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THE POINCARÉ–BIRKHOFF THEOREM AND SOME APPLICATIONS

Abstract. In this manuscript we present, from a personal perspective, an account of our long standing collaboration with Fabio Zanolin, with focus on the Poincaré–Birkhoff theorem. Starting with the PhD thesis of the second author, we go through the work about the modified version of the Poincaré–Birkhoff theorem that Fabio proposed us, and then pass to two works which originated from it, with different collaborators, in which some open questions were answered. Finally, we get to a recent paper with Fabio, where we provided a general framework in which the number of fixed points of area preserving maps is larger than the one obtainable with the Poincaré–Birkhoff theorem.

1. Introduction

The Poincaré–Birkhoff theorem was the subject of the PhD thesis of the second author, supervised by Fabio Zanolin. The first Zanolin's paper C. Rebelo read was the paper on periodic solutions of Duffing equations with superquadratic potential [15]. In this marvellous paper, T. Ding and F. Zanolin apply fixed point theorems in order to obtain harmonic and subharmonic solutions for second order ordinary differential equations with superquadratic potential. One of these fixed point theorems was the Poincaré–Birkhoff fixed point theorem in the version by W.Y. Ding [17, 18]. That was the first time the second author met the Poincaré–Birkhoff theorem and she remained fascinated by the theorem and its power in the search for periodic solutions of ordinary differential equations. More or less one year after, she began her PhD under the supervision of Zanolin and one of his suggestions of work was to study the proof of the Poincaré–Birkhoff theorem. This proof already had a quite long story which began in [43] where it was given for some cases. In that paper, Poincaré considered an area-preserving homeomorphism from an annulus onto itself and conjectured that it admits (at least) two fixed points when some "twist" condition is satisfied. Roughly speaking, the twist condition consists in rotating the two boundary circles in opposite angular directions. This concept will be made precise in what follows.

In 1913, Birkhoff in [3] published a complete proof of the existence of at least one fixed point. Indeed the reasoning guaranteeing the existence of the second point had a gap and the existence of a second fixed point was obtained only in the case that the first one has a nonzero index. Birkhoff detected this imprecision in his proof and corrected it in [4]. In that paper Birkhoff also weakened the hypothesis about the invariance of the annulus under the homeomorphism. Not asking the invariance of the boundaries is an important improvement in the statement of the theorem in what

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concerns the applications. He also generalized the area-preserving condition.

In 1977, Brown and Neumann decided to give a clear and detailed proof of the theorem in his classical version. More precisely in [8] the authors proved the following version of the theorem

THEOREM 1. *Consider an area preserving homeomorphism* $h : \mathbf{R}^2 \to \mathbf{R}^2$ *satisfying*

$$h(x,y) = (x - r_1, y), \ \ y \geq 1$$
$$h(x,y) = (x + r_2, y), \ \ y \leq 0$$
$$h(x + 2\pi, y) = h(x,y) + (2\pi, 0),$$

for some $r_1, r_2 > 0$. *Then* h *has two distinct fixed points* F_i, $i = 1,2$, *which are not from the same periodic family, that is,* $F_1 - F_2 \neq (2k\pi, 0)$ *for every* $k \in \mathbf{Z}$.

The homeomorphism h in this theorem corresponds to a lifting of an area-preserving twist homeomorphism of an annular region in the plane. Note that if we consider an homeomorphism which preserves the area $dxdy$ we can consider a lifting which preserves the same measure [12].

The proof of this result is done by contradiction using the notion of index of a curve with respect to an homeomorphism. This index is well defined for curves which do not contain fixed points of the homeomorphism and is invariant under homotopies which maintain fixed the extreme points of the curves and along which there are no fixed points of the homeomorphism. Using these facts, the authors of the proof assume that there is at most one family $(x^* + 2k\pi, y^*)$ of fixed points of h and arrive to a contradiction.

An important corollary of the proof of the previous theorem is the following (see [39, Remark 1.1.] and also [47]) where we denote by $ind(P)$ the fixed point index of the fixed point P

COROLLARY 1. *Under the conditions of the previous theorem, if there exists an isolated fixed point, then there exists one with a nonzero index. Also, if all the fixed points are isolated then they are in a finite number r and we have* $ind(F_i) \leq 1$, $\sum ind(F_i) = 0$ *and* $\sum (ind(F_i))^2 > 0$ *where* F_i, $i = 1, \ldots, r$, *are the fixed points.*

2. H. Jacobowitz and W.Y. Ding versions

An important restriction of the classical version of the Poincaré–Birkhoff theorem is the invariance of the boundaries. In fact, if we want to apply the theorem to the existence of periodic solutions for a system of ordinary differential equations in the plane, depending periodically on the variable t, it is natural to apply it to the associated Poincaré map. For that we must have a Poincaré map which is area-preserving, for example a Poincaré map associated to an Hamiltonian system (consequence of Liouville's theorem) and moreover if we want to apply the classical Brown and Neumann version of the theorem we should have invariance of the boundaries of the annular region considered. This is a very difficult (if not impossible in most of the equations) condition to satisfy.

As previously said, Birkhoff himself proposed a version of the theorem not requiring the invariance of the boundary. As its proof was not very clear, in [30, 31] Jacobowitz proposed a new version ot the theorem and its proof.

In order to state this version let us introduce some notation. Let $\Gamma_i = (\theta_i(\cdot), r_i(\cdot))$, $i = 1,2$, defined in $[0,1]$ be two simple curves such that $\theta_i(0) = -\pi$, $\theta_i(1) = \pi$, $\theta_i(s) \in (-\pi,\pi)$ for each $s \in (0,1)$ and $r_i(0) = r_i(1)$. Let us consider the 2π-periodic extensions which we still call Γ_i. Note that the Γ_i defined in this way are simple curves which are liftings of curves which we do not assume that are star-shaped around the origin. Denoting by A_i the region bounded by the curve Γ_i (included) and the axis $r = 0$ (excluded), Jacobowitz proved the following theorem

THEOREM 2. *Let $\psi : A_1 \to A_2$ be an area-preserving homeomorphism, defined by $\psi(\theta,r) = (\theta + g(\theta,r), f(\theta,r))$ where*

i) *g and f are 2π periodic in the first variable;*

ii) $g(\theta,r) < 0$ *on* Γ_1*;*

iii) $\liminf_{r\to 0} g(\theta,r) > 0$.

Then, ψ admits at least two fixed points, which do not differ by a multiple of $(2\pi,0)$.

In this theorem the invariance of the origin is assumed. W.Y. Ding generalized this result not assuming the invariance of the inner boundary. In fact, in [17, 18] he considered a topological annular region and proved the following theorem

THEOREM 3. *Let $\mathcal{A}$ be an annular region around the origin bounded by two simple curves, C_1 the inner and C_2 the outer. Denote by D_i the open region bounded by C_i, then $\mathcal{A} = \bar{D}_2 \setminus D_1$. Let $T : \mathcal{A} \to T(\mathcal{A}) \subset \mathbf{R}^2 \setminus \{(0,0)\}$ be an area-preserving homeomorphism. Assume that*

i) *the inner curve C_1 is star-shaped around the origin;*

ii) *T admits a lifting $\tilde{T}$ associated to the polar coordinates covering projection of the form*

$$\tilde{T}(\theta,r) = (\theta + g(\theta,r), f(\theta,r))$$

where f and g are continuous and 2π-periodic in θ and

$g(\theta,r) > 0$ *in the lifting of C_1 and* $g(r,\theta) < 0$ *in the lifting of C_2 (Twist condition)*;

iii) *T can be extended to an area-preserving homeomorphism $T_0 : \bar{D}_2 \to \mathbf{R}^2$ such that* $(0,0) \in T_0(D_1)$.

Then $\tilde{T}$ admits two fixed points which images under the projection are two distinct fixed points of T.

Ding based the proof of this theorem on the result of Jacobowitz.

At this point we come back to the proposal of Fabio Zanolin to the second author: to give a proof of the Ding's version of the Poincaré–Birkhoff theorem directly from the Brown and Neumann version. In fact, Jacobowitz' proof was not very easy to follow and some authors had doubts about its correctness. Fabio Zanolin revealed a great intuition suggesting this as we can understand below. Moreover, during the elaboration of that proof, Fabio's contribution was essential. In fact he designed the strategy of the proof, found several important references and gave a constant support. The result of this work is published in [45] where the proof of the Ding's version of the theorem was given but with an extra assumption: not only the star-shaped condition was required for the inner boundary but also for the outer one.

To ask for the star-shaped condition for the boundaries is not a strong restriction for the applications. In fact usually it is not difficult to prove it.

In his paper, W.Y. Ding mentioned that condition *i*) of the theorem was only a technical one. Some years later Martins and Ureña showed in [38] that this is not the case. Martins was introduced to the Poincaré–Birkhoff theorem by the authors of this manuscript and from the beginning questioned the need of the star-shaped condition. He even painted a picture of this theorem in which he showed the importance of the star-shaped condition. His intuition was correct. Not so many time after, Le Calvez and Wang [32] proved that also the external boundary should be star-shaped and hence Fabio was right in thinking about the importance of a new proof of this version of the theorem. For more details about these versions of the theorem and historical results see [12, 24].

3. Applications to the search of harmonic and subharmonic solutions

The extensions of the Poincaré–Birkhoff theorem allowing that the boundaries do not remain invariant under the homeomorphism were obtained having in mind the applications to ODEs. In the paper in which he introduced the generalization of the Poincaré–Birkhoff theorem [17], W.Y. Ding gives an application to the existence of infinitely many 2π-periodic solutions to

$$x'' + g(x) = p(t)$$

in the superlinear case, that is assuming $\lim_{|x|\to+\infty} \frac{g(x)}{x} = +\infty$, when p is a 2π-periodic function. Tun Ren Ding [13] applied the W.Y. Ding's version of the theorem to prove the existence of infinitely many periodic solutions in the case the derivative of g is bounded, allowing resonance. After these papers many others appeared, for example [7, 15, 16, 20, 22, 23, 40, 44, 46, 48, 49] to quote just a few, where the Poincaré–Birkhoff theorem was applied. Even if this is only a small sample of the large amount of papers which we could cite, we can observe that the theorem was applied along the years by many researchers.

The version of the theorem which does not require the invariance of the boundaries is a potent tool to prove the existence of periodic solutions to an Hamiltonian system of the form

$$\begin{cases} x' = -\frac{\partial H}{\partial y}(t,x,y) \\ y' = \frac{\partial H}{\partial x}(t,x,y) \end{cases} \tag{1}$$

where $H : \mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}$ is a continuous scalar function that we assume T-periodic in t and C^2 in $z = (x,y)$. Under these conditions we have existence and uniqueness of Cauchy problems associated to this system. Let us denote by $z(t;t_0,z_0) = z(t;t_0,x_0,y_0)$ the solution of (1) such that $z(t_0;t_0,z_0) = (x_0,y_0)$.

Assuming extra conditions which guarantee that the solutions of Cauchy problems with initial condition in some open set $\mathcal{D}$ are defined in $[0,T]$, we can consider the Poincaré operator

$$P : \mathcal{D} \to \mathbf{R}^2 \text{ defined by } P(z_0) = P(x_0,y_0) = z(T;0,z_0)$$

which is an homeomorphism onto its image. By Liouville theorem this homeomorphism is area preserving.

In order to apply the Poincaré–Birkhoff theorem it is useful to consider the lifting of maps associated to polar coordinates. More precisely, given $z : [0,T] \to \mathbf{R}^2 \setminus \{(0,0)\}$, we can consider its lifting associated to the polar coordinates $(\theta(t), r(t))$, $t \in [0,T]$. Also it is useful to consider the T-rotation number of $z(\cdot;0,z_0)$ defined by

$$Rot(z_0) = \frac{\theta(T) - \theta(0)}{2\pi}.$$

This rotation counts the counterclockwise turns described by $z(t)$ when t goes from zero to T.

With this setting we can state the following theorem [45] which gives a kind of recipe to prove the existence of T-periodic solutions using the Poincaré–Birkhoff theorem

THEOREM 4. *Let $\mathcal{A} \subset \mathcal{D}$ be an annular region surrounding $(0,0)$ and let Γ_i, $i = 1,2$, be their inner and outer boundaries which are strictly star-shaped around the origin. Assume that*

i) $z(t;t_0,z_0) \neq (0,0)$ $\forall t_0 \in [0,T[$, $\forall z_0 \in \Gamma_1$ and $t \in [t_0,T]$;

ii) there exist $m_i \in \mathbf{Z}$ with $m_1 \geq m_2$ such that

$$Rot(z_0) > m_1,\ \forall z_0 \in \Gamma_1 \text{ and } Rot(z_0) < m_2,\ \forall z_0 \in \Gamma_2.$$

Then for each integer ℓ with $\ell \in [m_2, m_1]$ there are two fixed points of the Poincaré map which correspond to two periodic solutions of the Hamiltonian system having the T-rotation number ℓ.

This method was already used for example in [15, Theorem 1]. Note that in that theorem uniqueness of solutions for Cauchy problems was not guaranteed and hence an approximation of the vector field by smooth functions was considered. As a consequence one T-periodic solution was obtained instead of two.

One of the important consequences of the theorem above is that we obtain multiplicity of T-periodic solutions because the Poincaré–Birkhoff theorem is applied to different liftings of the Poincaré map and hence we obtain solutions with different rotation numbers.

From the fact that the rotation around the origin of the solutions obtained by the application of the Poincaré–Birkhoff theorem is known, it is possible in many situations to guarantee the existence of subharmonic solutions. These solutions are solutions which are mT-periodic for $m > 1$ and are not kT-periodic for $k < m$. To prove the existence of subharmonic solutions, the idea is to consider an annular region around the origin and to obtain estimates for the rotation of the solutions in the time interval $[0, mT]$ and then prove by contradiction that the solutions obtained from the application of the Poincaré–Birkhoff theorem cannot be kT-periodic for $k < m$. In [14, Proof of Theorem 2.3], the authors proved this using the Prime Number Theorem which states that, denoting by $\pi(x)$ the number of positive prime numbers which are less or equal to x, we have

$$\lim_{x \to +\infty} \frac{\pi(x)\log(x)}{x} = 1.$$

As far as we know it is the first time that this theorem was used in this context. In fact, as a consequence of this theorem if $a < b$ then there exists $L > 0$ such that for each $x > L$ there exists a prime number ζ satisfying $ax < \zeta < bx$. Using this remark and the estimates for the rotations they guaranteed that the solutions obtained are subharmonic. For more details see [14, 15]. This technique was used in many papers after.

In what concerns applications in higher dimensions, some years after appeared the theorems by Ureña and Fonda in [25, 26] and by Gidoni and Fonda in [21]. For a version for non convex domains see [27]. These theorems are also useful in the plane when uniqueness is not guaranteed.

4. A generalized Poincaré–Birkhoff theorem

Some years after, Fabio proposed to both authors of this manuscript to study asymptotically linear Hamiltonian systems. In the following we describe the problem we analyzed. Let $H : \mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}$ be a continuous scalar function which is C^2 in the second variable and T-periodic in the first one. Assume $H'_z(t, 0) = 0$ and that there are positive constants A, B such that

$$|H'_z(t,z)| \leq A\|z\| + B, \text{ for each } t \in [0,T],\ z \in \mathbf{R}^2.$$

Under these conditions, given $t_0 \in \mathbf{R}$ and $z_0 \in \mathbf{R}^2$, the problem (1) has a unique solution $z(t;t_0,z_0)$ defined in $\mathbf{R}$ and satisfying $z(t_0;t_0,z_0) = z_0$. We also supposed that (1) is asymptotically linear at infinity and at zero to the T periodic systems $z' = A_\infty(t)z$ and $z' = A_0(t)z$, respectively.

For completeness we recall that system $x' = X(t,x)$ is asymptotically linear at infinity to the T-periodic system $x' = A(t)x$ if $X(t,x) = A(t)x + Y(t,x)$ and

$$\lim_{\|x\| \to +\infty} \frac{Y(t,x)}{\|x\|} = 0$$

uniformly for $t \in [0,T]$. Analogously the definition of asymptotically linear at zero.

The idea of Fabio was inspired by two remarks made by Amann and Zehnder in [1, Remark 1] and Conley and Zehnder in [10, Introduction, page 211]. In [1] the authors analyzed an asymptotically linear Hamiltonian system in $2N$-dimensions in which they considered constant linearizations at infinity and at zero. They introduced an index i which depended on the matrices of the linearizations and said that in the case of dimension 2 to say that the index is positive is equivalent to say that the twist condition of the Poincaré–Birkhoff theorem is satisfied. In a subsequent work, Conley and Zehnder, [10] considered asymptotically linear systems at infinity and at the origin once again in dimension $2N$. They defined the Maslov index for nonautonomous T-periodic linear T-nonresonant systems and proved that when the indexes of the linearizations at infinity and at zero are different there is at least one T-periodic solution, two if the first one is nondegenerate. The authors mentioned that the existence of at least one (or two in the nondegenerate case) solutions is in contrast with the Poincaré–Birkhoff theorem for the case $N = 1$ where $|j_0 - j_\infty|$ should be a measure of the lower bound of the number of T-periodic solutions of the asymptotically linear system. Fabio proposed us to prove the Conley and Zehnder conjecture. The first thing we had to do was to study the Maslov index. This task was not so easy for us, we recall some afternoons at Fabio's home the three of us reading about this index. Then we had to understand the relation between the Maslov index and the rotation of the solutions of the planar linear systems. Also this gave us some headaches and we recall perfectly that Fabio encouraged us to continue thinking about this because he had the feeling that an interesting result would appear. He was right, of course, and when we arrived to this result we understood that in order to prove the conjecture we had to generalize the Poincaré–Birkhoff theorem, more precisely, to prove the theorem under a weaker twist condition. And we managed to do it, and now we must say that at the beginning Fabio was not so convinced. Our theorem is the following [34, Corollary 2] (the liftings we mention are associated to the polar coordinate covering projection)

THEOREM 5. *Let Γ_1 be a circle of radius R around the origin and Γ_2 a simple curve surrounding the origin. Let $\mathcal{B}_i$, $i = 1,2$, be the finite closed domain bounded by Γ_i and $\mathcal{A}_i = \mathcal{B}_i \setminus \{(0,0)\}$. Let $\tilde{\mathcal{A}}_i$ and $\tilde{\Gamma}_i$ be the liftings of $\mathcal{A}_i$ and Γ_i.*

Let $\Psi : \mathcal{A}_1 \to \mathcal{A}_2$ be an area preserving homeomorphism. Assume that Ψ admits a lifting $\tilde{\Psi}(\theta,r) = (\theta + \Theta(\theta,r), \mathcal{R}(\theta,r))$ with Θ and $\mathcal{R}$ both 2π-periodic in θ. Moreover assume that $\Theta(\theta,R) > 0$ (or $\Theta(\theta,R) < 0$) for each θ and that there exists $\bar{\theta}$ such that

$\Theta(\bar{\theta},0) < 0$ *(respectively* $\Theta(\bar{\theta},0) > 0$*) (modified twist condition).*

Then $\tilde{\Psi}$ *has at least one fixed point in* $\tilde{\mathcal{A}}_1$ *which corresponds to a fixed point in* $\mathcal{A}_1$*. Moreover if this fixed point is unique it has nonzero index.*

The proof of this theorem was built adapting the proof in the Brown and Neumann result and hence using the index of a curve with respect to the homeomorphism $\tilde{\Psi}$. We assumed that there were no fixed points and arrived to a contradiction using this notion of index.

The result we obtained about the Conley-Zehnder conjecture using also this modified Poincaré–Birkhoff theorem was the following

THEOREM 6. *Suppose that system* (1) *is asymptotic at infinity and at zero to the* T*-periodic and* T*-nonresonant linear systems* $z' = A_0(t)z$ *and* $z' = A_\infty(t)$*, respectively. Denote by* i_T^0 *and* i_T^∞ *the corresponding* T*-Maslov indices. If* $i_T^0 \neq i_T^\infty$ *then* (1) *has at least* $\max\{1; 2\left\lfloor \frac{|i_T^0 - i_T^\infty|}{2} \right\rfloor\}$ *nontrivial* T*-periodic solutions. Moreover, if* i_T^0 *is even then* (1) *has at least* $2\left\lceil \frac{|i_T^0 - i_T^\infty|}{2} \right\rceil$ *nontrivial* T*-periodic solutions.*

In particular, Theorem 5 was used improve the number of solutions when the T-Maslov index i_T^0 is even, since in such case there is a lift of the Poincaré map which satisfies the modified twist condition at the origin, but not the usual twist condition of Theorem 4.

In order to obtain a sharper lower bound on the number of T-periodic solutions for the asymptotically linear Hamiltonian systems we needed a similar result to the previous one but with the roles of the origin and of Γ_1 interchanged. We thought about it but we were not able to obtain it. Hence, [34, Theorem 2] is not symmetric in what concerns the roles of the indexes at zero and at infinity.

The possible symmetric version of Theorem 5 remained one of our open questions for some time. Two years after, in collaboration with Juan Campos and Rogério Martins we obtained a counterexample for the symmetric case in [9], more precisely we proved the following

THEOREM 7. *Let* $\mathcal{C} = \{(\theta, r) :\ r \geq 0\}$*. There exists an area preserving homeomorphism* $\tilde{\Psi} : \mathcal{C} \to \mathcal{C}$ *of the form* $\tilde{\Psi}(\theta, r) = (\theta + \Theta(\theta, r), \mathcal{R}(\theta, r))$*, with* Θ *and* $\mathcal{R}$ *both* 2π*-periodic in* θ*, such that* $\mathcal{R}(\theta, 0) = 0$ *without fixed points and satisfying*

i) there exists $\bar{\theta}$ *such that* $\lim_{r\to+\infty}\Theta(\bar{\theta}, r) < 0$ *(respectively* $\lim_{r\to+\infty}\Theta(\bar{\theta}, r) > 0$*)*

ii) $\Theta(\theta, 0) > 0$ *(respectively* $\Theta(\theta, 0) < 0$*).*

The counterexample was obtained by constructing a suitable Hamiltonian function $\mathcal{H}(\theta, r)$, 2π periodic in θ and whose flow satisfies for any fixed positive time the assumptions of Theorem 7 but has no fixed points (see Figure 1). Such Hamiltonian was the result of the smooth gluing along a previously constructed curve $\alpha(\theta)$ of two Hamiltonian functions. The first, defined in the region $\mathcal{R}_2$, satisfies the twist condition

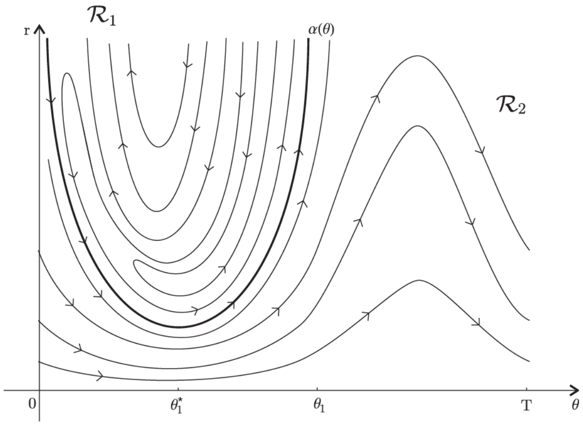

Figure 1: The Hamiltonian flow, represented in the period interval $[0, T = 2\pi]$, verifies the twist conditions in Theorem 7 but does not admit any fixed point. The points $\theta = 0$ and $\theta = \theta_1$ are vertical asymptotes of $\alpha(\theta)$.

at zero, the second, defined in the region $\mathcal{R}_1$, bounded from below by $\alpha(\theta)$, satisfies the weak twist condition at infinity.

In [9] there exists also an example of a case in which the fixed point mentioned in [34] is unique and a version of Theorem 5 for the half plane $\mathcal{C}$. Finally also an example in which the boundaries in Theorem 5 cannot be interchanged for a bounded annular region was given [9, Remark 1].

Theorem 7 rules out the possibility of getting a modified Poincaré–Birkhoff theorem just interchanging the twist conditions at zero and infinity. However, the Hamiltonian considered in the counterexample did not correspond to an Hamiltonian system which is asymptotically linear at infinity. Then, the possibility of obtaining a sharper result for this class of systems was still an open problem. This problem was solved fourteen years after, in 2018, when the second author and Paolo Gidoni, at the time post-doc in Lisbon, obtained the optimal lower bound for the number of T-periodic solutions of an asymptotically linear planar hamiltonian system in the joint paper [29]. More precisely, the result proved in that paper is the following

THEOREM 8. *Suppose that system* (1) *is asymptotic at zero and at infinity to the T-periodic and T-nonresonant linear systems $z' = A_0(t)z$ and $z' = A_\infty(t)$, respectively, and denote respectively with i_T^0 and i_T^∞ their T-Maslov indices. Then system* (1) *has at least $|i_T^0 - i_T^\infty|$ T-periodic solutions. Moreover, if i_0^T is even, with $i_T^0 \neq i_T^\infty$, then the number of solutions is at least $|i_T^0 - i_T^\infty| + 1$.*

This result was obtained by combining the Poincaré–Birkhoff theorem with a suitable topological degree theory that, when applied to an asymptotically linear system, detected the additional solution and its rotation number. This shed some light also on the counterexample in [9], in the sense that it gave a concrete evidence that the weak twist at infinity and the area preserving property are no longer sufficient, without further assumptions on the vector field, to guarantee the existence of a fixed point. In [29] several examples illustrating the sharpness of the lower bound were provided. The generalized notion of degree introduced in [29] was then developed and improved by P. Gidoni in [28], where it is used to give a shorter proof of Theorem 8, together with some other applications to the existence and multiplicity of periodic solutions of ODEs.

Before going to the next section, we refer that the modified version of the Poincaré–Birkhoff theorem became useful in the applications (see for example [5,33]).

5. Beyond Poincaré–Birkhoff: *a plethora of fixed points* *

And we arrive to our last work with Fabio (for now), still connected with the Poincaré–Birkhoff theorem, namely [37]. In some situations the rotational properties provided by the Poincaré–Birkhoff theorem are not sufficient to detect finer oscillatory properties of the periodic solutions. An example is given by the well known equation

$$x'' + a(t)g(x) = 0$$

with $g(0) = 0$, $sg(s) > 0$ for all $s \neq 0$ and the T-periodic weight function $a(t)$ a function that changes its sign. In fact, in this case the oscillatory behaviour is concentrated in the intervals where $a(t) > 0$, where the corresponding Poincaré maps behaves as a twist map, whereas on each interval where $a(t) < 0$ the corresponding Poincaré map is an expansive-contractive homeomorphism admitting at most one zero. Even if over a period the twist condition required by the Poincaré–Birkhoff theorem holds, it is not possible to say how the oscillations of the periodic solutions are divided among the intervals where $a(t)$ is positive. This is due to the fact that when we apply the Poincaré–Birkhoff theorem, we get just the total number of oscillations over a period. However, in the previous equation, the T-Poincaré map can be written as the composition of maps, where twist maps alternate with expansive contractive ones. This fact allows to enter the setting of the so called Stretching Along the Paths technique (SAP for short), which is a simple topological method to detect fixed points and chaotic dynamics in planar maps introduced in a joint work by Fabio and Duccio Papini in [67] (for a more recent exposition, see [42]).

During our long collaboration with Fabio, he had already proposed us to work together on the applications to ODEs of SAP method, and in 2010 and 2013 we wrote the papers [35] and [36], respectively. Even in this case, we had manner to appreciate the kindness, the deepness of thought, and encyclopedic mathematical culture of Fabio, as well as his incredible stamina. Of course, we also recall fondly the many strolls to the Rizzi to have some delicious panini for lunch, a very pleasant and needed pause from work.

*When we were discussing the title of the paper [37], with all the results already proved, we were in a very relaxed and happy mood, and Fabio playfully suggested to include the expression in italic in the title. At the end, we all opted for a more conventional title, but this seems just the right occasion to use it.

So, when some years later Fabio suggested to provide a general unifying framework for comparing the results obtainable with the Poincaré–Birkhoff theorem and the sharper multiplicity results which one can get using the SAP technique (see e.g. [67], [11]) [19]), we enthusiastically accepted, and we wrote the paper [37].

Borrowing from [37], skipping the many technicalities and definitions involved, the typical general results that we obtain are summarized as follows:

> *Assume that the Poincaré map F is of the form $F = \Phi \circ \Psi$ with Ψ a twist mapping of an annulus $A[r_0, R_0]$ into an annulus $A[r_1, R_1]$ and Φ is expansive along a direction and has a compressive effect along a complementary direction. If the expansive-contractive effect is sufficiently strong, then we obtain a number of fixed points which in the area preserving case doubles the typical number of fixed points obtainable with the Poincaré–Birkhoff theorem, and we can prove the existence of complex dynamics on two symbols for F^2. If*
>
> $$F := \Phi_n \circ \Psi_n \circ \Phi_{n-1} \circ \Psi_{n-1} \circ \cdots \circ \Phi_1 \circ \Psi_1$$
>
> *where Ψ_i and Φ_i are of the same type than Φ and Ψ, then F has at least $2^{2n} m_1 m_2 \ldots m_n$ fixed points in $A[r_0, R_0]$, where m_i is the twist corresponding to Ψ_i at the boundary of $A[r_0, R_0]$. Moreover, for each itinerary (orbit of a point through the sequence of maps $\Psi_1, \Phi_1, \cdots \Phi_n, \Psi_n$) it is possible to determine explicitly the integers which are rotation numbers associated to at least one of the corresponding fixed points.*

If we consider area preserving maps, we enter the setting of the Poincaré–Birkhoff theorem. We notice that by its application we could just prove the existence of $2TW_{\max}$ fixed points for F (at most), where

$$TW_{\max} := m_1 + m_2 + \cdots + m_n + \left\lceil \frac{n}{2} \right\rceil,$$

so that, as n increases, our result in [37] guarantees the existence of a much larger number of periodic solutions than the Poincaré–Birkhoff theorem.

In [37] the general results were then applied to the study of multiplicity of periodic solutions and complex dynamics of planar systems of the form

$$\begin{cases} x' = h(t, y) \\ y' = -[\lambda a^+(t) - \mu a^-(t)]g(x), \end{cases} \tag{2}$$

where $h, g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, locally Lipschitz in the second variable and T-periodic in the t-variable and such that

$$g(0) = 0,\ h(t, 0) \equiv 0 \ \text{ and } \ sh(t, s) > 0 \ \text{ for all } s \neq 0 \ \text{ and all } \ t \in \mathbb{R}$$

and the function $a(t)$ alternates n positive bumps with n negative ones in a period.

System (2) includes second order scalar equations, but also equations involving ϕ-Laplacian type nonlinear differential operators of the form

$$(\phi(x'))' + [\lambda a^+(t) - \mu a^-(t)]g(x) = 0$$

with $\phi :]-L, L[\to \mathbb{R}$ an odd strictly increasing homeomorphism (see, for instance [2,6] and the references therein).

All the results presented in [37] are obtained under suitable conditions for $g(x)/x$ and $h(t,x)/x$ at zero and infinity and for suitable choices of the positive parameters λ and μ (which control the amount of twist and stretch, respectively). In these results we provide the desired finer description of the oscillatory properties for the periodic solutions which correspond to the fixed points detected via SAP.

6. Final Remarks

The second author began her PhD in 1993, the first author began working with Fabio in 2000. Since then, they have had a very fruitful mathematical cooperation with Fabio†, which developed in a precious friendship involving also his wife Paola. We end with an open invitation to come to visit us to Lisbon with Paola, to continue our mathematical cooperation and to pass some nice moments all together, going to Colina after a day's work (where you will certainly eat "Carne de porco à Alentejana e mousse de chocolate") and going to Azenhas do Mar to look again to that funny waves (who knows, we could write a paper about it...).

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†Fabio also introduced them to the concept of persistence in dynamical systems and to the techniques used to prove it in systems of ODEs. But this is another story, and will be told somewhere else.

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