## Wilfried Sieg

## PROVABLY RECURSIVE FUNCTIONALS OF THEORIES WITH KOENIG'S LEMMA

## INTRODUCTION

The problems I want to consider and the results I want to discuss are PRIMA FACIE of a very traditional proof theoretic sort: they concern the reduction of subsystems of second and higher order arithmetic to constructively unproblematic theories. The arguments for these results seem also to be of a traditional proof theoretic sort: they use formal and semiformal sequent calculi and exploit the fact that these calculi allow the elimination of cuts. There are, however, fascinating and significant twists; namely, (i) the constructively unproblematic theories are all fragments of elementary number theory, (ii) the subsystems are nevertheless sufficiently strong to serve as formal frames for large parts of mathematical analysis and algebra, and (iii) the arguments use systematically "derivations as computations" through a form of Herbrand's theorem. The latter slogan can be taken as the theme of the paper. [1]

## A. PROBLEMS \& RESULTS.

Every finitely branching, but infinite tree has an infinite branch. That is Koenig's lemma, a most useful tool for mathematical and meta-mathematical investigations. The Heine-Borel covering theorem and Goedel's completeness theorem, to mention just two examples, can be proved using Koenig's lemma over a very weak subsystem of second order arithmetic. Fixing on a second-order framework, the principle can be stated in a variety of ways of strikingly different strengths.

A1. FORMULATIONS OF KOENIG's LEMMA. There is, first of all, the schematic formulation, where the tree is given by any second order formula. This formulation - as was pointed out by Howard - is equivalent to the full comprehension principle. Matters begin to get more delicate when we consider the abstract formulation $\boldsymbol{K L}$ :

$$
(\forall f)[T(f) \&(V x)(\exists y)(l b(y)=x \& f(y)=0) \rightarrow(\exists g)(\forall x) f(\bar{g}(x))=0]
$$

Here $T(f)$ abbreviates that $\{x \mid f(x)=0\}$ forms a finitely branching tree i.e.

$$
\begin{aligned}
(\forall x, y)\left[f\left(x^{*} y\right)=0\right. & \rightarrow f(x)=0] \&(\forall x)(\exists z)(\forall y)\left[f\left(x^{*}<y>\right)=0\right. \\
& \rightarrow(y<z \vee y=z)]
\end{aligned}
$$

In the presence of suitably strong set existence principles, e.g. the full arithmetical comprehension principle, $K L$ is equivalent to a bounded version $B K L$, in which a bound for the size of the immediate descendants of a node is given by a function.
$(V \bar{f}, g)[T(f, g) \&(\forall x)(\exists y)(l b(y)=x \& f(y)=0) \rightarrow(\exists b)(\forall x) f(\bar{b}(x))=0]$
Here $T(f, g)$ abbreviates

$$
\begin{aligned}
(\forall x, y)\left[f \left(x^{*} y=0\right.\right. & \rightarrow f(x)=0] \&(\forall x, y)\left[f\left(x^{*}<y>\right)=0\right. \\
& \rightarrow(y<g(x) \vee y=g(x))]
\end{aligned}
$$

Indecd, the bounding function can be taken to be the constant function with value 1 . Thus we are looking at trees of $0-1$-sequences. This special version of BKL is called WEAK KOENIG'S LEMMA WKL. What is the relative strength of these principles? To answer this question let ( $B T$ ) be the second order version of primitive recursive arithmetic ( $P R A$ ) together with the comprehension principle for quantifier-free formulas. Friedman observed in an unpublished paper (written in 1969) that over ( $B T+\Sigma_{1}^{0}-A C$ ) Koenig's lemma $K L$ is equivalent to the full arithmetical comprehension principle $\Pi_{\infty}^{0}-C A$. Thus we have two immediate results (the systems and principles
are formulated precisely in $B 1$ ):
(1) $\left(B T+\Sigma_{1}^{0}-A C+K L\right)$ is equivalent to $\left(\Pi_{\infty}^{0}-C A\right) \wedge$ and thus conservative over elementary number theory $(Z)$.
(2) $\left(B T+\Sigma_{1}^{0}-A C+\Pi_{\infty}^{1}-I A+K L\right)$ is equivalent to $\left(\Pi_{\infty}^{0}-C A\right)$ and thus NOT conservative over ( $Z$ ).

That $W K L$ is weaker than $K L$ is witnessed by the following result due to Kreisel; see [Kreisel e.a.]:
(3)

$$
\begin{equation*}
(K):=\left(B T+\Sigma_{1}^{0}-A C+\Pi_{\infty}^{1}-I A+W K L\right) \text { is conservative over }(Z) . \tag{3}
\end{equation*}
$$

What is possibly the interest of such theories? One reason for strong interest is simple: the bulk of classical analysis can be developed in these or other, related conservative extensions of $(Z)$. That is the outcome of work by Takeuti, Feferman, and Friedman. [2] Their investigations lie in a rather long tradition of persisten efforts to pursue mathematical analysis by "restricted" means. The work of constructivists like Kronecker, Brouwer, and Bishop is clearly part of that tradition. Predicatively inclined mathematicians contributed also significantly; indeed, Weyl's Das Kontinuum is an early landmark in this kind of research. Detailed investigation with sharp logical - mathematical focus, finally, were prodded by the foundational concerns of the Hilbert school. Hilbert, in the early twenties, showed in lectures how to develop classical analysis straightforwardly in a theory that is equivalent to full second order arithmetic. When the consistency problem even for $(Z)$ turned out to be much more recalcitrant than had been expected, it was very natural to be concerned with subsystems of second order arithmetic in two complementary ways: to prove their consistency by constructive means AND to establish their significance by developing substantial parts of mathematical analysis in them. That the latter can be done already in conservative extensions of $(Z)$ is, prima facie, surprising and satisfying.

A2. WEAK SUBSYSTEMS. It is now utterly trivial to observe that such a development can be carried out partially in ad-hoc subsystems that are proof theoretically equivalent to. proper fragments of $(Z)$. The interesting question is this:

Can such a development be given for cobesive parts of mathematical practice in fixed subsystems that are conservative over informative fragments?

Friedman's theory $\left(W K L_{0}\right)$ is a weak theory and actually adequate for a good deal of ordinary mathematical practice in analysis and algebra. The latter claim has been substantiated by detailed work of Friedman and Simpson. [3] For example, $\left(W K L_{0}\right)$ proves that every continuous, real-valued function on the unit interval is uniformly continuous, has a supremum and actually attains it; the Heine-Borel theorem and the Cauchy-Peano existence theorem for ordinary differential equations can be established in this theory. To mention one example from algebra, $\left(W K L_{0}\right)$ proves the existence of prime ideals in countable commutative rings. Friedman showed that the theory is conservative over primitive recursive arithmetic (PRA) for $\Pi_{2}^{0}$-sentences. (The system ( $F$ ) considered here is equivalent to ( $W K L_{0}$ ), but more amenable to proof theoretic investigation.)
(4) $(F):=\left(B T+\Sigma_{1}^{0}-A C+\Sigma_{1}^{0}-I A+W K L\right)$ is conservative over (PRA) for $\Pi_{2}^{0}$ - sentences.

This theorem was strengthened by Harrington as follows:
(5) ( $F$ ) is conservative over $\left(\Sigma_{i}^{0}-I A\right)+$ for $\Pi_{1}^{1}-$ sentences.

Notice that (4) follows immediately, as $\left(\Sigma_{1}^{0}-I A\right)+$ is conservative over (PRA) for $\Pi_{2}^{0}$ - sentences; that is a result of Parsons's [4] - I was very much interested in Friedman's result and a related one of Minc's, for reasons I will spell out in a minute; so I gave an alternative, elementary proof theoretic argument for their results. (Friedman's proof had been model theoretic; Minc's argument was not quite correct and concerned with a somewhat more restricted theory [5]). Clearly, looking at the results (3) - (5) one may wonder whether they captured isolated phenomena or whether they were aspects of quite general connections. The quite diverse model theoretic arguments did not give any indication; my proof theoretic argument, however, can be extended to establish a general theorem that implies the earlier results as special cases. In its formulation II $_{n}$ denotes the class of prenex formulas in the language of analysis whose prefix is of length at most $n$ and starts with an existential quantifier.

MAIN THEOREM. $\left(F_{n}\right):=\left(B T+\Sigma_{1}^{0}-A C+I_{n}-I A+W K L\right)$ is conservative over $\left(\Sigma_{n}^{0}-I A\right)+$ for $\Pi_{1}^{1}$-sentences, $\mathrm{n}>0$. The same holds
for the finite type extension $\left(F_{n}^{\omega}\right)$ :
It is the proof of this theorem I want to present in outline, i.e. I will focus on second order arithmetic; but let me emphasize that I consider the extension to finite types of great significance. After all, finite type theories allow a more direct formalization of mathematical practice: there is no need to work with codes of higher type objects. But before sketching arguments I give two reasons for interest in these weak theories.

A3. FOUNDATIONAL \& COMPUTATIONAL INTEREST. The foundational interest of Friedman's conservation result - assuming it is established by elementary means - is clear: it yields a direct finitist justification of that part of mathematics that can be systematically developed in ( $F$ ). The second point of interest has been emphasized by Kreisel in such contexts since the fifties; it is brought out best by a particular answer to the question

What more than its truth do we know, if we bave proved a theorem in a weak formal theory?

The answer is given for $\Pi_{2}^{0}$ theorems; namely, if $(\forall x)(\exists y) R x y$ is provable in $T$, then there is a recursive function f , such that $(\forall x) R x f(x)$ is also provable in $T$. As a matter of fact, if $T$ is weak, then $f$ may be in a computationally significant class of recursive functions. Taking Friedman's ( $F$ ) as $T$, all the provably recursive functions are actually primitive recursive. Obviously, further refinements are called for, if one wants to extract systematically the algorithmic content of derivations of $\Pi_{2}^{0}$ - theorems. One such refinement was established by me [Sieg, 1985] for theories in which the Kal-mar-elementary functions take the place of the primitive recursive ones. (A closely related result is found in [Simpson and Smith]).

THEOREM. $\left(E T+\Sigma_{1}^{0}-A C+W K L\right)$ is conservative over (KEA) for $\Pi_{2}^{0}-$ sentences.

The real challenge consists in extending the mathematical as well as the metamathematical work to theories whose provably recursive functions are contained in a small class of subelementary, "feasible" functions. [6] I am convinced that proof theory can be fruitfully applied here; i.e. a proof theory that pays attention to details of the coding of syntactic objects and to the complexity of syntactic operations.

## B. LANGUAGES \& CALCULI.

I am going to describe now the formalisms to be used and sketch the ideas underlying my proof of Friedman's conservation result: the main theorem is established by suitably extending this sketch. Let me emphasize once more that I am restricting myself to second order theories throughout.

B1. SECOND ORDER ARITHMETIC; FINITARY SEQUENT CALCULI. The language $L(P R A)$ of primitive recursive airthmetic is contained in the language of every theory I am going to consider. It is expanded to the language $L(Z)$ of number theory by adjoining number quantifiers. Parameters for one-place number theoretic functions may be added; the resulting languages are denoted by $L(P R A)+$ and $L(Z)+$. The language $L+$ of analysis is $L(Z)+$ extended by function quantifiers. - Formulas are built up from atomic and negated atomic formulas using $\&, V$ as logical connectives and, if appropriate, the quantifiers, $V, \exists$. Negation for complex formulas is defined; that can be done, as only classical theories are considered. Conditionals and biconditionals are also defined in the usual way.

Theories formulated in these languages will always contain particular base theories. In the case of the language $L(P R A)$ it is simply primitive recursive arithmetic; namely, the axiom $\rightarrow\left(0^{\prime}=0\right)$, the defining equations for all primitive recursive functions, and the induction principle for (quantifierfree) formulas. The latter is given, equivalently, either as the axiom schema

$$
F 0 \&(V x<a)\left(F x \rightarrow F x^{\prime}\right) \rightarrow F a
$$

or as the rule


When considering the language $L(Z)$ we denote the base theory by ( $Q F-I A$ ); it contains the axioms of ( $P R A$ ), except that the induction principle $I A$ (for quantifier-free formulas) is given in the usual form

$$
F 0 \&(V x)\left(F x \rightarrow F x^{\prime}\right) \rightarrow F a .
$$

( $Q F-I A$ ) is, as can be seen quite readily, conservative over $(P R A) \cdot\left(\Sigma_{n}^{0}-\right.$ $I A$ ) is the theory containing the induction principle for $\Sigma_{n}^{0}$-formulas; ( $\Sigma_{\infty}^{0}-I A$ ) is then evidently full elementary number theory ( $Z$ ). In the case of $L+$, the base theory is called ( $B T$ ); it contains the axiom of ( $Q F-I A$ ), possibly with function parameters in the defining equations of primitive recursive function (al)s and the formulas used in It also contains the schema for the explicit definition of functions $E D$ :

$$
(\nexists f)(\forall x) f(x)=t_{a}[x] \text {. }
$$

This clearly allows to establish each instance of the comprehension principle for quantifier-free formulas; the other function-existence principles to be considered are $W K L$ and the axiom of choice $A C$ in the form

$$
(V x)(\exists y) F x y \rightarrow(\exists g)(\forall x) F x g(x),
$$

where $F$ may contain number - and function - parameters. With this, the languages and principles for all the theories are described.

The logical calculi underlying these theories are always sequent calculi in Tait's form. Their fundamental properties are well-known: invertibility of logical rules for $\&, V$, normalizability, and the subformula property of normal derivations. (For a very nice presentation, see [Schwichtenberg, 1977].)

B2. SKETCH OF PROOF THEORETIC ARGUMENT (for Friedmans's conservation result). The fundamental properties of sequent calculi are exploited to prove the conservation result of Friedman's: $(F)$ is conservative over (PRA) for $\Pi_{2}^{0}$-sentences. I want to sketch this argument to motivate the further considerations. As ( $B T$ ) is trivially conservative for $\Pi_{2}^{0}$-sentences over (PRA), we just have to show that every $\Pi_{2}^{0}$-sentence provable in $\left(B T+\Sigma_{1}^{0}-I A+\Sigma_{1}^{0}-A C+W K L\right)$ is already provable in ( $B T$ ). So consider, first of all, $(B T)$ extended by $\Sigma_{1}^{0}-A C$, or equivalently $Q F-A C$, and $W K L$. That this is indeed a conservative extension of ( $B T$ ) for $\Pi_{\mathbf{2}}^{\mathbf{0}}$ - sentences follows immediately from two lemmata.
$Q F-A C$ - ELIMINATION. Let $\Delta$ contain only $A$-formulas; if $D$ is a normal derivation of $\Delta[\neg Q F-A C]$, then there is a normal derivation $E$ of $\Delta[\neg Q F-\lambda A]$.

Here $\Delta[\ldots]$ is the sequent consisting of $\Delta$ and instances (and instantiations) of the schema ... . $Q F-\lambda A$ is the purely universal version of $E D$; namely,

$$
(\forall x) \lambda y .\left(t_{a}[y]\right)(x)=t_{a}[x] .
$$

$\exists$-formulas are purely existential. The second lemma concerns the elimination of $W K L$.

WKL-Elimination. Let $\Delta$ contain only $\exists$-formulas; if $D$ is a normal derivation of $\Delta[\neg W K L]$, then there is a normal derivation $E$ of $\Delta$ $[\neg Q F-\lambda A]$.

At the heart of matters and also the reason for the condition that $\Delta$ must contain only $\exists$-formulas is the restricted possibility of $\exists$-inversion, a form of Herbrand's theorem.
$\exists$ - Inversion. Let $\Delta$ contain only $\quad \exists$-formulas and let Fa be quantifierfree; if $D$ is a normal derivation of $\Delta,(\exists x) F x$, then there is a finite sequence of terms $t_{1}, \ldots, t_{n}$ and a normal derivation $E$ of $\Delta, F t_{1}, \ldots$, $F t_{n}$.

The proof of the $\exists$-inversion lemma proceeds inductively on the length of normal derivations. So do the arguments for the elimination lemmata, though they contain crucial steps in which a negation of the principle at hand is being analyzed. More precisely, let us consider first $\neg Q F-A C$; i.e. a normal derivation of

$$
\Delta[\neg Q F-A C],(\forall x)(\exists y) F x y \&(\forall g)(\exists x) \neg F x g(x),
$$

where the indicated instance of $\neg Q F-A C$ has been introduced in the last step of the derivation. By $\&-$ and $\quad V$-inversion there are (shorter) derivations of

$$
\Delta[\neg Q F-A C],(\exists y) F c y \text { and } \Delta[\neg Q F-A C],(\exists x) \neg F x u(x),
$$

where $c$ and $u$ are number - and function-parameters. By induction hypothesis there are derivations $D 1$ and $D 2$ of

$$
\Delta[\neg Q F-\lambda A],(\exists y) F c y \text { and } \Delta[\neg Q F-\lambda A],(\exists x) \neg F x u(x) .
$$

Exploiting D1 via $\exists$-inversion as a computation, one can define with $Q F-\lambda A$ a particular choice-function $b$, such that

$$
\Delta[\neg Q F-\lambda A], F c b(c)
$$

has a normal derivation. Analogously, one obtains from $D_{2}$ a term $t$ and a normal derivation of

$$
\Delta[\neg Q F-\lambda A], \neg F t u(t)
$$

Making appropriate substitutions there are normal derivations of

$$
\Delta[\neg Q F-\lambda A], F t b(t) \text { and } \Delta[\neg Q F-\lambda A], \neg F t b(t)
$$

and thus of $\Delta[\neg Q F-\lambda A] .-$ In a similar setting one exploits "the provability" of $\neg W K L$, i.e. of

$$
T(b) \&(\forall x)(\exists y)(l b(y)=x \& b(y)=0) \&(\forall g)(\nexists x) \neg b(\bar{g}(x))=0
$$

to obtain a contradiction between the first two conjuncts (that guarantee the existence of arbitrarily long branches in the binary three b) and the third conjunct (that expresses the well-foundedness of b). Here, incidentally, one makes use of a majorization technique of [Howard, 1974].

But how can we deal with the $\Sigma_{1}^{0}$-induction scheme? For that one appeals to a fact established essentially by Parsons. I want to formulate it as follows:

LEMMA. Let $\Delta$ consist of axioms of $(B T)$ and let $F$ be a $\Pi_{2}^{0}$-sentence; if $\neg \Delta\left[\neg \Sigma_{1}^{0}-I A\right], F$ has a normal derivation $D$, then there is a normal derivation $E$ of $\neg \Delta^{*}, F$, where $\Delta^{*}$ contains $\Delta$ and possibly additional $B T$-axioms.

With these elimination lemmata it is quite easy to establish the conservation result by induction on (the lenght of) normal derivations in $(F)$. The critical questions are these: can one extend the above considerations to the ( $F_{n}$ ) with $\quad n>1$ ? and, can one enlarge the class of conserved sentences from $\Pi_{2}^{0}$ - to $\Pi_{1}^{1-}$ sentences? The second question has a simple answer: "Yes, by means of Herbrand's theorem!" The first question has also a simple answer: "No!" The obstacle is due to the restricted $\exists$-Inversion together with the impossibility of reducing the stronger induction principles to the quantifierfree one. Posing the problem in this way suggests a classical solution: ELIMINATE THE INDUCTION PRINCIPLE BY THE $\omega$-RULE and PROVE SUITABLE VERSIONS OF I -INVERSION AND THE ELIMINATION LEMMATA FOR THE INFINITARY SEMI-FORMAL SYSTEM. This strategy leads indeed to the desired goal. (Notice that all references to well-orderings are references to segments of the standard well-ordering of type $\epsilon_{0}$.)

B3. SEMI-FORMAL SYSTEM; EMBEDDING. The infinitary system ( $B T_{\infty}$ ), into which the $\left(F_{n}\right)$ can be embedded, has not only infinitary derivations, but also infinitary terms. The latter are necessary to obtain an appropriate form of the $\exists$-inversion lemma- [7] Thus the language of $\left(B T_{\infty}\right)$ is that of (BT) extended by terms $\left\langle t_{i}\right\rangle$, where the subscript $i$ is always assumed to range over $N$.

Definition. (number- and function-terms with depth)
1.(i) All individual constants and parameters are N -terms of depth 1; (ii) all function constants and parameters are F-terms of depth 1.
2. (application)

If $t, t_{1}, \ldots, t_{n}$ are terms of the appropriate kinds and of depth $|t|,\left|t_{i}\right|$, $\ldots,\left|t_{n}\right|$, then $t\left(t_{1}, \ldots, t_{n}\right)$ is an $N$-term of depth $\max \left(|t|,\left|t_{1}\right|, \ldots\right.$, . $\left|t_{n}\right| \mid+1$.
3. ( $\lambda$-abstraction)

If $t$ is an $N$-term of depth $|\mathrm{t}|$, then $\lambda .(t)$ is an $F$-term of depth $|t|+1$.
4. (sequencing)

If $t_{i}, i \in N$, form a sequence of $N$ - terms and the $t_{i}$ are of depth $\left|t_{i}\right|$, then $\left\langle t_{i}\right\rangle$ is a (unary) $F$-term of depth $\sup \left(\left|t_{i}\right|+1\right)$.

The calculus for $\left(B T_{\infty}\right)$ is obtained from the finitary one by adding the $\omega$-rule in the form

$$
\frac{\Gamma, \Delta(\underline{n})}{\Gamma, \Delta(\underline{a})} \text { for all } \underline{n} \in N^{k}
$$

The axioms are those of [Schwichtenberg, 1977], admitting function parameters. In addition we have $\lambda$-conversion, i.e. $Q F-\lambda A$, and $<>$-conversion in the form

$$
<t_{i}>(n)=t_{n} .
$$

The ordinal theoretic measures of complexity - $\lg (E), c r(E), t d(F), t d(E)$ are defined as usual. - The ( $F_{n}$ ) can be embedded into ( $B T_{\infty}$ ) in such a way that the cutrank of the infinitary derivation is less than or equal to $\dot{n}+1$, i.e. it is determined solely by the complexity of the formulas in the induction schema.

EMBEDDING LEMMA. Let $\Gamma$ be any set of formulas, let $\Delta$ contain only $B T$-axioms (but no instances of $Q F-I A$ ), and let $F$ be an arbitrary formula; if $D$ is a finitary normal derivation of

$$
\Gamma, \neg \Delta\left[\neg Q F-A C, \neg W K L, \neg \mathbb{T}_{n}-I A\right], F,
$$

then there is an infinitary derivation $E$ in $\left(B T_{\infty}\right)$ of

$$
\Gamma[\neg Q F-A C, \quad W K L], F .
$$

Furthermore we have: $\lg (E)<\omega^{2}, \operatorname{cr}(E)<n+2$, and $t d(E)<\omega$.
The cut-elimination theorem for $\left(B T_{\infty}\right)$ can be established as usual. But here I am only interested in transforming special derivations into quasinormal ones. (A derivaton is called quasi-normal if its cut-rank is at most

1 ; i.e. it is either normal or its cut formulas are atomic.)

QUASI-NORMALIZATION. Let $n>0$ and let $D$ be a $B T_{\infty}$-derivation of $\Gamma$ with $\lg (D) \leqslant \alpha<\omega^{2}, t d(D)<k<\omega$, and $\operatorname{cr}(D)<n+2$; then there is a quasi-normal derivation $E$ of $\Gamma$ such that $\lg (E) \leqslant 2_{n}^{\alpha}<\omega_{n}^{\omega}$ and $t d(E) \leqslant 2_{n}^{k}<\omega$. [8]

That concludes the elementary considerations preparing the ground for the crucial elimination lemmata. They will allow us to remove instances of the axiom of choice and Koenig's lemma from infinitary derivations - WITHOUT increasing the length and term-depth of the derivations essentially.

## C. INVERTING \& ELIMINATING.

As before, the fundamental fact is a form of Herbrand's theorem - adapted to the infinitary context. I will formulate that first and then the elimination lemmata.

C1. LEMMATA. The $\exists$-inversion lemma is a straightforward generalization of that for the finitary calculus. Clearly, one has to provide suitable bounds for the lenght and termdepth of derivations.
$\exists$-INVERSION LEMMA. Let $\Delta$ contain only $\exists$-formulas and let Fbaf be quantifier-free; if $D$ is a quasi-normal derivation of $\Delta,\left(\Psi_{x}\right)$ Fxaf with $\lg (D) \leqslant \alpha$ and $t d(D) \leqslant \beta$ then there is an $N$-term $t$ and a quasinormal derivation $E$ of $\Delta$, Ftaf, such that $\lg (E) \leqslant k(\alpha+1)$ and $t d(E) \leqslant \beta+k(\alpha+1)$. ( $k$ is a fixed natural number determined from the proof.)

The proof proceeds by induction on the length of $D$ and does not hide any surprises; when analyzing the $\omega$-rule one exploits the possibility of forming infinite terms. For the proof of the "substantive" elimination lemmata a more specialized lemma is needed, for the very technical reason of keeping the bound on the termdepth "in bounds".

COROLLARY (of the proof). Let $\Delta$ contain only $\exists$-formulas and let Fbaf be quantifier-free; if $D$ is a quasi-normal derivation of $\Delta$, ( $\mathcal{\Psi x}$ ) Fxaf with $\omega \leqslant l g(D) \leqslant \alpha, t d(D) \leqslant \beta$, and $\exists$-inferences to $\Gamma,(\exists x) G x$ only from $\Gamma$, Gs with $t d(s) \leqslant l<\omega$; then there is an $N$-term $t$ and a quasi-normal derivation $E$ of $\Delta$, Ftaf with $\lg (E) \leqslant k(\alpha+1), t d(t) \leqslant$ $\leqslant k(\alpha+1)$, and $t d(E) \leqslant \max (\beta, k(\alpha+1))$. ( $k$ is a fixed natural number at least as great as $l$ and is determined from the proof.)

The crucial ideas for eliminating the axiom of choice and weak Koenig's lemma from infinitary derivations are similar to those needed in the finitary case; one just has to keep track of the ordinal complexity of derivations and terms (and extend, in particular, the majorization techniques to the present context).
$Q F-A C$ - ELIMINATION. Let $\Delta$ contain only $\exists$-formulas; if $D$ is a quasi-normal derivation of $\Delta[\neg Q F-A C]$ with $\lg (D) \leqslant \alpha, t d(D) \leqslant \beta$, and $\exists$-inferences to $\Gamma,(\exists x) G x$ only from $\Gamma$,Gs with $t d(s) \leqslant l<\omega$; then there is a quasi-normal derivation $E$ of $\Delta$ with $\lg (E) \leqslant \omega \cdot \alpha$ and $t d(E) \leqslant \max \left(\beta, \omega \cdot \alpha^{2}\right)$. ( $E$ still satisfies the side-condition on $\exists$-inferences).

The formulation of the $W K L$-elimination lemma is completely analogous to that for the elimination of the quantifier-free axiom of choice.

WKL-ELIMINATION. Let $\Delta$ contain only $\exists$-formulas; if $D$ is a quasinormal derivation of $\Delta[\neg W K L]$ with $\lg (D) \leqslant \alpha, \operatorname{td}(D) \leqslant \beta$, and $\exists$-inferences to $\Gamma,\left(\exists_{x}\right) G x$ only from $\Gamma$, Gs with $t d(s) \leqslant l<\omega$; then there is a quasi-normal derivation $E$ of $\Delta$ with $\lg (E) \leqslant \omega \cdot \alpha$ and $t d(E) \leqslant \max \left(\beta, \omega \cdot \alpha^{2}\right)$. ( $E$ still satisfies the side-conditon on $\mathcal{A}$-inferences.)

C2. ARGUMENT for the main theorem. All the ingredients for establishing the main theorem have been presented now; they just have to be brought together. Recall that we want to show for any $\Pi_{1}^{1}$-sencence $G$, provability in $\left(F_{n}\right)$ implies provability in $\left(\Sigma_{n}^{0}-I A\right)+$. Assume, without loss of generality, that $G$ is an arithmetic formula containing possibly function parameters and such that

$$
\left(F_{n}\right) \vdash G .
$$

Then, clearly, $\left(F_{n}\right) \vdash G^{H}$, where $G^{H}$ is the Herbrand normal form of $G$, and there is a normal derivation of the sequent

$$
\neg \triangle\left[\neg Q F-A C, \neg W K L, \neg \neg \mathbb{I}_{n}-I A\right], G^{H},
$$

where $\Delta$ contains only $B T$-axioms, but no instances of $Q F-I A$, as those are subsumed under $\quad \mathcal{H I}_{n}-I A$. Quasi-normalization of the derivation obtained from the embedding lemma yields a quasi-normal derivation of length less than $\omega_{n}^{\omega}$ and termdepth less than $\omega$ of the sequent

$$
[\neg Q F-A C, \neg W K L], G^{H} .
$$

The last two elimination lemmata, used in a single inductive argument, provide us with a quasi-normal derivation of

$$
G^{H}
$$

whose length and termdepth are both bounded by $\omega_{n}^{\omega}$. Now we assume that $G^{H}$ is of the form ( $\exists x$ ) Fxaf and use the $\bar{\exists}$-inversion lemma to finally get a term $t$ and a quasi-normal derivation of

Ftaf
whose length and termdepth is still bounded by $\omega_{n}^{\omega}$. This is the final step in transforming the finitary derivation of $G$ into an infinitary derivation of an instance of $G$ 's Herbrand normal form. The considerations involved in this transformation can be formalized in $\left(\Sigma_{n}^{0}-I A\right)+$. The reflection principle for quantifier-free formulas with termdepth less than $\omega_{n}^{\omega}$ and derivations of length and termdepth less than $\omega_{n}^{\omega}$ allows us to conclude that

$$
\left(\Sigma_{n}^{0}-I A\right)+\vdash G^{H} .
$$

Herbrand's theorem [9] guarantees the ultimate conclusion; namely,

$$
\left(\Sigma_{n}^{0}-I A\right)+\vdash G .
$$

If $G$ is purely arithmetic, it is provable in the fragment of arithmetic ( $\Sigma_{n}^{0}-I A$ ).

C3. CONSEQUENCES. Complementing theme and method by topical results, we can obtain a characterization of the provably recursive functionals of the $\left(F_{n}\right)$ and, consequently, of the $\left(\Sigma_{n}^{0}-I A\right)+$. [10] Note, that here as above $n$ is always assumed to be greater than 0 . Some well-known facts concerning classes of recursive function(al)s follow immediately. - Let me first mention the characterizations of two theories with Koenig's Lemma, full second order number theory and the theory of arithmetic properties: the provably recursive functionals of the former are Spector's bar - recursive functionals, those of the latter the $<\epsilon_{0}$-recursive functionals. [11] In the argument for the main theorem I made implicit use of the fact that the (unnested) $<\omega_{n}^{\omega}$-recursive functionals can be introduced in $\left(\Sigma_{n}^{0}-I A\right)+$; namely, when claiming that certain considerations can be formalized in $\left(\Sigma_{n}^{0}-I A\right)+$. [12] In turn, the argument yields: if $\left(F_{n}\right)$ proves ( $\forall x$ ) ( $\exists y) R x y$, where $R$ is primitive recursive (in function parameters), then there is a $<\omega_{n}^{\omega}$-recursive function(al) $f$, such that the statement $(\forall x) R x f(x)$ is provable in $\left(\Sigma_{n}^{0}-I A\right)+$. Consequently we have:

THEOREM. The provabły recursive functionals of $\left(F_{n}\right)$ and $\left(\Sigma_{n}^{0}-I A\right)+$ are exactly the $<\omega_{n}^{\omega}$-recursive functionals.

A straightforward refinement of Schwichtenberg's argument for the introducibility of the nested $-<\epsilon_{0}$-recursive functionals in $(Z)+$ allows us to introduce the nested $-<\dot{\omega}_{n}^{\omega}$-recursive functionals in $\left(\Sigma_{n+1}^{0}-I A\right)+$ together with the quantifier-free axiom of choice. Thus the earlier conservation theorem implies:

THEOREM. All nested $-<\omega_{n}^{\omega}$-recursive functionals can be introduced in $\left(\Sigma_{n+1}^{0}-I A\right)+$.

The theorems have as a corollary a special (and especially interesting) case of a theorem in [Tait, 1965].

COROLLARY. Nested $-<\omega_{n}^{\omega}$-recursion is reducible to unnested $-<\omega_{n+1}^{\omega}$ recursion.

## NOTES

1. This is the stylistically improved, but substantively unchanged text I had prepared for my talk at the workshop "Logic and Computer Science: New Trends and Applications" on October 14, 1986. However, the research for this paper was carried out to a large extent in the spring and summer of 1984. The results concerning second order systems were presented to the European Summer Meeting of the ASL in Paris, July 1985; see the abstract for that meeting in the Journal of Symbolic Logic, vol. 52, 1987, pp. 342-343.
2. See [Takeuti], [Feferman, 1977 and 1985], and [Friedmanj.
3. To get some impressions of the work on "Reverse Mathematics" consult the book Harvey Friedman's Research on the Foundations of Mathematics, edited by Harrington, Morley, Scedrov, and Simpson, North-Holland, Amsterdam, 1985.
4. The result (for pure number theory) was announced by Parsons in Reduction of inductions to quantifier-free induction, Notices AMS 13 (1966), p. 740; a proof is in On a number-theoretic choice schema and its relation to induction. The latter paper was published in: Intuitionism and Proof Theory, edited by Kino, Myhill, and Vessley, North-Holland, Amsterdam, 1977, pp. 459-473.
5. See the discussion in my paper Fragments of arithmetic on p. 65.
6. Recent work of Buss, Takeuti, and Clote establishes most interesting relationships between fragments of arithmetic and complexity classes.
7. Such a strategy was also pursued in Feferman and Sieg, Proof-theoretic equivalences between classical and constructive theories, Lecture Notes in Mathematics 897, Springer Verlag, Berlin, 1981, pp. 78-142.
8. $\beta_{0}^{\alpha}=\alpha ; \beta_{n+1}^{\alpha}=\beta^{\beta_{n}^{\alpha}}$.
9. As formulated for example in Shoenfield, Mathematical Logic, AddisonWesley, Reading, 1967.
10. The characterization of the provably recursive functions for fragments of arithmetic was given by Parsons in Ordinal recursion in partial systems of number theory, Notices AMS 13 (1966), pp. 857-858; it follows from results below, when taking into account that $\left(\Sigma_{n}^{0}-I A\right)+$ is conservative over ( $\Sigma_{n}^{0}-I A$ ).
11. [Kreisel, 1951/52] gives the provably recursive functionals of $(Z)+$. In his introduction to the Stanford Report Kreisel wrote: "Spector's schemata provide the most perspicuous description to date of the provably recursive functions and functionals of classical analysis." Spector's paper Provably recursive functions of analysis: a consistency proof of analysis by an extension of principles formulated in current intuitionistic mathematics was published in volume 5 of Proceedings of Symposia in Pure Mathematics, 1962, pp. 1-27.
12. This fact can be established using techniques of [Kreisel, 1951/2] and [Parsons, 1973].

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WILFRIED SIEG - Department of Phiiosophy - Carnegie Mellon University - Pittsburgh, PA 15213

