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FUNCTIONALS DEPENDING ON CURVATURES

0. General introduction

In this talk I shall consider a class of problems in the Calculus of Variations. Roughly speaking, these problems are of the type:

\[
\text{minimize a functional } F(M), \text{ defined on a class } M \text{ of submanifolds } M \text{ of a given ambient manifold } M, \text{ and depending on the area and on the curvatures of } M.
\]

I shall begin with a general introduction to the problem, then I shall review some known results, and finally I shall report on some recent joint work [AST] with Raul Serapioni and Italo Tamanini of Trento University.

For simplicity, I shall consider mostly functionals defined on 2-dimensional surfaces in \( \mathbb{R}^3 \), however it will be clear that the ideas, and many of the results, that I am going to expound carry over to the study of submanifolds of general dimension and codimension in a Riemannian manifold.

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1. Introduction to the problem and review of known facts.

We denote by $M$ the class of the manifolds $M$ such that $M$ is a 2-dimensional compact, oriented, embedded submanifold with boundary, of class $C^2$, in $\mathbb{R}^3$. At each point $x \in M \sim \partial M$, the two principal curvatures $k_1(x), k_2(x)$ are defined. Typical functionals depending on curvatures are

\begin{align}
(1.1) \quad F_0(M) &= \int_M \left\{ 1 + k_1(x)^2 + k_2(x)^2 + (k_1(x) k_2(x))^2 \right\}^{1/2} d\mathcal{H}^2(x) \\
(1.2) \quad \tilde{W}(M) &= \int_M (k_1^2 + k_2^2) d\mathcal{H}^2 \\
(1.2) \quad W(M) &= \int_M H^2 d\mathcal{H}^2
\end{align}

where $\mathcal{H}^2$ is the 2-dimensional Hausdorff measure, and $H(x) = k_1(x) + k_2(x)$ is twice the mean curvature of $M$ at $x$.

More generally, one may consider functionals of the type

\begin{equation}
(1.4) \quad F(M) = \int_M \psi(x, \nu_M(x), A_M(x)) d\mathcal{H}^2
\end{equation}

where $\nu_M(x)$ is a unit normal vector field that orients $M$, and $A_M(x)$ is the second fundamental form of $M$ at $x$. Of course, some assumptions shall have to be made on the function $\psi(x, \nu, A)$, but we will discuss this matter later on.

Now, one may consider the problem of minimizing the functional $F(M)$ subject to various side conditions. For instance, we may consider a boundary value problem as follows:

\begin{equation}
(1.5) \quad \text{given a surface } M_0 \in \mathcal{M}, \text{ minimize } F(M) \text{ among the surfaces } M \in \mathcal{M} \text{ such that} \\
\partial M = \partial M_0 \\
\nu_M(x) = \nu_{M_0}(x) \quad \forall x \in \partial M_0
\end{equation}

Or we may prescribe the topological type:

\begin{equation}
(1.6) \quad \text{given a non-negative integer } g, \text{ minimize } F(M) \text{ among the closed surfaces in } \mathcal{M} \text{ having genus } g.
\end{equation}

Or we may consider problems with constraints:
(1.7) minimize $F(M)$ among the closed surfaces that include some fixed set (or some fixed volume), or which have some fixed area.

Of course, we may also consider several combinations of these problems. Now, why should one be interested in problems like the ones above? Of course, this is partly a matter of taste, however, it seems to us that functionals depending on curvatures are quite natural objects, from an analytical, geometrical and a physical point of view. In fact, such functionals have been considered at various places and times in mathematics. Probably, the first appearance is in the eighteenth century, in the work by James Bernoulli and Euler on the problem of elastica. Let me recall that by this name goes the problem of finding the possible equilibrium shapes taken by a flexible elastic rod, whose end points are displaced at prescribed points in space. One possible model of this problem is that the equilibrium shapes are described by curves $\gamma : [0, 1] \rightarrow \mathbb{R}^3$ minimizing the energy functional

\begin{equation}
E(\gamma) = \int_0^1 k^2 dH^1
\end{equation}

where $k$ is the curvature of $\gamma$, subject to the constraints

\begin{align}
\gamma(0) &= a, \quad \gamma(1) = b \\
\text{lengths of } \gamma &= \text{constant}
\end{align}

For more information on this problem, I refer to the historical paper [Tr] and to the references listed there.

A second appearance is in 1811, again in connection with a problem in elasticity. This time functionals depending on $k_1 + k_2$ and $k_1 k_2$ are proposed by Sophie Germain as reasonable forms of the bending energy for an elastic plate. Again I refer to [Tr] for more information. In more recent times, T.Willmore [Wil], for compact surfaces $M$, suggested the study of immersions $u : M \rightarrow \mathbb{R}^3$ which minimize, or which are at least stationary points, for the functional $W(M)$ in (1.3). This functional is usually called Willmore functional and its stationary points are called Willmore surfaces. Clearly, among surfaces of a same genus, because of the Gauss-Bonnet theorem, the two functionals $W(M)$ and $\tilde{W}(M)$ differ by a constant term and have the same minimizers and the same stationary points.

Since Willmore's paper, much work has been devoted by differential geometers to the study of Willmore surfaces. For more information, I refer to the very readable recent survey [PS] and to the bibliography quoted there.
On the other hand, as far as I know, not very much work has been done on Willmore's functional (or on other functionals depending on curvatures) from the point of view of the Direct Method in the Calculus of Variations, and this is in fact the aspect of the problem that I would like to consider here. More precisely, I am interested in the following question:

(a) what weak compactness properties have the minimizing sequences for functionals of the type \((1.4)\), provided reasonable coerciveness assumptions are made on the function? We expect that there is some minimal space that contains \(C^2\) surfaces, and that contains also all possible limits of minimizing sequences. This space should be the natural ambient where the existence of minimizers can be obtained.

(b) what are the regularity properties of the “surfaces” which are limits of smooth surfaces with uniformly bounded norms of the curvatures? We expect these surfaces to enjoy weak differentiability properties of first and second order, i.e. existence of approximate tangent planes and approximate differentiability of these planes. Moreover we expect to be able to define “curvatures” of our limit surfaces in term of these approximate differentials.

(c) for use in boundary value problems we expect the normal \(\nu\) to our surfaces to have trace properties on 1-dimensional regular slices.

(d) for use in problems with a prescribed topological type, we would like our surfaces to have some weakly defined notion of genus, moreover we have to investigate how this genus behaves with respect to the convergence in which the minimizing sequences converge.

(e) what is a good extension of the classical functional to the larger space where the existence can be obtained? The method of relaxation yields a good extension, but, is there a good representation formula for the relaxed functionals?

(f) what is the further regularity of the minimizers, provided suitable smoothness and “ellipticity” assumptions are made on the integrands?

We shall discuss the questions above in section 2. Here we remark that there are in fact a few papers [Hu1], [Hu2], Si2 where functionals depending on curvatures are studied from the point of view of the Direct Method, and we end this section reporting on some results contained in these papers.

J.Hutchinson [Hu1] develops the notion of curvature varifolds, i.e. generalized surfaces having a weakly defined p-summable second fundamental form. Then he studies minimum problems for functional of the type
(1.4), where \( \Psi(x, \nu, A) \) is continuous, non negative, convex in \( A \) and has some superlinear growth in \( A \). Hutchinson proves a compactness result for curvature varifolds and then obtains an existence theorem for the problem of minimizing \( F(M) \) subject to some boundary conditions. These boundary conditions consist of the prescription of the boundary of \( M \) and of some other condition, that seems however unrelated to the prescription of the normal vector \( M \) on \( \partial M \). Hutchinson obtains also regularity properties [Hu2] for those m-dimensional curvature varifolds whose second fundamental form is \( p \)-summable with \( p > m \). In the papers quoted above, Hutchinson does not deal with problems with prescribed topological type and does not study the regularity of the minimizers.

In L. Simon's paper [Si2], two problems are considered:

(a) minimize the Willmore functional among the embedded compact 2-dimensional surfaces without boundary of a prescribed genus in \( \mathbb{R}^n \)

(b) minimize more general functionals of the type (1.4) in homotopy classes of immersions \( u : M \rightarrow N \), where \( M \) is a 2-dimensional and \( N \) is an n-dimensional riemannian manifold

Here we shall discuss briefly Simon's treatment of problem (a) above. First, Simon takes a minimizing sequence of surfaces \( \{\Sigma_k\}_{k \in \mathbb{N}} \) of a same genus \( g \). Because the Willmore functional is obviously invariant under translation, and because it is also invariant under conformal transformations of the ambient space, the minimizing sequence can be taken such that

\[
0 \in \Sigma_k \quad \text{for all } k
\]

\[
\mathcal{H}^2(\Sigma_k) = 1 \quad \text{for all } k
\]

Then, for each \( k \), Simon considers the positive Radon measure \( \mu_k = \mathcal{H}^2 \llcorner \Sigma_k \), which is defined as

\[
\mu_k(E) = \mathcal{H}^2(E \cap \Sigma_k) \quad \text{for all Borel sets } E \text{ in } \mathbb{R}^n
\]

Now the measures \( \mu_k \) are uniformly bounded (by (1.11)) and, possibly taking a subsequence, one has

\[
\mu_k \rightharpoonup \mu \quad \text{as measures}
\]

for some positive Radon measure \( \mu \). From the fact that the surfaces \( \Sigma_k \) have the same area, and \( W(\Sigma_k) \leq \text{const} \) for all \( k \), Simon obtains a uniform bound
on the diameters of $\Sigma_k$:

$$c^{-1} \leq \text{diam } \Sigma_k \leq c \quad \text{for all } k$$

and, by (1.10), it follows that all the measures $\mu_k$ and also $\mu$, have a same compact support. Next come the difficult tasks of:

- showing that the measure $\mu$ corresponds to a smooth surface $\Sigma_0$ that minimizes $W(\Sigma)$ among the surfaces of the same genus as $\Sigma_0$
- estimating the genus $g_0$

This is done by L.Simon by using arguments both from Differential Geometry and PDE, and using heavily the special features of Willmore's functional in euclidean space.

We emphasize that Simon is able to show the regularity of the measure $\mu$ without passing through weak differentiability properties of $\mu$. The regularity of $\mu$ is obtained because $\mu$ is the limit of a minimizing sequence of $W(M)$ and because of the special structure of the Willmore functional.

Finally, we remark that both J.Hutchinson and L.Simon consider integrands $\Psi(x, \nu, A)$ which are convex functions of the second fundamental form $A$, hence $\Psi$ cannot be a function of the Gauss curvature, for 2-dimensional surfaces, or a function of higher invariants of $A$, for higher dimensional surfaces.

2. Generalized surfaces with curvatures measures and minimization of functional.

Our approach to the problems outlined in (a), ... , (f) of section 1 is founded on the following very simple observation: if $M$ is a $C^2$ surface in $\mathbb{R}^3$, then all the information about the curvatures of $M$ must be contained in the graph

$$G = \{(x, \nu(x)) \in \mathbb{R}^3 \times \mathbb{R}^3 | x \in M\}$$

of the Gauss map $\nu(x) : M \rightarrow S^2 \subset \mathbb{R}^3$ of the surface; in particular, the area of $G$ and the curvatures of $M$ are linked by the simple relation

$$\mathcal{H}^2(G) = \int_M \{1 + (k_1^2 + k_2^2) + (k_1k_2)^2\}^{1/2} d\mathcal{H}^2.$$ 

In fact, the tangent plane $\Pi(x, \nu(x))$ to $G$ at a point $(x, \nu(x))$ is determined by the tangential derivatives of $\nu(x)$ at $x$ (that is by the second fundamental
form of \( M \) at \( x \), and (2.1) follows easily from the area formula [Fe2], [Si1]. From (2.1), we obtain immediately the compactness result contained in theorem 1 below, but first we set some notation.

**DEFINITION 1.** For all surfaces \( M \in \mathcal{M} \) we define the number \( \|M\|_{\text{curv}} \) as follows

\[
\|M\|_{\text{curv}} = \mathcal{H}^2(M) + \int_M (k_1^2 + k_2^2)^{1/2} \, d\mathcal{H}^2 + \int_M |k_1 k_2| \, d\mathcal{H}^2
\]

We remind to the reader that to every oriented \( C^1 \) surface \( M \) is associated a 2-dimensional current \([M]\), i.e. a linear functional on 2-forms, defined by

\[
[M](\varphi) = \int_M \langle \varphi(x), \tau(x) \rangle \, d\mathcal{H}^2(x)
\]

where \( \tau(x) \) is any field of unit tangent 2-vectors to \( M \), as for instance \( \tau = *\nu \).

**THEOREM 1.** Let \( M_0 \in \mathcal{M} \) be fixed and let \( M_j \in \mathcal{M} \) be a sequence of surfaces such that

\[
\text{(2.3,i)} \quad \|M_j\|_{\text{curv}} \leq \text{const} < +\infty
\]

\[
\text{(2.3,ii)} \quad \begin{cases}
\partial M_j = \partial M_0 \\
\nu_j(x) = \nu_0(x) & \forall x \in \partial M_0
\end{cases}
\]

for all \( j \). Then, possibly taking a subsequence, the current \([G_j]\) carried by the graphs \( G_j \) of the Gauss maps \( \nu_j : M_j \to \mathbb{R}^3 \) converge weakly to a current \( \Sigma \) in \( \mathbb{R}^3 \times \mathbb{R}^3 \). The current \( \Sigma \) is a 2-dimensional integer multiplicity rectifiable current and

\[
\partial \Sigma = \partial [G_j] = \partial [G_0] .
\]

**Proof.** Because of (2.3,i) and (2.1), the mass of \([G_j]\) is uniformly bounded. On the other hand, (2.3,ii) implies that \( \partial [G_j] = \partial [G_0] \) for all \( j \), hence the mass of \( \partial [G_j] = \partial [G_j] \) is also uniformly bounded. Then the theorem follows by the well known compactness results for integral currents [Fe2], [Si1], [Whi].

**COROLLARY 1.** Let \( F(M) \) be a functional defined on \( \mathcal{M} \) and assume that

\[
\text{(2.4)} \quad F(M) \geq \|M\|_{\text{curv}} \quad \text{for all} \quad M \in \mathcal{M} .
\]
Then for every minimizing sequence for problem (1.5) one can extract a subsequence $M_j$, such that the corresponding currents $[G_j]$ converge to a current $\Sigma$ as in theorem 1.

**DEFINITION 2.** If (2.4) holds, we say that the functional $F(M)$ is coercive on the curvatures of $M$.

**REMARK.** If $F(M)$ is an integral functional as in (1.4), then the coerciveness condition (2.4) may be ensured by a positive growth condition on the integrand, as

$$(2.5) \quad \psi(x, \nu, A) \geq c_0[1 + |A| + |\det A|].$$

Now that we have a compactness result, we shall proceed in two directions:

(2.7) obtain the existence of minimizers for the problems described in section 1.

(2.8) study the structure and the weak differentiability properties of the currents $\Sigma$ obtained as limits of minimizing sequences; in particular, we would like such $\Sigma$ to be Gauss maps, at least in some weak sense, of surfaces in $\mathbb{R}^3$.

Before pursuing (2.7) and (2.8), we make a few comments. First, the idea of using the graph of the Gauss map of a surface $M$ as the support of a measure in $\mathbb{R}^3 \times S^2$ carrying information about $M$ – and the consequent idea of thinking of measures in $\mathbb{R}^3 \times S^2$ as generalized surfaces – are by no means new. In fact, these are basically L.C. Young ideas [You1], [You2], [You3], and on very similar ideas are based also F.J. Almgren theory of varifolds [Alm] [All] and Yu.G. Reshentnyak’s work [Res] on functions of measures. However, the authors just mentioned were interested in functionals that control only the area of the surfaces, as in Plateau’s problem, hence in functionals that control only part of the area of $G$. Precisely, the projection on $\mathbb{R}^2$ of that area. On the other hand, we are studying functionals that control the derivatives of the normal, hence we may have control of the whole area of $G$, and we have to consider vector valued, rather than just positive, measures on $\mathbb{R}^3 \times S^2$.

Second, the idea of using the current associated to the graph of a map $u: \mathbb{R}^n \to \mathbb{R}$ as a geometrical object carrying information on $Du$, and the idea of considering the graph, instead of the map, as a generalized minimizer of a functional $\int f(Du)$, are also not new. In particular they lie under all the
study of non-parametric minimal surfaces and of functionals of growth one
[Giù], [DM]. Even for vector valued maps $u$ it was well known that functional
of growth one in the minors of the jacobian matrix $Du$ could be seen as
functional on the graph of $u$

However, it seems to be only with the paper [GMS1] that the current
carried by the graph of a map $u : \mathbb{R}^n \to \mathbb{R}^k$ is used to study functionals
$\int f(x, u, Du)$ where $f$ is a convex function in the minors of $Du$, with arbitrary
superlinear growth. The papers [GMS1] is concerned mainly with problems
motivated by non-linear elasticity [Ba], but it contains ideas that may be
useful for many geometric problems, and some of these ideas are used here,
too.

Now we state a few results in the direction outlined in (2.8). For proofs
we refer to [AST]. Let $\Sigma$ be a 2-dimensional integer multiplicity rectifiable
current in $\mathbb{R}_x^2 \times \mathbb{R}_y^2$, and assume that $\Sigma$ is the weak limit of the graphs of the
Gauss map of surfaces $M_j$ having uniformly bounded areas and curvatures,
as in theorem 1. Let $\Sigma$ be carried by the 2-dimensional countably rectifiable
set $R$, and let $\xi$ be a Borel measurable field of unit 2-vectors tangent to $R$.
Denote $p$ the projection $\mathbb{R}_x^2 \times \mathbb{R}_y^2 \to \mathbb{R}_x^2$. Call $P$ the rectifiable set $pR$ and let
$\nu(x)$ be a Borel measurable unit normal field to $P$.

Then we have

**THEOREM 2.** (i) For almost all points $(x, y) \in R$ one has that

(2.12) the 2-dimensional plane in $\mathbb{R}_x^2 \times \mathbb{R}_y^2$, that corresponds to the 2-vector
$\xi(x, y)$ is orthogonal to the vector $(y, 0)$.

(ii) For almost all points $x \in P$, one has

$$p^{-1}(\{x\}) \subset \{(x, \nu(x), (x, -\nu(x))$$

(iii) One has the decomposition

$$R = R^{(0)} \cup R^{(1)} \cup \left( \bigcup_{j=1}^{\infty} R_j \right)$$

where

- $\mathcal{H}^2(R^{(0)}) = 0$
- for all $(x, y) \in R \sim R^{(0)}$ there exists the approximate tangent plane
to $R$ at $(x, y)$
\(- \mathcal{R}^{(1)} = \{(x, y) \in R \sim \mathcal{R}^{(0)} \mid (p \land p)\xi(x, y) = 0\} \)
\(- \mathcal{H}^2(p\mathcal{R}^{(1)}) = 0 \)

(iv) the sets \( R_j \) are mutually disjoint and, for all \( j \), one has

\[ R_j \subset W_j \]

where \( W_j \) is the graph over a \( C^1 \) 2-dimensional oriented manifold \( N_j \subset \mathbb{R}^2 \) of a \( C^1 \) function

\[ f_j : N_j \rightarrow \mathbb{R}^3 \]

(v) \( P \subset P^{(0)} \cup \left( \bigcup_{i=1}^{\infty} N_j \right) \), \( \mathcal{H}^2(P^{(0)}) = 0 \)

(vi) for all \( j \) and for all \( (x, y) \in R_j \), one has

\[ y = f_j(x) = \nu_j(x) \]

where \( \nu_j(x) \) is the continuous unit vector field that orients \( N_j \), moreover the point \( x \) is of density one for \( P \) on \( N_j \)

(vii) for all \( j \) and for all \( x \in pR_j \subset N_j \) the map \( \nu_j \) is approximately differentiable on \( pR_j \).

Theorem 2 above describes quite completely the structure of the sets \( R \) and \( P \) and their differentiability properties of first and second order. From this theorem one can obtain also trace properties for the functions \( \nu_j \); we do not enter the subject and refer to [AST] for more information.

For the current \( \Sigma \), or, if we like, for the rectifiable set \( R \), we may define a second fundamental form, and all the curvatures we want. To this purpose, we write the tangent 2-vector \( \xi(x, y) \) to \( R \) as a combination of base 2-vectors. We take the standard base \( \{e_1, e_2, e_3\} \) in \( \mathbb{R}^3 \) and the base \( \{e_1, e_2, e_3\} \) in \( \mathbb{R}^3_y \).

Then we have

\[ \xi(x, y) = \sum_{r,s=1,2,3} \xi^{(0)}_{rs}(x, y)e_r \land e_s + \sum_{r,\alpha=1,2,3} \xi^{(1)}_{r\alpha}(x, y)e_r \land e_\alpha + \]

\[ + \sum_{\alpha<\beta} \xi^{(2)}_{\alpha\beta}(x, y)e_\alpha \land e_\beta \]

DEFINITION 3. The vector measure \( A = \{A_{r\alpha}\}_{r,\alpha=1,2,3} \), where

\[ A_{r\alpha} = \xi^{(1)}_{r\gamma}(x, y)\tau_{\alpha\gamma}(x, y) \]
and
\[ \tau_{\alpha \gamma}(x, y) = (e_\alpha \wedge e_\gamma, \tau(x, y)) = (e_\alpha \wedge e_\gamma, *y) \]
is said to be the second fundamental form of $Z$.

The measure
\[ H = \sum_{\alpha=1}^{3} A_{\alpha} \tau_{\alpha} \]
is said to be the mean curvature of $Z$.

The measure
\[ K = (\xi_{12}^{(2)}(x, y)y_3 + \xi_{23}^{(2)}(x, y)y_1 + \xi_{31}^{(2)}(x, y)y_2) \mathcal{H}^2 \subset R \]
is said to be the Gauss curvature of $\Sigma$.

We remark that if $\Sigma$ is the graph of Gauss map of a $C^2$ surface, then the projections on $M$ of the measures $A_{\alpha}, H, K$, are just the integrals on $M$ of the classical second fundamental form, mean curvature, Gauss curvature. Moreover, if one has $\Sigma = \lim_{j \to a} [G_j]$ and
\[ \int_{M_j} |A_j|^p d\mathcal{H}^2 \leq \text{const} < +\infty \quad \forall j \]
for some fixed $p > 1$, then the measure
\[ V = |\xi^{(0)}| \mathcal{H}^2 \subset R \]
in $\mathbb{R}^2 \times S^2$ is a curvature varifold in the sense of J. Hutchinson [Hu1], and the density $\frac{dA}{dV}$ coincides to the second fundamental form of $V$ in Hutchinson’s sense.

Motivated by theorem 2 and definition 3 above, we should like to say that the current $\Sigma$, or the rectifiable set $R$, or may be the pair $\{R, P\}$, is a (generalized) surface with curvature measures. Now, the notion of “curvature measures”, for certain subsets of euclidean space, and for suitable metric spaces, has been introduced before by various authors, as in [Ale], [Fel], [Pog], in connection with volume of tubes, integral geometry, real Monge-Ampère equations. Although our approach is different, and motivated by the Calculus of Variations, it is possible that our point of view become useful also in the just mentioned subjects, and specially in the study of prescribed curvature problems.

Now we introduce a notion of genus for our surfaces with curvature measures. We consider the projection $q : \mathbb{R}^3_x \times \mathbb{R}^3_y \to \mathbb{R}^3_y$. If $M$ is a regular
surface in $\mathbb{R}^3$ and $G \subset \mathbb{R}^3_\times S^2 \subset \mathbb{R}^3_\times \mathbb{R}^3_\times$ is the graph of the Gauss map of $M$, then $Q = q_1[G]$ is a 2-dimensional integer multiplicity current in $S^2$. Moreover, if $\partial M = 0$, one has also $\partial Q = 0$ and, by the constancy theorem [Fe2], it follows that

$$Q = \gamma [S^2]$$

for some integer $\gamma$. In such a case, it is easy to see that

$$4\pi \gamma = \int_M K(x) dH^2$$

so that $2\gamma$ is just the Euler-Poincaré characteristic $\chi(M)$ of $M$. On the other hand, for any 2-dimensional integer multiplicity current $\Sigma$ in $\mathbb{R}^3_\times S^2$ with $\partial \Sigma = 0$ one has

$$q_1 \Sigma = \gamma [S^2]$$

for some integer $\gamma$, and we decide to say that $2\gamma$ is the Euler-Poincaré characteristic of $\Sigma$, denoted $\chi(\Sigma)$.

If $M_j, G_j, \Sigma$ are as in theorem 1, for all $j$ one has $\partial M_j = 0$, $\chi(M_j) = \gamma_0$, and all the $M_j$ are supported in a same compact set then one has also

$$q_1 \Sigma = \gamma_0 [S^2]$$

so that our notion of Euler-Poincaré characteristic is continuous with respect to the convergence of the minimizing sequences (with a same support). This is useful when considering problems with prescribed topological type. It would be desirable to have some combinatorial invariant defined on the set $R$, related to $\chi(\Sigma)$ by a Gauss-Bonnet type theorem.

Finally we pass to discuss the problem of the existence of minimizers for functionals depending on curvatures. The best approach should be by relaxation. For example, consider the Dirichlet problem (1.5), where $F(M)$ is some functional coercive on the curvatures. We may consider the functional $\tilde{F}$ defined on the space $I$ of all 2-dimensional integer multiplicity rectifiable currents in $\mathbb{R}^3_\times \mathbb{R}^3_\times$ as

$$\tilde{F}(T) = \begin{cases} F(M) & \text{if } T = G_M \text{ and } \partial M = \partial M_0, \nu_M = \nu_0 \text{ on } \partial M_0, \\ +\infty & \text{otherwise} \end{cases}$$

then we may consider the functional $\overline{F} = I \to \mathbb{R}$ defined as

$$\overline{F} = \sup\{\Phi : I \to \mathbb{R} | \Phi \leq \tilde{F}, \Phi \} \text{ l.s.c. with respect}$$
to the weak convergence of currents.

The functional $\overline{F}$ is the relaxed functional of $\tilde{F}$ and is the natural object to study [ET], [Bu]. In fact, if $F$ is coercive, then $\overline{F}$ has a minimum point; moreover, the minimum value of $\overline{F}$ coincides with the infimum of $F$. However, the approach becomes really useful when one has some simple characterization of the set

$$\{T \in I | \overline{F}(T) < +\infty\}$$

and some integral representation formula for $\overline{F}$. Unfortunately such characterizations are still missing. The corresponding questions are open also for the polyconvex functionals studied in [GMS1].

Another approach is to consider the space

$$\text{curv} (M_0) = \{T \in I | \partial T = \partial [G_0], T \text{ satisfies (2.12)}\}$$

and to consider functionals $F : \text{curv} (M_0) \rightarrow \mathbb{R}$ of the type

$$(2.13) \quad F(T) = \int \Psi(\overline{T}(x,y)) d||T||$$

where $\psi$ is convex. For instance, if we take $\psi(\alpha) = |\alpha|$ then for all regular surfaces $M \in \mathcal{M}$ one has that the functional in (2.13) coincides with the functional $F_0$ in (1.1). It is easy [AST] to obtain existence results for such functionals, however, in general, the minima of these functional in $\text{curv}(M_0)$ need not to coincide to the infima in $\mathcal{M}$ of the same functionals.

We end this long talk with two remarks.

The first remark is about regularity. It is expected that the 2-dimensional currents $\Sigma$ in $\mathbb{R}_x^3 \times \mathbb{R}_y^3$, which are obtained as minimizers, are indeed the Gauss map of good surfaces, at least under reasonable assumptions. To show this is the question of the regularity. This question is in fact a collection of different problems, i.e.: partial regularity and complete regularity of the current $\Sigma$ upstairs, the same for the projection $p_1 \Sigma$, occurrence of singular points in the projection and others. The study of partial regularity for minimizers of polyconvex functional, using the currents carried by the graph of the map, has been started in [GMS2].

The second remark is about non completely coercive functionals. In fact, one might consider functionals that do not control all the curvatures of $M$, so that they do not control the whole area of $G$. For instance, one may consider the functional

$$F^{(0,1)}(M) = \mathcal{H}^2(M) + \int_M \sqrt{k_1^2 + k_2^2} d\mathcal{H}^2$$
which controls the area of the current

\[ [G]^{(0,1)} = \left\{ \sum_{r,s=1,2,3} \xi_r^{(0)}(x,y)e_r \wedge e_s + \right. \\
\left. + \sum_{r,\alpha=1,2,3} \xi_r^{(1)}(x,y)e_r \wedge e_\alpha \right\} \mathcal{H}^2 \subseteq G \]

but does not control the area of \([G]\). Now, if \(M_j\) is a sequence of surfaces such that the numbers \(F^{(0,1)}(M_j)\) are bounded, then, possibly taking a subsequence, one has

\[ [G_j]^{(0,1)} \rightharpoonup \Sigma \]  

(weakly as measures)

for some current \(\Sigma\) representable by integration. While \(\Sigma\) is still a candidate to be a minimizer, it is no longer to be expected that \(\Sigma\) is the \((0,1)\) part of some rectifiable current. However \(\Sigma\) must have some special properties. This is similar to what happens for polyconvex functional \(\int f(x,u,Du)\) which are only partially coercive. The study of the vector measures which are limits of graphs of functions \(u : \mathbb{R}^n \to \mathbb{R}^k\) with uniformly bounded partial areas has been started in [An] only for the case when the norm

\[ \|u\|_{BV} = \int |u| + \int |Du| \]

is bounded. This study has been useful to understand better certain problems of relaxation in spaces of vector valued \(BV\) functions [AMT]. Similar results are expected for functionals depending on curvatures.

REFERENCES


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