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TRIHARMONIC CURVES IN BIANCHI-CARTAN-VRANCEANU SPACES

Abstract. In this paper, we study triharmonic curves in Bianchi-Cartan-Vranceanu 3- dimensional spaces. We characterize the triharmonic curves in terms of their curvature and torsion.

1. Introduction.

In the theory of curves in Riemannian manifolds, one of the most and to give characterizations of a regular curve.
First we should recall some notions and results related to the harmonic and the Polyharmonic ($r-$ harmonic $r \geq 1$) maps between Riemannian manifolds.

Harmonic maps $\psi : (M,g) \to (N,\tilde{g})$ between Riemannian manifolds are the critical points of the energy functional

$$E_1 : C^\infty(M,N) \to \mathbb{R},\ \ E_1(\psi) = \frac{1}{2}\int_M |\mathrm{d}\psi|^2 v_g,$$

and are characterized by the vanishing of the first tension field

$$\tau_1(\psi) = -\mathrm{d}^*\mathrm{d}\psi = trace\nabla \mathrm{d}\psi,$$

where d is the exterior differentiation and d^* is the codifferentiation.

We remind that the bienergy of ψ is given by

$$E_2 : C^\infty(M,N) \to \mathbb{R},\ \ E_2(\psi) = \frac{1}{2}\int_M |\tau(\psi)|^2 v_g,$$

and the bitension field $\tau_2(\psi)$ has the expression

$$\tau_2(\psi) = -\Delta^\psi \tau(\psi) - trace_g R^N(\mathrm{d}\psi, \tau(\psi))\mathrm{d}\psi,$$

where $\Delta^\psi = -trace(\nabla^\psi)^2 = -trace(\nabla^\psi\nabla^\psi - \nabla^\psi_\nabla)$.

A smooth map ψ is *biharmonic* if it satisfies the following biharmonic equation

$$\tau_2(\psi) = 0.$$

Biharmonic maps are the critical points of the bienergy functional E_2. We call proper biharmonic the non-harmonic biharmonic maps. Biharmonic curves ψ of a Riemannian manifold are the solutions of the fourth order differential equation

$$\nabla^3_{\phi'}\phi' - R(\phi', \nabla_{\phi'}\phi')\phi' = 0. \tag{1}$$

Eells and Lemaire [5] proposed the problem to consider the polyharmonic ($r-$ harmonic $r \geq 1$) maps of order r, these are critical points of the $r-$ energy functional defined by

$$E_r(\psi) = \int_M e_r(\psi) v_g, \ \ r \geq 1, \tag{2}$$

where $e_r(\psi) = \frac{1}{2}\|(\mathrm{d}+\mathrm{d}^*)^r\psi\|^2$ for smooth maps ψ.
A map ψ is $r-$ harmonic if it is a critical point of the functional $E_r(\psi)$ defined in (2).

Every harmonic map is a solution of the polyharmonic map equation, see [1] for a recent classification result. In [12], S.B. Wang studied the first variation formula of the k- energy E_k , whose critical maps are called k- harmonic maps. In [8], S. Maeta showed the second variation formula of the k- energy. Triharmonic curves with constant curvature in space forms were studied by Maeta in [8]. In [9], the authors study triharmonic curves in homogenous spaces proving the existence of triharmonic curves with nonconstant curvature and given the classification of triharmonic helices.

The aim of this paper is to continue the study of curves, more precisely triharmonic curves. We characterize the triharmonic curves in terms of their curvature and torsion.

2. Preliminaries

Let m and l be two real numbers and denote by $\mathcal{D}_{m,3}$ be the domain of $\mathbb{R}^3$ defined by

$$\mathcal{D}_{m,3} = \{(x,y,z) \in \mathbb{R}^3 : \ \delta = 1 + m(x^2+y^2) > 0\}.$$

By considering on $\mathcal{D}_{m,3}$ the 2- parameters family of homogeneous Riemannian metrics:

$$g_{BCV} = \mathrm{d}s^2_{m,l} = \left(\frac{\mathrm{d}x}{\delta}\right)^2 + \left(\frac{\mathrm{d}y}{\delta}\right)^2 + \left(\mathrm{d}z + \frac{l}{2}\frac{x\mathrm{d}y - y\mathrm{d}x}{\delta}\right)^2,$$

we obtain a 2- parameters family of 3- dimensional Riemannian manifolds $\mathcal{M}^3_{m,l} = (\mathcal{D}_{m,3}, g_{BCV})$, called Bianchi-Cartan-Vranceanu spaces (BCV-spaces, in short). In particular, if $l \neq 0$ and $m = 0$, $\mathcal{M}^3_{0,l}$ is the Heisenberg group $\mathbb{H}_3$ endowed with a left invariant metric. The complete classification of BCV-spaces can be found in [3, 11].
An orthonormal basis is composed by

$$E_1 = \delta\frac{\partial}{\partial x} - \frac{l}{2}y\frac{\partial}{\partial z}, \ \ E_2 = \delta\frac{\partial}{\partial y} + \frac{l}{2}x\frac{\partial}{\partial z}, \ \ E_3 = \frac{\partial}{\partial z}$$

with the dual basis is composed by

$$\omega_1 = \frac{\mathrm{d}x}{\delta}, \ \ \omega_2 = \frac{\mathrm{d}y}{\delta}, \ \ \omega_3 = \mathrm{d}z + \frac{l}{2}\frac{x\mathrm{d}y - y\mathrm{d}x}{\delta}.$$

PROPOSITION 1. *The Levi Civita connection* ∇ *of* $\mathcal{M}^3_{m,l}$ *with respect to this frame is given by the following relations:*

$$
\begin{aligned}
\begin{pmatrix} \nabla_{E_1}E_1 \\ \nabla_{E_1}E_2 \\ \nabla_{E_1}E_3 \end{pmatrix} &= \begin{pmatrix} 0 & 2my & 0 \\ -2my & 0 & \frac{l}{2} \\ 0 & -\frac{l}{2} & 0 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} \\
\begin{pmatrix} \nabla_{E_2}E_1 \\ \nabla_{E_2}E_2 \\ \nabla_{E_2}E_3 \end{pmatrix} &= \begin{pmatrix} 0 & -2mx & -\frac{l}{2} \\ 2mx & 0 & 0 \\ \frac{l}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} \\
\begin{pmatrix} \nabla_{E_3}E_1 \\ \nabla_{E_3}E_2 \\ \nabla_{E_3}E_3 \end{pmatrix} &= \begin{pmatrix} 0 & -\frac{l}{2} & 0 \\ \frac{l}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}.
\end{aligned} \tag{3}
$$

Also, we obtain the bracket relations

$$[E_1,\, E_2] = (-2my)E_1 + (2mx)E_2 + (l)E_3, \quad [E_2,\, E_3] = 0, \quad [E_1,\, E_3] = 0. \tag{4}$$

We shall adopt the following notation and sign convention. The Riemannian curvature operator is given by

$$\mathrm{R}(X,\, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,\,Y]}Z. \tag{5}$$

The Riemannian curvature tensor is given by

$$\mathrm{R}(X,\, Y,\, Z,\, W) = g_{BCV}(\mathrm{R}(Y,\, X)Z,\, W) = -g_{BCV}(\mathrm{R}(X,\, Y)Z,\, W), \tag{6}$$

where X,Y,Z,W are smooth vector fields on $\mathcal{M}^3_{m,l}$.
The nonvanishing components of the Riemann-Christoffel are

$$\mathrm{R}_{1212} = 4m - \frac{3}{4}l^2,\;\; \mathrm{R}_{1313} = \frac{l^2}{4} = \mathrm{R}_{2323}. \tag{7}$$

3. Polyharmonic curves in $\mathcal{M}^3_{m,l}$

3.1. Biharmonic curves in $\mathcal{M}^3_{m,l}$

In [3] and [6], the authors studied biharmonic curves in BCV-spaces and they obtained interesting classification results. In [2] the authors proved that any proper biharmonic curve in the Heisenberg group $\mathbb{H}_3$ is a helix and gave its explicit parametrization.

Let $\psi : I \to \mathcal{M}^3_{m,l}$ be a differentiable curve parametrized by arc length and let $\{T,N,B\}$ be the orthonormal moving Frenet frame along the curve ψ in $\mathcal{M}^3_{m,l}$ such that $T = \psi'$ is the unit vector field tangent to ψ, N is the unit vector field in the direction $\nabla_T T$ normal to ψ (principal normal) and $B = T \wedge N$ (binormal vector). Then we have the following Frenet equations

$$\left(\begin{array}{c}\nabla_T T\\ \nabla_T N\\ \nabla_T B\end{array}\right)=\left(\begin{array}{ccc}0 & k & 0\\ -k & 0 & \tau\\ 0 & -\tau & 0\end{array}\right)\left(\begin{array}{c}T\\ N\\ B\end{array}\right), \tag{8}$$

where

$$k^2 = g_{BCV}(\nabla_T T, \nabla_T T),$$

is the squared curvature of ψ and τ is its torsion.

The planes spanned by $\{T,N\}$, $\{T,B\}$ and $\{N,B\}$ are respectively known as the osculating, the rectifying and the normal plane.

From (8) we have

$$\nabla_T^3 T = (-3kk')T + (k'' - k^3 - k\tau^2)N + (2k'\tau + k\tau')B, \tag{9}$$

where $k' = \frac{dk}{ds}$, $k'' = \frac{d^2k}{ds^2}$, $\tau' = \frac{d\tau}{ds}$ and s is the arc length parameter.
Writing

$$\left\{\begin{array}{l} T = T_1e_1 + T_2e_2 + T_3e_3\\ N = N_1e_1 + N_2e_2 + N_3e_3\\ B = T\wedge N = B_1e_1 + B_2e_2 + B_3e_3, \end{array}\right. \tag{10}$$

and using (7), we have that

$$\left\{\begin{array}{l} \mathrm{R}(T,N,T,N) = \sum_{1\le i,j,k,l\le 3} T_iN_jT_kN_l\mathrm{R}_{ijkl} = (4m-l^2)B_3^2 + \frac{l^2}{4}\\ \mathrm{R}(T,N,T,B) = \sum_{1\le i,j,k,l\le 3} T_iN_jT_kB_l\mathrm{R}_{ijkl} = (l^2-4m)N_3B_3\\ \mathrm{R}(B,N,T,N) = \sum_{1\le i,j,k,l\le 3} B_iN_jT_kN_l\mathrm{R}_{ijkl} = (l^2-4m)T_3B_3\\ \mathrm{R}(B,N,T,B) = \sum_{1\le i,j,k,l\le 3} B_iN_jT_kB_l\mathrm{R}_{ijkl} = (4m-l^2)T_3N_3\\ \mathrm{R}(B,T,T,B) = \sum_{1\le i,j,k,l\le 3} B_iT_jT_kB_l\mathrm{R}_{ijkl} = (l^2-4m)N_3^2 - \frac{l^2}{4}. \end{array}\right. \tag{11}$$

THEOREM 1 ([3]). *Let $\psi : I \to \mathcal{M}^3_{m,l}$ be a differentiable curve parametrized by arc length. Then ψ is a proper non-geodesic biharmonic curve if and only if*

$$\left\{\begin{array}{l} k = constant \neq 0\\ k^2 + \tau^2 = (4m-l^2)B_3^2 + \frac{l^2}{4}\\ \tau' = (l^2-4m)N_3B_3. \end{array}\right. \tag{12}$$

3.2. Triharmonic curves in $\mathcal{M}^3_{m,l}$

To study the triharmonic curves in $\mathcal{M}^3_{m,l}$, we shall use their Frenet vector fields and equations.

Let us denote by $\psi : I \to \mathcal{M}^3_{m,l}$ an arc length parametrized curve in $\mathcal{M}^3_{m,l}$. Assume that ψ is non-geodesic.

If $r=2t,\ t\geq 1$, then (2) takes the form [7, 12]

$$E_{2t}(\psi)=\frac{1}{2}\int_M<\underbrace{(\mathrm{d}^*\mathrm{d})...(\mathrm{d}^*\mathrm{d})}_{t\ times}\psi,\ \underbrace{(\mathrm{d}^*\mathrm{d})...(\mathrm{d}^*\mathrm{d})}_{t\ times}\psi>v_g. \tag{13}$$

If $r=2t+1$, then (2) takes the form

$$E_{2t+1}(\psi)=\frac{1}{2}\int_M<\mathrm{d}\underbrace{(\mathrm{d}^*\mathrm{d})...(\mathrm{d}^*\mathrm{d})}_{t\ times}\psi,\ \mathrm{d}\underbrace{(\mathrm{d}^*\mathrm{d})...(\mathrm{d}^*\mathrm{d})}_{t\ times}\psi>v_g. \tag{14}$$

The Euler-Lagrange equations of (13) and (14), reduces to the equation

$$\tau_r(\psi)=\nabla_T^{2r-1}T+\sum_{s=0}^{r-1}(-1)^s R(\nabla_T^{2r-3-s}T,\ \nabla_T^s T)T,\ r\geq 1. \tag{15}$$

Solutions of $\tau_r(\psi)=0$ are called r- harmonic curves.

REMARK 1. We say that a $r-$ harmonic curve is proper if it is not harmonic. Any harmonic curve is a $r-$ harmonic curve, for any $r\geq 1$.

An arc length parametrized curve $\psi: I\to M^n$ from $I\subset\mathbb{R}$ to a Riemannian manifold M^n of dimension n is called *triharmonic* if [4]

$$\nabla_T^5 T+R(\nabla_T^3 T,\ T)T-R(\nabla_T^2 T,\ \nabla_T T)T=0. \tag{16}$$

PROPOSITION 2. *Let $\psi: I\subset\mathbb{R}\to M^3$ be a differentiable curve parametrized by arc length. Then ψ is triharmonic curve if and only if*

$$\begin{cases}\xi_1(s)=0\\ \xi_2(s)-\xi_4(s)R(N,T,T,N)-\xi_5(s)R(B,T,T,N)+\xi_6(s)R(B,N,T,N)=0\\ \xi_3(s)-\xi_4(s)R(N,T,T,B)-\xi_5(s)R(B,T,T,B)+\xi_6(s)R(B,N,T,B)=0,\end{cases} \tag{17}$$

where

$$\begin{aligned}
\xi_1(s) &= -10k'k''-5kk^{(3)}+5kk'(2k^2+\tau^2)+5k^2\tau\tau'\\
\xi_2(s) &= k^5+k^{(4)}-15kk'^2-10k^2k''+k\tau^2(2k^2+\tau^2)-6\tau^2k''\\
&\quad -12k'\tau\tau'-3k\tau'^2-4k\tau\tau''\\
\xi_3(s) &= 4\tau k^{(3)}+k\tau^{(3)}-9k^2k'\tau-4k'(\tau^3-\tau'')+6\tau'(k''-k\tau^2)-\tau'k^3\\
\xi_4(s) &= k''-k(2k^2+\tau^2)\\
\xi_5(s) &= 2k'\tau+k\tau'\\
\xi_6(s) &= k^2\tau.
\end{aligned}$$

Proof. From (8) we have

$$\nabla_T^2 T = (-k^2)T + (k')N + (k\tau)B, \tag{18}$$

$$\nabla_T^3 T = (-3kk')T + (k'' - k(k^2+\tau^2))N + (2k'\tau + k\tau')B, \tag{19}$$

$$\nabla_T^5 T = \xi_1(s)T + \xi_2(s)N + \xi_3(s)B, \tag{20}$$

By (16) we see that ψ is a triharmonic curve if and only if

(21)
$$\xi_1(s)T + \xi_2(s)N + \xi_3(s)B + \xi_4(s)R(N,T)T + \xi_5(s)R(B,T)T - \xi_6(s)R(B,N)T = 0.$$

Using (6), we have (17).

□

THEOREM 2. *Let $\psi : I \to \mathcal{M}_{m,l}^3$ be a differentiable curve parametrized by arc length. Then ψ is a proper non-geodesic triharmonic curve if and only if*

(22)
$$\left\{\begin{array}{c} \xi_1(s) = 0 \\ \xi_2(s) + \left((4m-l^2)B_3^2 + \frac{l^2}{4}\right)\xi_4(s) + \xi_5(s)(l^2-4m)N_3B_3 + \xi_6(s)(l^2-4m)T_3B_3 = 0 \\ \xi_3(s) + (l^2-4m)N_3B_3\xi_4(s) - \xi_5(s)\left((l^2-4m)N_3^2 - \frac{l^2}{4}\right) + \xi_6(s)(4m-l^2)T_3N_3 = 0. \end{array}\right.$$

Proof. Combining (11) and (17), it is obtained (22). □

COROLLARY 1. *If $\tau = 0$, $l^2 - 4m \neq 0$ and $N_3B_3 \neq 0$. Then, $k = 0$.*

Proof. From (22), we obtain

$$\left\{\begin{array}{c} \xi_1(s) = 0 \\ \xi_2(s) + \left((4m-l^2)B_3^2 + \frac{l^2}{4}\right)(k'' - 2k^3) = 0 \\ (l^2-4m)N_3B_3(k''-2k^3) = 0. \end{array}\right. \tag{23}$$

Solving system of equations (23) we have $k = 0$.

□

4. Triharmonic helices in $\mathcal{M}_{m,l}^3$

Helix (circular helix) is a geometric curve with non-vanishing constant curvature and non-vanishing constant torsion. Now, for any helix in $\mathcal{M}_{m,l}^3$, the system (22) becomes

$$\left\{\begin{array}{l} (k^2+\tau^2)^2 - (2k^2+\tau^2)\left((4m-l^2)B_3^2 + \frac{l^2}{4}\right) - k\tau(4m-l^2)T_3B_3 = 0 \\ (4m-l^2)N_3((2k^2+\tau^2)B_3 + k\tau T_3) = 0. \end{array}\right. \tag{24}$$

THEOREM 3. *Let* $\psi : I \to \mathcal{M}^3_{m,l}$ *be a non-geodesic triharmonic helix parametrized by arc length. If* $N_3 = 0$*, then* $T_3 = constant$.

Proof. Let ψ be a non geodesic curve parametrized by arc length. Then from the Frenet equations (8) we obtain

$$g_{BCV}(\nabla_T B, e_3) = -\tau N_3.$$

Put $B = B_1 e_1 + B_2 e_2 + B_3 e_3$. Then

$$\begin{aligned}
\nabla_T B &= (B_1' - 2myT_1B_2 + 2mxT_2B_2 + \frac{l}{2}(T_3B_2 + T_2B_3))E_1 \\
&+ (B_2' + 2myT_1B_1 - 2mxT_2B_1 - \frac{l}{2}(T_1B_3 + T_3B_1))E_2 \\
&+ (B_3' + \frac{l}{2}(T_1B_2 - T_2B_1))E_3.
\end{aligned}$$

Using the definition of covariant derivative, we get

$$g_{BCV}(\nabla_T B, e_3) = B_3' + \frac{l}{2}(T_1B_2 - T_2B_1).$$

Since $N_3 = 0$ and $T_1B_2 - T_2B_1 = -N_3$, we have

$$g_{BCV}(\nabla_T B, e_3) = B_3'.$$

Comparing these two equations we have $B_3' = 0$.
From the definition of helix and (24), we have $T_3 = constant$. □

THEOREM 4. *Let* $\psi : I \to \mathcal{M}^3_{m,l}$ *be a non-geodesic triharmonic helix parametrized by arc length. If* $N_3 \neq 0$*, then*

$$\tau^2 = \frac{l^2 - 8k^2 - |l|\sqrt{l^2 - 16k^2}}{8}, \quad or, \quad \tau^2 = \frac{l^2 - 8k^2 + |l|\sqrt{l^2 - 16k^2}}{8}. \tag{25}$$

Proof. Since $N_3 \neq 0$, from (24), we obtain

$$N_3(4(k^2 + \tau^2)^2 - (2k^2 + \tau^2)l^2) = 0. \tag{26}$$

Hence (26) implies (25). □

5. General helix in $\mathcal{M}^3_{m,l}$

In 1845, de Saint Venant first proved that a space curve is a general helix if and only if the ratio of curvature to torsion be constant (see [10] for details).

DEFINITION 1. *Let ψ be a curve in $\mathcal{M}^3_{m,l}$ and $\{T,N,B\}$ be the Frenet frame on $\mathcal{M}^3_{m,l}$ along ψ.*

1) *If both k and τ are constant along ψ, then is called circular helix with respect to Frenet frame.*

2) *A curve ψ such that*

$$\frac{\tau}{k} = c,\ c \in \mathbb{R}, \tag{27}$$

is called a general helix with respect to Frenet frame.

If k =constant $\neq 0$ and $\tau = 0$, then the curve ϕ is a circle.

THEOREM 5. *Let $\psi : I \to \mathcal{M}^3_{m,l}$ be a non-geodesic triharmonic general helix parametrized by arc length. If $N_3 = 0$, then ψ is a circular helix.*

Proof. From (27), we have

$$\begin{cases} \xi_1(s) = -10k'k'' - 5kk^{(3)} + 10k^3k'(c^2+1) \\ \xi_2(s) = k^5(c^2+1)^2 + k^{(4)} - 15kk'^2(c^2+1) - 10k^2k''(c^2+1) \\ \xi_3(s) = -c\xi_1(s) \\ \xi_4(s) = k'' - k^3(c^2+1) \\ \xi_5(s) = 3ck'k \\ \xi_6(s) = ck^3. \end{cases} \tag{28}$$

By using equations (28) in (22), equation (22), we can obtain a system of three differential equations characterizing triharmonic general helix in $\mathcal{M}^3_{m,l}$

$$\begin{cases} \xi_1(s) = -10k'k'' - 5kk^{(3)} + 10k^3k'(c^2+1) = 0 \\ \xi_2(s) + ((4m-l^2)B_3^2 + \frac{l^2}{4})\xi_4(s) + (l^2-4m)B_3(N_3\xi_5 + T_3\xi_6) = 0 \\ \xi_5(s)((4m-l^2)N_3^2 + \frac{l^2}{4}) + (l^2-4m)N_3(B_3\xi_4(s) + \xi_6(s)T_3) = 0. \end{cases} \tag{29}$$

Substituting $N_3 = 0$ into the third equation in (29) we have $k'k = 0$, which implies k =constant and hence τ =constant. Then ψ is a circular helix.

□

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