cdot D^{m-1}RR(Y_2, \dots, Y_{m+4}) \\
&= \Big((S'_{\nabla_{X_1} Y_1} - [\nabla_X, S'_{Y_1}]) \cdot D^{m-1}R\Big)(Y_2, \dots, Y_{m+4})
\end{aligned}$$

since $\nabla_X D^{m-1}R = 0$. Now, recall that

$$(\nabla_X S')_Y = [\nabla_X, S'_Y] - S'_{\nabla_X Y} . \tag{3.6}$$

Then, the proposition follows at once from $\nabla S' = 0$.

As an immediate corollary we get

PROPOSITION 3.2. *Let ∇ be an AS-connection on (M,g). Then, the difference tensor field $S' = \nabla - D$ satisfies*

$$D_X(D^m R) = -S'_X \cdot D^m R ,$$

for all $m \geq 0$.

A simple computation also gives

PROPOSITION 3.3. *Let ∇ be an AS-connection on (M, g) and let A be any endomorphism of T_pM. Then*

$$\begin{aligned}&\Big(A\cdot(D^{m+1}R)_p\Big)(X_0,X_1,\dots,X_{m+4})\\&=\Big(([A,S'_{X_0}]+S'_{AX_0})\cdot(D^mR)_p\Big)(X_1,\dots,X_{m+4})\\&+\Big(S'_{X_0}\cdot\Big(A\cdot(D^mR)_p\Big)X_1,\dots,X_{m+4}\Big),\end{aligned}$$

for all tangent vectors $X_0, X_1, \dots, X_{m+4}$ at p.

Finally, we have

PROPOSITION 3.4. *Let ∇ be an AS-connection, $S' = \nabla - D$ and (T, K) its infinitesimal model on $V = T_pM$. Let A be an element of $\mathfrak{h} = \{A \in \mathfrak{so}(v)/A\cdot T = A \cdot K = 0\}$, then $A\cdot(D^mR)_p = 0$ for all $m \geq 0$.*

Proof. Let S be the value of S' at p. If A is an element of $\mathfrak{h}$, from (3.3) and $A \cdot T = A \cdot (T'_p) = 0$, we get

$$A \cdot S = 0\,.$$

On the other hand, $A \cdot K = A \cdot R'_p = 0$. Then, (3.4) implies

$$A \cdot R_p = 0\,.$$

Hence, Proposition 3.3 and an inductive argument give $A\cdot(D^mR)_p = 0$ for all $m \geq 0$.

4. The canonical connection

Among the AS-connections of a locally homogeneous Riemannian manifold (M, g) there exists one which is canonically defined. In order to describe this connection we need some more concepts.

For each integer $r \geq 0$, let $\mathfrak{g}(p, r)$ be the Lie subalgebra of $\mathfrak{so}(T_pM)$ defined by

$$\mathfrak{g}(p,r) = \{A \in \mathfrak{so}(T_pM)/A\cdot D^mR_p = 0\,,\ 0\leq m\leq r\}\,, \tag{4.1}$$

where $D^0 R = R$. These Lie algebras give rise to a filtration of $\mathfrak{so}(T_pM)$. Let $k(p)$ be the first integer for which

$$\mathfrak{g}\Big(p, k(p)\Big) = \mathfrak{g}\Big(p, k(p) + 1\Big) . \tag{4.2}$$

In general, $k(p)$ varies from point to point. But, since (M, g) is locally homogeneous, $k(p) = k$ is a *constant*. We call this integer the *Singer invariant* of $(M, g)(k < \frac{3}{2}n$, see [1] p.165).

Denote by $\mathfrak{h}(p)$ the Lie algebra $\mathfrak{g}(p, k) = \mathfrak{g}(p, k + 1)$. Let $\mathfrak{h}(p)^{\perp}$ be the orthogonal complement of $\mathfrak{h}(p)$ in $\mathfrak{so}(T_pM)$ w.r.t. the scalar product induced by the metric g_p. In other words,

$$\mathfrak{h}(p)^{\perp} = \{A \in \mathfrak{so}(T_pM)/\mathrm{trace}(AB) = 0 \text{ for all } B \in \mathfrak{h}(p)\} .$$

Note that $\mathfrak{h}(p)^{\perp}$ is $ad_{\mathfrak{h}(p)}$-invariant, since

$$\mathrm{trace}([A, B], B') = -\mathrm{trace}([B, B'], A)$$

for all A, B, B' in $\mathfrak{so}(T_pM)$.

THEOREM 4.1. ([5]) *Let (M, g) be a locally homogeneous Riemannian manifold. Then, there exists a unique AS-connection ∇_0 such that, for all p in M,*

$$S_0 \in \mathfrak{h}(p)^{\perp} ,$$

where $S_0 = \nabla_0 - D$ is the difference tensor field between ∇_0 and the Levi Civita connection D.

This connection is called the *canonical AS-connection* of (M, g).

The Lie algebra $\mathfrak{g}_o$ associated with the canonical connection by the Nomizu construction has several remarkable properties.

Denote by (T_0, K_0) the infinitesimal model on $V = T_pM$ associated with $\nabla_0\Big(T_0 = (T_0')_p, K_0 = (R_0')_p\Big)$.

Then, we have

PROPOSITION 4.2. *The subalgebra $\mathfrak{g}_0 = \{A \in \mathfrak{so}(v)/A \cdot T_0 = A \cdot K_0 = = 0\}$ of $\mathfrak{g}_0$ coincides with $\mathfrak{g}(p)$.*

Proof. By Proposition 3.4 we have $\mathfrak{h}_0 \subseteq \mathfrak{h}(p)$. Thus, it remains to prove that $\mathfrak{h}(p)$ is contained in $\mathfrak{h}_0$. Let A be an element of $\mathfrak{h}(p)$. Since $\mathfrak{h}(p) = \mathfrak{g}(p,k) = \mathfrak{g}(p,k+1)$, from Proposition 3.3 we get

$$\Big([A,(S_0)_x] + (S_0)_x\Big) \cdot (D^m R)_p = 0\,,$$

for all $m \leq k$, where $S_0 = (S_0')_p$. Therefore,

$$[A,(S_0)_x] + (S_0)_{Ax} \in \mathfrak{h}(p)\,.$$

On the other hand, $(S_0)_{AX}$ is an element of $\mathfrak{h}(p)^\perp$.

Moreover, $\mathfrak{h}(p)^\perp$ is $ad_{\mathfrak{h}(p)}$-invariant. Hence, also $[A,(S_0)_x]$ is an element of $\mathfrak{h}(p)$. It follows that

$$(A \cdot S_0)_X = [A,(S_0)_X] + (S_0)_{AX} = 0\,.$$

Then, (3.2) and (3.4) imply that $A \in \mathfrak{h}_0$ as claimed.

As a corollary we have

PROPOSITION 4.3. *If $A \in \mathfrak{h}(p) = \mathfrak{g}(p,k) = \mathfrak{g}(p,k+1)$, then*

$$A \cdot (D^m R)_p = 0$$

for all $m \geq 0$ (and not only for $m \leq k+1$).

Following K. Nomizu [7], we say that a pair (X, A), where X is a tangent vector at p and $A \in \mathfrak{so}(T_pM)$, is a *Killing generator* at p if

$$A \cdot (D^m R)_p = -\Big(D_X(D^m R)\Big)_p\,, \tag{4.3}$$

for all $m \geq 0$. The set $\mathfrak{k}(p)$ of the Killing generators at p is a Lie algebra w.r.t. the following bracket

$$\Big[(X,A),(Y,B)\Big] = \Big[A(Y) - B(X), (R_p)_{XY} + [A,B]\Big]\,. \tag{4.4}$$

Of course, this notion is meaningful for any Riemannian manifold (M,g). In such a case, $\mathfrak{k}(p)$ may vary with p. But, if (M,g) is locally homogeneous, then all the Lie algebras $\mathfrak{k}(p)$ are isomorphic and we have

Proposition 4.4. *The Lie algebra $\mathfrak{g}_0$ associated with the canonical connection ∇_0 of a locally homogeneous Riemannian manifold (M,g) is isomorphic to the Lie algebra $\mathfrak{k}(p)$ of the Killing generators at some point p of M.*

Proof. Proposition 3.2 implies that $\left(X,(S_0)_X\right)$ is a Killing generator at p, for each tangent vector X.

Hence, if (X,A') is another Killing generator at p,

$$A = A' - (S_0)_X$$

in an element of $\mathfrak{h}(p) = \mathfrak{h}_0$. Therefore, $X + A$ is a well-defined element of $\mathfrak{g}_0 = V \oplus \mathfrak{h}_0$. Define

$$\phi(X,A') = X + A\,. \tag{4.5}$$

Then, ϕ is an *injective* linear map of $\mathfrak{k}(p)$ into $\mathfrak{g}_0$. This map is also *surjective*. In fact, if $A \in \mathfrak{h}(p) = \mathfrak{h}_0$, then $\left(X, A + (S_0)_X\right)$ is a Killing generator at p, since

$$\left(A + (S_0)_X\right)\cdot(D^m R)_p = -(S_0)_X \cdot (D^m R)_p$$

for all $m \geq 0$ (Proposition 4.3). Then,

$$\phi\left(X, A + (S_0)_X\right) = X + A\,.$$

Finally, ϕ is a *morphism of Lie algebras*. Namely, if $\phi(X,A') = X + A$, $\phi(Y,B') = Y + B$, we have

$$\begin{aligned}&\phi\left(\left[(X,A'),(Y,B')\right]\right)\phi\left(A'(Y) - B'(X),(R_p)_{XY} + [A',B']\right)\\ &= \phi\Big(A(Y) - B(X) - (S_0)_X Y + (S_0)_Y X,\ (R_p)_{XY} + \left[(S_0)_X,(S_0)_Y\right]\\ &- \left[A,(S_0)_Y\right] + \left[B,(S_0)_X\right] + [A,B]\Big)\,.\end{aligned}$$

Since $A \cdot S_0 = 0$, we get

$$\begin{aligned}(S_0)_{AY - BX - (S_0)_X Y + (S_0)_Y X} &= \left[A,(S_0)_Y\right]\\ &- [B,(S_0)_X] - (S_0)_{(T_0)_X Y}\,.\end{aligned}$$

Then,

$$\phi\Big(\big[(X,A'),(Y,B')\big]\Big) = \Big(A(Y) - B(X) - (T_0)_X Y\,,$$
$$(k_0)_{XY} + [A,B]\Big)$$

by (3.4). Hence,

$$\phi\Big(\big[(X,A'),(Y,B')\big]\Big) = \big[\phi(X,A'),\phi(Y,B')\big]$$

as claimed.

For next applications, the following proposition will be useful.

PROPOSITION 4.5. *If (M,g) is a simply-connected homogeneous Riemannian manifold, then the Lie algebra of its isometry group (i.e., the Lie algebra of the Killing vector fields) is isomorphic to the Lie algebra of the Killing generators $\mathfrak{k}(p)$.*

This result follows at once from the Corollary at page 117 of [7], by taking in account that all the points of a homogeneous Riemannian manifold are "regular".

5. Local and global homogeneity

The infinitesimal model associated with the canonical AS-connection ∇_0 of a locally homogeneous Riemannian manifold (M,g) will be called *canonical infinitesimal model*. Then, we have

THEOREM 5.1. ([10]) *If (M,g) is locally isometric to a globally homogeneous Riemannian manifold (M',g'), then its canonical infinitesimal model is regular.*

Proof. It is not restrictive to suppose M' simply connected.

Then, $M' = G/H$, where G is the connected component of its isometry group and H is the isotropy group of some point O of M'. Of course, H is closed in G.

Let G' be the universal covering group of G. Denote by π the projection of G' onto G. Then, G' acts isometrically on M' as follows

$$G' \times M' \longrightarrow M',\ (a', p') \longrightarrow \pi(a')(p).$$

The isotropy group H' of O in G' is $\pi^{-1}(H)$. Hence, H' is closed in G and connected, since G' and H' are simply connected. G and G' have isomorphic Lie algebras and also H and H'. Because of Theorem 4.5, the Lie algebra of G' is isomorphic to the Lie algebra of the Killing generators $\mathfrak{k}'(0)$ of (M', g').

On the other hand, $\mathfrak{k}'(0)$ is isomorphic to $\mathfrak{k}(p)$, since (M,g) and (M',g') are locally isometric. Therefore, the Lie algebra of G' is isomorphic to $\mathfrak{g}_0$ (Proposition 4.4). Moreover, the Lie algebra of H' is isomorphic to the Lie subalgebra of $\mathfrak{k}(p)$, whose elements are the Killing generators at p of type $(0, A)$. Hence, it is isomorphic to the subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}$.

Since H' is closed in G', the canonical infinitesimal model is regular.

The previous result can be extended to each AS-connection. In fact we have

THEOREM 5.2. *Let ∇ be any AS-connection on (M,g). If (M,g) is locally isometric to a globally homogeneous Riemannian space, then the infinitesimal model associated to ∇ is regular.*

Proof. We shall get a contradiction by supposing that the infinitesimal model associated to ∇ is not regular.

Let $\mathfrak{g}$ and $\mathfrak{g}_0$ be the Lie algebras associated to ∇ and ∇_0 by the Nomizu construction. We have

$$\mathfrak{g} = V \oplus \mathfrak{h} \subseteq \mathfrak{g}_0 = V \oplus \mathfrak{h}_0,$$

since $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{h}_0$ (Proposition 3.4 and Proposition 4.2).

Let G_0 be the simply connected Lie group whose Lie algebra is $\mathfrak{g}_0$. Denote by $\underline{G}$ the connected Lie subgroup of G_0 corresponding to the subalgebra $\mathfrak{g}$. Let G be the universal covering group of $\underline{G}$ and π the projection of G on $\underline{G}$ (covering map). Let H be the connected subgroup of G, whose Lie algebra is $\mathfrak{h}$. The situation is summarized by the following diagram:

$$\begin{array}{ccc} H & \subset & G \\ \pi\downarrow & & \downarrow\pi \\ \underline{H} = \pi(H) & \subset & \underline{G} \subset G_0 \end{array}$$

If H is not closed, then $\underline{H}$ is connected but not closed in $\underline{G}$. Otherwise $\pi^{-1}(\underline{H})$ and H should be closed in G. Namely, π is continous and H is just the connected component of the identity of $\pi^{-1}(\underline{H})$. On the other hand, the topology of $\underline{G}$ is finer than the induced topology. Therefore, the subgroup $\underline{H}$ cannot be closed in G_0 too.

Denote by H^{G_0} and $H^{\underline{G}}$ the closures of $\underline{H}$ in G_0 and in $\underline{G}$, respectively (i.e., the smallest closed subgroups of G_0 and $\underline{G}$ containing H). Of course, these groups are connected. Let H_0 be the connected subgroup of G_0, whose Lie algebra is $\mathfrak{h}_0$. H_0 is closed since the canonical infinitesimal model is regular. Then we have

$$\underline{H} \subset H^{\underline{G}} \subseteq H^{G_0} \cap \underline{G} \subseteq H^{G_0} \subseteq H_0 \subseteq G_0 \,,$$

where the first inclusion is proper. By considering the corresponding Lie algebras, we get

$$\mathfrak{h}_0 \supseteq \text{ Lie } (H^{\underline{G}}) \supset \mathfrak{h}$$

and

$$\mathfrak{g} \supseteq \text{ Lie } (H^{\underline{G}}) \supset \mathfrak{h} \,.$$

Therefore, $\mathfrak{h} = \mathfrak{h}_0 \cap \mathfrak{g} \supset \mathfrak{h}$, which is a contradiction.

6. Construction of the examples

It follows from Theorem 5.2 that it is sufficient to produce a non-regular infinitesimal model in order to get an example of locally homogeneous Riemannian manifold which cannot be locally isometric to any Riemannian homogeneous space.

But the construction of the Lie algebra $\mathfrak{g} = V \oplus \mathfrak{h}$ starting from the infinitesimal model (T, K) could not be an easy task. In fact, it requires the computation of $\mathfrak{h}$, which may be unpleasent.

In order to avoid these problems, and to get simple examples, we follow an idea of O. Kowalski (see [3]).

Let (T, K) be any infinitesimal model on V. From 2.4 we get

$$[K_{XY}, K_{ZV}] = K_{K_{XY}ZV} + K_{ZK_{XY}V} \,. \tag{6.1}$$

Therefore, the operators K_{XY} span a subalgebra $\mathfrak{h}'$ of $\mathfrak{h} = \{A \in \mathfrak{so}(v)/A \cdot T = A \cdot K = 0\}$. Hence

$$\mathfrak{g}' = V \oplus \mathfrak{h}'$$

is a subalgebra of $\mathfrak{g}$. This is called the *transvection algebra* of (T, K).

Let G' be the simply connected Lie group whose Lie algebra is $\mathfrak{g}'$ and H' its connected subgroup corresponding to $\mathfrak{h}'$. Then, we have

THEOREM 6.1. *If the locally homogeneous Riemannian manifold (M, g) associated with the infinitesimal model (T, K) is locally isometric to a Riemannian homogeneous space (i.e., if (T, K) is regular), then H' is closed in G'.*

This theorem is essentially due to O. Kowalski [3].

We provide here a direct proof.

Proof. The infinitesimal model is regular. Therefore, (M, g) is locally isometric to G/H, where G is the simply connected Lie group corresponding to the Lie algebra $\mathfrak{g} = V \oplus \mathfrak{h}$ and, as usual, $\mathfrak{h} = \{A \in \mathfrak{so}(v)/A \cdot T = A \cdot K = 0\}$. G/H carries an AS-connection ∇ whose torsion and curvature at the origin O coincide with T and K (V is identified with $T_0(G/H)$ in the usual way). It is the "canonical" connection corresponding to the reductive decomposition $\mathfrak{g} = V \oplus \mathfrak{h}$ (see section 2). Then, also the *transvection group* of ∇ and its universal covering are acting transitively on G/H by isometries. The universal covering of the transvection group of ∇ is isomorphic to G'. Moreover, H' is closed in G' as showed by the Lemma 1.19, p.28 of [11] (see also the remark at page 31 which characterizes the Lie algebra of the group G') and therefore the theorem is proved.

We shall finish by producing a simple example.

Let G be the product $SU(2) \times SU(2)$ and H its (one-parameter) subgroup of the diagonal matrices of the form

$$\begin{pmatrix} e^{i\alpha t} & & & \\ & e^{-i\alpha t} & & \\ & & e^{i\beta t} & \\ & & & e^{-i\beta t} \end{pmatrix},$$

where α, β are constant and $\beta \neq 0$. If the ratio α/β is *irrational*, then H is *not closed* in G.

Let $\mathfrak{g} = \mathrm{su}(2) \oplus \mathrm{su}(2)$ be the Lie algebra of G and $\mathfrak{h}$ its subalgebra corresponding to H.

Let V be the orthogonal complement of $\mathfrak{h}$ w.r.t. the Killing form of $\mathfrak{g}$. Define T and K by

$$T_X Y = [X,Y]_V \,, \tag{6.2}$$

$$K_{XY} = ad_{[X,Y]_{\mathfrak{h}}} \,, \tag{6.3}$$

where $[X,Y]_V$ and $[X,Y]_{\mathfrak{h}}$ denote the projections of the bracket $[X,Y]$ on V and $\mathfrak{h}$, respectively.

It is easy verified that (T,K) is an *infinitesimal model* on V. Moreover, the *transvection algebra* $\mathfrak{g}'$ associated to (T,K) in the sense described above is isomorphic to $\mathfrak{g}$. Therefore, (T,K) is *non-regular* if α/β is *irrational*.

This example can be easily generalized. Namely, let G be a simply connected Lie group. Let H be a connected subgroup of G such that there exists a reductive decomposition $\mathfrak{g} = V \oplus \mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ of G ($\mathfrak{h}$ is the Lie algebra of H and $[\mathfrak{h}, V] \subseteq V$). Suppose that there exists an $Ad(H)$-invariant inner product $<>$ on V. Moreover, let

$$ad : \mathfrak{h} \longrightarrow ad\mathfrak{h},\ X \longrightarrow ad_X \,,$$

be an isomorphism. If we define T and K by (6.2) and (6.3), we get an infinitesimal model on V. Its transvection algebra is isomorphic to $\mathfrak{g}$ and therefore (T,K) is non-regular as soon as H is not closed in G.

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Franco TRICERRI
Dipartimento di Matematica "U. Dini"
Università di Firenze
Viale Morgagni 67/A, 50134 Firenze, Italy.

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