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GINO FANO'S LATE INVESTIGATIONS ON FANO-ENRIQUES THREEFOLDS

Abstract. This paper investigates Gino Fano's contributions to the study and classification of a special class of three-dimensional algebraic varieties: the so-called Fano-Enriques threefolds (i.e. Fano threefolds whose general hyperplane section is an Enriques surface). We focus on his 1938 memoir with the aim of analysing it from an historical-critical perspective, also in the light of its quite limited reception at that time and the subsequent rediscovery in the 1980s. Besides the weak links with foreign mathematical traditions, we consider the positioning of Fano's work in the Italian geometric context. Starting from this case-study, we discuss some distinctive features of the 'late' Italian School of algebraic geometry which, combined with a scenario of cultural autarky, determined its decline.

1. The historical issue.

Gino Fano was an outstanding mathematician of his time and a significant protagonist of the so-called Italian School of algebraic geometry. Born in Mantua in 1871, his life mostly developed in Turin where he came as university student in 1888 and where he was full professor of projective and descriptive geometry from 1901 to 1938.[1] His name is associated to the golden age of Italian algebraic geometry for his realizations in several research fields: foundations of geometry, algebraic curves, continuous groups in projective and birational geometry, line geometry, K3 and Enriques surfaces and their automorphisms. Despite the wide spectrum of fields of study, the figure of Fano is specially linked to the study of special three-dimensional algebraic varieties, known today as Fano threefolds, i.e. smooth projective varieties V whose anticanonical system $|-K_V|$ is ample or, equivalently, whose first Chern class is ample. Starting from the analysis of the cubic hypersurfaces in the complex four-dimensional projective space, Fano opened a new line of investigations. In an effort to prove the irrationality of some of these threefolds, he dealt with their classification for over forty years, playing a pioneering role.

Gino Fano's contribution to the study of another class of varieties, the so-called Fano-Enriques threefolds, is lesser known but equally interesting from the historical point of view. Fano-Enriques threefolds are irreducible three-dimensional varieties V' whose general hyperplane sections are Enriques surfaces[2] – i.e. algebraic surfaces S with irregularity $q = 0$, such that $K_S \neq 0$ and $2K_S = 0$ in $Pic(S)$, where K_S is the canonical divisor associated with S and $Pic(S)$ is its Picard group – and such that V' is not a cone over S.[3]

[1] On Fano's biography see [13].

[2] At that time, they were merely designated as 'surfaces of genus zero and bigenus one', as expressed in the title of Fano's memoir.

[3] This definition can be rewritten in more modern terms as follows: a threefold V' is a Fano-Enriques threefold of genus $-\frac{1}{2}K_{V'}^3+1$ and degree $-K_{V'}^3$ if it has canonical singularities, $-K_{V'}$ is not a Cartier divisor

Fano firstly faced the study of Fano-Enriques threefolds in 1938, in the paper *Sulle varietá algebriche a tre dimensioni le cui sezioni iperpiane sono superficie di genere zero e bigenere uno*, published in the *Memorie della Societá dei XL*. Here he proved that the general V' is birational to a Fano threefold and he gave a geometric classification of these threefolds, which was however based on a restrictive hypothesis on the singular points of V'. Even if Fano's work did not bring to definitive results, it exerted an influence on following research in algebraic geometry and it deserves to be examined for two basic reasons. Firstly, the memoir *Sulle varietá algebriche...* was the last publication by Fano (who was a Jew) before the enactment of racial laws and his subsequent relocation in Switzerland. Consequently, its study opens a new perspective over the last period of Fano's life (from 1939 to 1950), a phase very little studied till now. Secondly, this paper is highly informative on the development of the Italian School of algebraic geometry because on one hand it represents a good synthesis of the culture of this research School and on the other hand it is a litmus paper of its decline in the Thirties.

2. Introduction to the study of Fano-Enriques threefolds.

The starting point of Fano's study is a result by Lucien Godeaux [45, p. 134]:

> Every three-dimensional normal algebraic variety, non-cone, whose hyperplane sections are regular surfaces of genus zero and bigenus one with bicanonical curve of order zero ($p_a = p_g = 0$, $P_6 = 1$) contains a linear system of surfaces of genera one ($p_a = p_g = P_2 = 1$) whose dimension increased by one unit equals the dimension of the ambient space. [33, p. 41] [4]

Let p be the genus of the curve-sections. Assuming that the linear system of these curves is complete in relation to the genus over the surface-sections of W_3^{2p-2}, this variety is a Fano-Enriques threefold $W_3^{2p-2} \subset \mathbb{P}^p$ of order $2p-2$ and its curve-sections are non-special normal curves $C_p^{2p-2} \subset \mathbb{P}^{p-1}$. W_3^{2p-2} contains a linear system of dimension $p-1$ and degree $2p-6$: it is composed by surfaces ϕ of genus one and it has curve-intersections of genus $p-2$. Godeaux had proved that the surfaces that are hyperplane sections F^{2p-2} of W_3^{2p-2} contain, in addition to the linear system $|C_p^{2p-2}|$ of curves of their hyperplane sections, the linear adjoint system $|\gamma_p^{2p-2}|$ to the first, having the same characters, deriving that $|2C| = |2\gamma|$. Fano followed this approach but added:

and $-K_{V'} \sim_{\mathbb{Q}} H$ for some ample Cartier divisor H on V'. Moreover, it must be remembered that an Enriques surface is the quotient of a K3 surface by a group of order two without fixed points. In the same way, a Fano-Enriques threefold is the quotient of a Fano three-dimensional variety by an involution with eight fixed points.

[4] Ogni varietá algebrica normale a tre dimensioni, non cono, le cui sezioni iperpiane sono superficie regolari di genere zero e bigenere uno con curva bicanonica di ordine zero ($p_a = p_g = 0$, $P_6 = 1$) contiene un sistema lineare di superficie di generi uno ($p_a = p_g = P_2 = 1$) la cui dimensione aumentata di un'unitá eguaglia quella dello spazio ambiente.

> this result may be better detailed [...] as well as further developed, determining the different cases of the variety in question, that is the individual values of p (very few) whereby these varieties W_3^{2p-2} actually exist. [33, p. 42] [5]

The goal of his memoir is therefore to find

> the representations of these varieties [...] in S_3, and thus all the so far unknown birational distinct types of simple linear systems of surfaces with genus zero and bigenus one, having a bicanonical curve of order zero. [33, p. 42] [6]

Under the assumptions of the theorem stated by Godeaux, p must be[7] greater than or equal to 4. Fano distinguished between the case $p = 4$, which corresponds to an Enriques threefold, and the cases $p \geq 5$. To analyse them, he built up a peculiar birational map between $W_3^{2p-2} \subset \mathbb{P}^p$ and a Fano threefold of lower order $M_3^{2p-6} \subset \mathbb{P}^{p-1}$. This construction is the mainstay of his memoir, even if Fano did not provide it explicitly, nor he proved its birationality. The result would have been rigorously demonstrated only in [16] for $p > 5$ under certain hypothesis, some of which, although via purely geometric method, had been stated by Fano himself.
From the properties of M_3^{2p-6}, it is possible to deduce the characteristics of the corresponding Fano-Enriques threefolds, thus outlining their classification. With this in view, Fano supposed that the linear system $|\phi|$ consisting of the surfaces of genus one was 'simple', namely he assumed that the passage of the surface for a generic point of the variety did not involve its passage for other variable points too. In that case,

> The system $|\phi|$ will represent in the usual sense a variety M_3^{2p-6} in S_{p-1}, referable to a variety W_3^{2p-2}, with surface-sections of genera one, which are images of ϕ, and canonical curve-sections of genus $p-2$ (in S_{p-3} spaces). [33, p. 42] [8]

On every hyperplane section F^{2p-2} of W_3^{2p-2}, $|2C| = |2\gamma|$ and therefore the linear systems $|2F|$ and $|2\phi|$ cut into equivalent curves over the hyperplane sections. Consequently, $|2F| = |2\phi|$ over W_3^{2p-2}, up to fundamental surfaces of the system $|F|$. But these fundamental surfaces of W_3^{2p-2} shall be multiple points, whose sum of multiplicities is equal to 32. Identifying with $2f$ and 2φ the images of $2F$ and 2ϕ in M_3^{2p-6},

[5] Questo risultato puó essere meglio precisato [...], nonché ulteriormente sviluppato, determinando i diversi casi delle varietá in parola, cioé i valori singoli di p (pochissimi) per cui tali varietá W_3^{2p-2} effettivamente esistono.

[6] Le rappresentazioni di queste varietá [...] sullo spazio S_3, e quindi tutti i tipi birazionalmente distinti, finora non noti, di sistemi lineari semplici di superficie di genere zero e bigenere uno, a curva bicanonica di ordine zero. Fano indicated by S_n the projective complex n-dimensional space $\mathbb{P}^n$.

[7] Indeed, if $p = 3$ the variety W_3^{2p-2} would be made up of a quadruple projective space $\mathbb{P}^3$.

[8] Il sistema $|\phi|$ rappresenterá allora nel consueto senso una varietá M_3^{2p-6} di S_{p-1}, riferibile a W_3^{2p-2}, con superficie sezioni di generi uno, immagini delle ϕ, e curve sezioni canoniche di genere $p-2$ (in spazi S_{p-3}).

the difference between their orders (which is equal to 8) will be the total order of the surfaces of M_3^{2p-6} images of the multiple points of W_3^{2p-2}. Due to the birational invariance of the genus, the fundamental surfaces of $|f|$ laying down in M_3^{2p-6} shall intersect 2φ in curves of genus zero. For this to happen, the relevant surfaces must necessarily be eight planes. So, Fano concluded:

> Over M_3^{2p-6} the difference $2f-2\varphi$ consists of eight fundamental planes of $|2f|$; over W_3^{2p-2} the corresponding difference $2F-2\phi$ consists of eight quadruple points (with multiplicity $\frac{32}{8}=4$), which are images of those planes. The variety W_3^{2p-2} with surface-sections of genus zero and bigenus one, thus has eight quadruple points; this fact of a W_3 bound only to have surface-sections of a particular type, resulting in certain isolated points, is definitely noteworthy. [33, p. 44] [9]

In addition, each of the eight planes of M_3^{2p-6} can be put in correspondence with the neighbourhood of the relative quadruple point of W_3^{2p-2}. In doing so, the conics identified by the surfaces 2φ on the plane correspond to sections with surfaces 2ϕ in the given neighbourhood, that is with quadrics passing simply through the given point. The cones tangent to W_3^{2p-2} at the eight quadruple points are cones in $\mathbb{P}^6$ that project Veronese surfaces. Fano then remarked that two of the eight planes could not have a line in common, whereas they could have a point in common. If this occurs, the aforementioned point is a double point of M_3^{2p-6} and the line through the two corresponding quadruple points of W_3^{2p-2} lies in W_3^{2p-2}. In the present case, Fano called 'congiunti' (modernly, 'associated') the two points involved. If two of these singular points P_i and P_j ($i,j=1,\ldots,8$ and $i\neq j$) are associated, the corresponding planes π_i and π_j of M_3^{2p-6} have a point in common which is the contraction of the line P_iP_j. The first part of the memoir is particularly important since, ahead of time, Fano identified the birational map λ between W_3^{2p-2} and M_3^{2p-6} that, in contemporary terms, makes the following diagram commutative.

$$\begin{tikzcd} & \widetilde{W} \arrow[dl] \arrow[dr] & \\ W_3^{2p-2} \subset \mathbb{P}^p \arrow[rr, dashed, "\lambda"'] & & M_3^{2p-6} \subset \mathbb{P}^{p-1} \end{tikzcd}$$

Here $\tilde{W}$ is the blow-up of a Fano-Enriques threefold in its eight singular points, whose images under the birational map are the eight planes over M_3^{2p-6}.

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[9]Sopra M_3^{2p-6} la differenza $2f-2\varphi$ é costituita da 8 piani, fondamentali per $|2f|$; sopra W_3^{2p-2} la differenza corrispondente $2F-2\phi$ é costituita da 8 punti quadrupli (di multiplicitá $\frac{32}{8}=4$), immagini di quei piani. La varietá W_3^{2p-2} a superficie sezioni di genere zero e bigenere uno ha dunque 8 punti quadrupli; e questo fatto di una W_3 vincolata soltanto ad avere superficie sezioni di un tipo determinato, e che ha di conseguenza certi punti multipli isolati, é senza dubbio particolarmente notevole.

3. The classification of Fano-Enriques threefolds for $6 \leq p \leq 13$.

In his paper of 1938, Fano limited himself to the analysis of the general case in which the possible pairs of incident planes between the eight planes of M_3^{2p-6} intersect in different points and every plane 'behaves' in the same way with the remaining ones. Fano immediately eliminated the case $p = 5$, although with a not completely rigorous argument: if $p = 5$ then a variety $W_3^8 \subset \mathbb{P}^5$ occurs; it would be projected from two of its quadruple points, even if they were associated, in a simple space $\mathbb{P}^3$ with at least six double points, which is not possible.[10]

The first case discussed in detail by Fano is $p = 6$: the Fano variety $M_3^6 \subset \mathbb{P}^5$ corresponding to $W_3^{10} \subset \mathbb{P}^6$ has canonical curve-sections of genus 4 so that it turns out to be the intersection of a quadric Q and a cubic hypersurface C of $\mathbb{P}^5$ which contains eight planes intersecting two-by-two. Thanks to some results previously obtained, Fano shows that Q is not singular (in particular, it is not a cone), and that M_3^6 is rational. The hyperplane sections of W_3^{10} are Enriques surfaces of a special type: the so-called Reye congruences. More than 30 years before, Fano himself had realised the first that Reye congruences are Enriques surfaces [23, pp. 77-78], relying on the representation of congruences of lines by surfaces in $\mathbb{P}^5$. The variety W_3^{10}, that is obtained by parametrising the dual conics of $\mathbb{P}^3$ degenerating into a couple of points, had already been partially studied by him in a paper published in the *Rendiconti del Circolo matematico di Palermo* in 1910. At his turn Godeaux had analysed a special case in [44, pp. 920-922].

There follows the treatment of case $p = 7$: here $M_3^8 \subset \mathbb{P}^6$ is the complete intersection of three quadric hypersurfaces and has canonical curve-sections of genus 5. On the basis of his former production [30–32], the author is able to exclude the possibility that all the eight planes of M_3^8 intersect two-by-two. As a consequence, over the corresponding $W_3^{12} \subset \mathbb{P}^7$ each of the eight points is not associated to at least one of the remaining. From every pair of points of this kind, W_3^{12} is projected in a variety of $\mathbb{P}^5$ with six double points. Hence, each of the eight planes of M_3^8 intersects exactly other six; this means that the eight planes can be divided into four pairs, in such a way that two of them intersect or are skew, depending on whether they belong or not to different pairs. Using a result stated in his 1937 paper *Su alcune varietá algebriche a tre dimensioni aventi curve sezioni canoniche* [31] Fano proved that, if a threefold of order 8 in $\mathbb{P}^6$ with canonical curve-sections contains a plane, the variety can be projected from this plane to $\mathbb{P}^3$. Thus, there is a correspondence between its hyperplane sections and surfaces ψ^4 of the 4^{th} order, passing through a curve η_9^9 of order 9, which is in turn contained in a ψ^3. By applying this property to M_3^8, it follows that the curve η_9^9 splits into a cubic curve δ and the six edges of a tetrahedron, whose vertices are, respectively, the projections of the four planes. As for M_3^8 also the existence of the variety W_3^{12} derives directly from the correspondent linear system by which is represented in $\mathbb{P}^3$. Starting out from the example of a particular involution, Fano reached the following conclusion: the variety W_3^{12} is the image of $\mathbb{P}^3$ by the linear system of all sextic surfaces containing

[10] If the hyperplane sections are not singular, it is possible to justify this statement in another way: under some basic assumptions, the degree d of a smooth Enriques surface in $\mathbb{P}^4$ must satisfy the equation $d^2 - 10d + 12 = 0$. Since this equation has no integer solution, it follows that $p > 5$.

a fixed plane cubic curve and going doubly through the edges of a tetrahedron.
As for the cases $p > 7$ are concerned, Fano observed that each of the eight planes of M_3^{2p-6} intersects at most four of the others. Firstly, he assumed that each plane intersects exactly four of the others, thus being skew with the remaining four. This implies that the eight planes can be divided into two quadruples, in such a way that two planes of the same quadruple are not intersecting, whilst two planes of different quadruples necessarily intersect. Fano asserted:

> All the 8 planes lay in a space Σ of dimension ≤ 8, identified by 3 of them of the same quadruple (because that space clearly contains all planes of the other quadruple, and the further plane of the first quadruple as a consequence). Indeed, I say that this system of 8 planes and the variety M_3^{2p-6} itself must belong to the space S_8, so that it will be $p-1=8$, $p=9$. [33, p. 54] [11]

However, he did not strictly justify this assertion but concentrated to show that the two inequalities $p \geq 9$ and $p \leq 9$ occur. To prove the first, he projected the variety W_3^{2p-2} from three points of the same quadruple. The opposite inequality was obtained showing that the space Σ is not a hyperplane of $\mathbb{P}^{p-1}$, neither is contained in it. As for the case $p = 9$ is concerned, a variety $M_3^{12} \subset \mathbb{P}^8$ is obtained, already taken into account by Fano in 1936, which is the intersection of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ with a quadric containing four planes. It is rational since it contains two congruences of conics with unisecant planes. Relying on the equivalence relations between different systems of surfaces appearing in the construction, Fano showed that there exists a linear system over M_3^{12}, composed by regular surfaces of genus zero and bigenus one. Such system allows to represent M_3^{12} as a $W_3^{16} \subset \mathbb{P}^9$ provided of eight quadruple points, images of the eight planes of M_3^{12}. These quadruple points are divided into two quadruples, such as two points of the same quadruple are always not associated. As a consequence, taking into account three points of the same quadruple, it is possible to project the threefold W_3^{16} univocally onto $\mathbb{P}^3$: in doing so, the image of the six projection centres will consist of six planes, split into two trihedrons, the images of the other quadruple points of W_3^{16} being the two vertices of the trihedrons. Conversely, the system consisting of the surfaces of order 4 passing simply through the edges of the two trihedrons and having their vertices as double points represents the $M_3^{12} \subset \mathbb{P}^8$. Fano concluded the analysis of the case $p = 9$ acknowledging that, as early as 1930, Godeaux had studied a threefold $V_3^{16} \subset \mathbb{P}^9$, image of the variety ∞^3 made up by the pairs of the involution of $\mathbb{P}^3$ defined by $x_i' = \frac{1}{x_i}$. However, he claimed the major generality of his work, noting that Godeaux's V_3^{16} is a special case of the variety W_3^{16}, having three double points and a hyperplane section consisting of four Veronese surfaces [44, p. 917].
Pursuing the study, Fano pointed out that over each other $W_3^{2p-2} \subset \mathbb{P}^p$ three of the eight quadruple points are not associated two-by-two and the fourth is associated to no

[11] Gli 8 piani stanno tutti in uno spazio Σ di dimensione ≤ 8, determinato da 3 fra essi di una stessa quaderna (poiché tale spazio contiene evidentemente tutti i piani dell'altra quaderna, e per conseguenza anche il piano ulteriore della prima quaderna). Dico anzi che questo sistema di 8 piani e la stessa varietá M_3^{2p-6} in parola devono appartenere allo spazio S_8, sicché sará $p-1=8$, $p=9$.

more than one of the first. Considering the projection of the variety W_3^{2p-2} from these four points, Fano concluded that necessarily $p \geq 11$.[12] In this case, the corresponding four planes of M_3^{2p-6} non-intersecting two-by-two lay in a projective space $\mathbb{P}^{11}$ which also contains the other four planes and six quadrics. They make up a surface of order 20, with curve-sections of genus 11. Since this surface must be a complete section of $M_3^{2p-6} \subset \mathbb{P}^{p-1}$, the dimension of the ambient space must be ≤ 12, hence $p \leq 13$. If $p = 13$, then a $W_3^{24} \subset \mathbb{P}^{13}$ is found, projecting from its eight quadruple points in a $W_3^4 \subset \mathbb{P}^5$ containing eight planes, images of the quadruple points. Each plane intersects three of the other seven planes in lines, other four in a single point, and it is skew in relation to the remaining. This W_3^4 is identified in $\mathbb{P}^5$ by the linear system of quadric hypersurfaces circumscribed in a tetrahedron, whose faces and vertices are images of the eight planes of W_3^4. Fano then concluded that there exists a $W_3^{24} \subset \mathbb{P}^{13}$ of this kind, represented in $\mathbb{P}^3$ by the linear system of surfaces of order 6 going doubly through the edges of a tetrahedron. The corresponding variety $M_3^{20} \subset \mathbb{P}^{12}$ is rather identified by the system of surfaces of the 4^{th} order passing simply through the edges of the same tetrahedron. Afterwards, under the previous conditions, Fano excluded the existence of the W_3^{2p-2} and of the corresponding M_3^{2p-6} for $p = 11, 12$. Nevertheless, varieties of this type are not included in Sano and Bayle's modern classification [1,65] and their existence is still today an open problem. Fano then aimed to demonstrate that

> Except for the cases already considered, there do not exist other M_3^{2p-6} of S_{p-1}, containing 8 planes, or W_3^{2p-2} of S_p, respectively, with 8 quadruple points, as requested. [33, p. 60] [13]

Despite being the weakest part of the work and containing flawed results (for instance, varieties of this type actually exist for $p = 8$ and $p = 10$) this section of the 1938 memoir is significant for the purely synthetic approach, so dear to the Italian School of algebraic geometry, with whom Fano addressed the issue. Firstly, he excluded that there were other cases in which every plane of M_3^{2p-6} intersected three of the remaining ones, as it would be $p = 13$. Then he tried to show that each of the eight planes could not meet only one or two of the others, using daring, and sometimes reckless, projective arguments.

4. A special case: the Enriques threefold $W_3^6 \subset \mathbb{P}^4$.

Two cases still have to be examined. The first is $p = 4$, in which W_3^6 is a classic Enriques threefold, while the variety $M_3^2 \subset \mathbb{P}^3$ is the double projective space $\mathbb{P}^3$.
The second consists of the eventual cases $p > 4$ in which the linear system $|\phi|$ representing M_3^{2p-6} belongs to an involution: since this cannot happen, this is immediately

[12] Here Fano's argument is not entirely accurate: he states that W_3^{2p-2} is projected in a variety in $\mathbb{P}^{p-4}$ of order $\geq 2p - 2 - 15 = 2p - 17$ from the four points previously considered. For this reason, it must be $p - 4 \geq (2p - 17) + 3 - 1$, namely $p \geq 11$. This conclusion turns out to be wrong.

[13] All'infuori dei casi giá considerati, non esistono altre M_3^{2p-6} di S_{p-1}, contenenti 8 piani, o rispettivamente W_3^{2p-2} di S_p con 8 punti quadrupli, come richiesto.

excluded.
If $p = 4$, the variety $W_3^6 \subset \mathbb{P}^4$ with surface-sections $F^6 \subset \mathbb{P}^3$ of genus zero and bigenus one, goes doubly through the edges of a tetrahedron. W_3^6 has six double planes which are obtained by the intersections, two-by-two, of four spaces $\mathbb{P}^4$ and all of them pass through the same point. Fano chose the latter as fundamental point [0] of the coordinate system and assumed that the four spaces $\mathbb{P}^4$ are identified by the equations $x_i = 0$ $(i = 1, \dots, 4)$. According to a remark formulated by Godeaux [45], W_3^6 is defined by an equation of the type:

$$x_1x_2x_3x_4 \left\{x_0^2 + 2x_0f_1(x_1,x_2,x_3,x_4) + f_2(x_1,x_2,x_3,x_4)\right\} +$$

$$+\varphi_2(x_2x_3x_4, x_1x_3x_4, x_1x_2x_4, x_1x_2x_3) = 0$$

where f_1 is a linear homogeneous polynomial in x_i, f_2 is a second order homogeneous polynomial in x_i and φ_2 is a quadratic form containing terms with just squares of the expressions $x_2x_3x_4, \dots, x_1x_2x_3$. [0] is a quadruple point of W_3^6 such as the lines departing from it cut pairs of an involution I_2 over the variety. Besides, W_3^6 is projected from this point in a double space $\mathbb{P}^3$, that we can assume defined by the equation $x_0 = 0$, having a branched surface of order 10. It is composed of four planes, traces of the spaces $x_i = 0$ $(i = 1, \dots, 4)$, and of the surface F^6, intersection between W_3^6 and the space $x_0 + f_1 = 0$. Moreover, the spaces $\mathbb{P}^3$ going through [0] intersect W_3^6 in surfaces F^6 of genus zero and bigenus one, having [0] as quadruple point and six double lines passing through this point.
At this stage, Fano claimed that W_3^6 is not rational and not even unirational. Fano was rather correct in claiming the non-rationality of W_3^6; however, the path towards a rigorous proof according to modern standards would be troubled. Leonard Roth, after proving in 1955 the unirationality [64, p. 97], will try to demonstrate the non-rationality of the generic Enriques threefold W_3^6 giving an argument involving Severi torsion. A short time later, J.P. Serre (1959) and J.A. Tyrrel (1960) will criticize Roth's argument, actually invalidating it. Only in the 1980s, the definitive proof of the irrationality of W_3^6 has been given, using more advanced tools than those utilized by Fano.
Fano's 1938 paper ends with a synthesis of its three key points: there exists a birational map between W_3^{2p-2} and M_3^{2p-6}; Fano-Enriques threefolds are always rational for $p \geq 6$; the list of these special varieties is complete. Unfortunately, the proofs of all the statements are incomplete, and the last statement is wrong. Fano's results about the rationality/non-rationality would have been rigorously demonstrated more than fifty years later, the existence of additional Fano-Enriques threefolds compared to those listed in 1938 would be proven from the 1980s onward, while the question of providing a final classification in the general case is still open.

5. Fano's later studies connected with Fano-Enriques threefolds.

Fano examined V_3^{10}, a very special case of Fano-Enriques threefold in $\mathbb{P}^6$ whose hyperplane sections consist of four planes and three quadric surfaces, in the paper *Osservazioni varie sulle superficie regolari di genere zero e bigenere uno*, published in

1944 in the *Revista de Matemàtica y física teórica* of Tucumàn, edited by his former colleague and friend A. Terracini. Here, Fano dealt with the issue from a completely different perspective with respect to the 1938 memoir. His study started from the analysis of the general Enriques surface of order six F^6 having the edges of a tetrahedron as double lines. He observed that its equation can be written as

$$x_1x_2x_3x_4f(x)+x_2^2x_3^2x_4^2+x_1^2x_3^2x_4^2+x_1^2x_2^2x_4^2+x_1^2x_2^2x_3^2=0$$

where $f(x)$ is a quadratic form of x_i $(i=1,\dots,4)$, whose ten coefficients can be considered as parameters (the so-called 'moduli'). Moreover, F^6 is represented over a surface of order 10 $F^{10}\subset\mathbb{P}^5$, in a way that the six double lines of F^6 correspond to six plane cubic curves over F^{10}. Fano then passed to analyse some particular cases of F^6, depending on less than ten moduli. To do this, he introduced the linear system Σ of dimension three made up of quadric surfaces. Considering the lines that belong to a whole pencil of quadrics of Σ they constitute a Reye congruence which represents a surface F^6 depending on nine moduli. If Σ holds a pencil of cones with the same vertex, the lines through this point and contained in the unique tangent plane passing through it correspond to a F^6 with eight moduli. Σ, however, could contain more than a pencil of cones, up to a maximum of ten pencils. In this case, the Enriques surface F^6 depends on four moduli and the corresponding F^{10} has ten lines intersecting two by two. In the second part of the paper, Fano introduced the linear system of adjoint cubic surfaces $|F^3|$ that would allow him to come to the Fano-Enriques threefold $V_3^{10}\subset\mathbb{P}^6$. Within the system $|F^3|$, there is a birational involutory transformation given by $u_i'=\frac{1}{u_i}$ and, more specifically, two homologous F^3 form together a special Enriques surface having the same double lines of the starting F^6. At the same time, $|F^3|$ intersects the surface F^6 in skew curves of order 4, equally concerned in the aforementioned involution. Fano called Λ the system of quadric surfaces that join the homologous pairs of curves (which is strictly related to the Fano-Enriques threefold) and focused his attention on the reducible F^3. These cubic surfaces 'break' into a face of the fundamental tetrahedron and a cone from the opposite vertex or, alternatively, into two faces of that tetrahedron and a plane passing through the opposite edge. Denoting with $|L|$ the linear system of all the quadrics which are circumscribed in the fundamental tetrahedron, Fano pointed out that the quadric surfaces of Λ contained in $|L|$ laid down, in turn, over the reducible F^3. Hence, the two types of F^3 previously found constitutes the complete intersection of Λ with the linear system $|L|$. But this means that Λ, seen as system of dimension three of quadric surfaces, is a variety of order 10, namely the $V_3^{10}\subset\mathbb{P}^6$. This different construction allowed Fano to conclude that the general hyperplane section of the Fano-Enriques threefold considered, made up of four planes and three quadrics, is the image of the two-dimensional systems of quadric surfaces in common to Λ and $|L|$. This achievement represents the most significative element of novelty in the 1944 work that ends with two results concerning Enriques surfaces of order 6: there exists 56 planes which intersects F^6 in pairs of cubic curves; the general F^6 contains 20 plane cubic curves which are adjoint two by two.

Alongside the 1944 publication, the lecture *Les transformations de contact birationelles dans le plan* that Fano held at the *Cercle Mathématique* of Lausanne on

February 10^{th}, 1944, upon the invitation of G. De Rham[14], is significant in connection with his late research interests. As far as Fano-Enriques threefolds are concerned, the drafts of the conference show the permanence of Fano's interest in the key concept of birational transformation, that he had extensively exploited in the 1938 memoir. At the *Cercle*, Fano introduced the concept of 'linear element': it is composed by a point of coordinates (x,y) and a line through this point in its neighborhood with angular coefficient $p = \frac{dx}{dy}$. A 'union' is the ensemble of these elements such that the curve locus of the points (x,y) and the envelope of the corresponding lines coincide. All this is necessary to introduce the contact birational transformations of the plane, which are algebraic bijective maps sending unions in unions. From a modern point of view, denoting with $\mathbb{P}\Omega^1_{\mathbb{P}^2}$ the projectivized cotangent bundle of $\mathbb{P}^2$, these transformations are birational automorphisms a such that $p \cdot a = p \cdot b$, where $p : \mathbb{P}\Omega^1_{\mathbb{P}^2} \to \mathbb{P}^2$ is the natural projection and $b \in Bir(\mathbb{P}^2)$. It is worth noting that $\mathbb{P}\Omega^1_{\mathbb{P}^2}$ is a three-dimensional variety, since it is the projectivization of a rank 2 vector bundle over $\mathbb{P}^2$. Hence, p turns out to be a map between a threefold and $\mathbb{P}^2$. Accordingly, a contact birational transformation of the plane is a peculiar automorphism of such threefold. In Lausanne, Fano dealt with two main issues: to characterize the systems of curves that, under these special transformations, correspond to two lines and to determine the decompositions of such maps into products of simpler operations. Besides the mathematical content, this lecture helps to highlight that in Switzerland Fano resumed the thread of his mathematical investigations in the field, as declared on this occasion.[15]

6. A first glimpse to the late Fano.

Fano's biography is quite well known as far as the youth phase is concerned: the studies in Turin, under the guidance of C. Segre and G. Castelnuovo, the post-doc stay in Germany in 1893-94, F. Klein's invitation to occupy a chair in Göttingen in 1899. On the contrary, his maturity is generally liquidated by saying that since 1901, after his arrival in Turin, Fano devoted himself for 37 years consecutively to the teaching of descriptive and projective geometry, and to research in classical algebraic geometry. Even more fragmentary is the reconstruction of his personal and professional life after 1938 when Fano's trajectory crossed the dramatic period of racial persecutions and war. On the grounds of racial laws, Fano was removed from his post in November 1938. Unable to tolerate the loss of rights and the collapse of the three pillars of his life (family, country, profession) he decided to leave Italy and moved to Switzerland in January 1939, settling in Lausanne. According to the recollections of his sons Roberto [36, p. 3] and Ugo [37, p. 137], at that moment he interrupted his scientific and teaching activities.
Previous historical research has already documented that this information was unproper. Fano resumed lecturing in November 1943, when the engineer Gustavo Colonnetti, his friend and former colleague, charged of the direction of the Lausanne Interment Camp for university Italian students, asked him to deliver the courses in descrip-

[14] Fano Archive, Special Mathematical Library, University of Turin, *Scritti. 3*, fols. 1-6.
[15] Fano Archive, Special Mathematical Library, University of Turin, *Scritti. 3*, fol. 2.

tive and projective geometry [54].
The line of geometrical investigation opened by Fano in 1938 and developed in the paper *Osservazioni varie sulle superficie regolari di genere zero e bigenere uno* and in the lecture *Les transformations de contact birationelles dans le plan*, allow to claim that in Lausanne Fano not even interrupted research activity: on the contrary Fano threefolds and Fano-Enriques threefolds were the main (namely the unique) new subject of research for him during the years of World War II. In this sense, the 1938 memoir marked the top of his scientific output, all his next engagement being aimed at improving and extending some points and fable aspects of his previous works on three dimensional varieties. Fano continued to work on these themes also after returning to Italy in May 1945 and until the death, so much so that B. Segre wrote in his obituary:

> We are aware that Fano was planning on drafting a global exposure of these last works, in which, hopefully, some hypotheses of work (it is not known whether and to what extent restrictive ones) made by him should have been removed [...]. However, we do not know to what extent he was able to carry out this project; and it would undoubtedly be of the greatest interest if, among His papers, something conclusive in this regard could be found. [67, p. 263] [16]

Recent analysis of some manuscripts kept in Fano's archive at the Turin University endorses such first-hand testimony [17]. Although incomplete and disordered, these notes show Fano's commitment to the study of birational models of Fano-Enriques threefolds. In particular, among the three-dimensional varieties here considered, there are four Fano threefolds birationally equivalent to the Fano-Enriques varieties obtained for $p = 4, 6, 7, 9$. In addition to describing their individual geometries, Fano focuses on the invariant Ω_2, the virtual arithmetic genus of the canonical surface. For these four M_3 - with a singular parallelism with the value p of the corresponding Fano-Enriques threefold - it is respectively equal to $4, 6, 7, 9$.
Fano was probably not satisfied by his results on Fano-Enriques threefolds. As a matter of fact, when in 1950 the Turin Mathematical Seminar organized a jubilee conference on the occasion of his being declared emeritus, Fano was invited to deliver a lecture on his main scientific achievements along forty years of academic activity and he merely touched on Fano-Enriques threefolds. Only while he was talking of the rationality of $M_3^8 \subset \mathbb{P}^6$, Fano mentioned the argument through which he had justified the rationality of the Fano-Enriques threefold in the case $p = 7$ [35, pp. 25 and 29].

[16] Ci consta che il Fano aveva in animo di redigere un'esposizione d'assieme di questi ultimi lavori, nella quale, possibilmente, avrebbero dovuto essere rimosse certe ipotesi di lavoro (non si sa se ed in qual misura limitative) da Lui ammesse [...]. Non ci é peró noto fino a qual punto Egli abbia potuto dare corso a tale progetto; e sarebbe indubbiamente di grandissimo interesse se, fra le Sue carte, si potesse ritrovare qualcosa di conclusivo al riguardo.

[17] Fano Archive, Special Mathematical Library, University of Turin, *Scritti vari*, fols. 45-52, 128-133.

7. The 1938 memoir's positioning in the milieu of the Italian geometric School.

Fano was basically the first to address a systematic study of Fano-Enriques threefolds. However, his work is to be placed in a precise context, that of the Italian geometric tradition, the so-called Italian School of algebraic geometry. This thesis is supported by two types of considerations: the first related to Fano's cultural roots and sources, namely the literature he used to construct the 1938 work; the second inherent to the positioning of his essay within the overall Italian geometric research program and the reasons that pushed him to undertake such a study at that particular historical junction. From the point of view of literature, through all the paper *Sulle varietá algebriche a tre dimensioni...* explicit citations, both in the footnotes and in the text, are a tiny handful and the analysis of the citational network clearly accounts for a phenomenon of cultural endogamy. This means that, in the very majority of cases, references are to publications of the Italian geometers. First of all, there are the works by Federigo Enriques: *Introduzione alla geometria sopra le superficie algebriche* (1896) and *Sopra le superficie algebriche di bigenere uno* (1906). In the first Enriques had given one of the first examples of non-rational algebraic surface of genus zero: it is a smooth normalization of a non-normal surface of degree six in $\mathbb{P}^3$ that passes, with multiplicity two, through the edges of the coordinate tetrahedron [19]. In the second, he had characterized this surface by the numerical invariants $p_a = P_3 = 0, P_2 = 1$ and proved that any Enriques surface is birationally isomorphic to the double cover of $\mathbb{P}^2$ with branched curve of degree eight [20]. In addition to these texts, Fano referred to the works of his 'beloved Masters' Segre and Castelnuovo about some numerical invariants[18], and about congruences of the third order [7, 70–72]. At the third place, the masterpieces of the Italian School of geometry, that had brought it to a *Fuhrende Stellung* at international level: Enriques and Castelnuovo again, on the theory of surfaces, Fano's papers on congruences of lines, Severi's on algebraic varieties (1906), even an old publication by Pasquale Del Pezzo (1884-87) concerning the surfaces of order n. At the international level, by contrast, Fano mentioned just two names: S. Junski for a paper on cubic complexes Γ (1913) and Godeaux for three of his juvenile works (1926, 1930, 1933). Clearly, the literature selection made by Fano – who was a cultivated mathematician – is rather unusual: almost entirely Italian and made up of all works published at the end of the XIX and in the first decade of the XX century, with the only exception of Godeaux's ones.
From this point of view, the analysis of Fano's mathematical heritage has been particularly useful. His personal library, preserved in the Special Mathematical Library at the University of Turin, Department of Mathematics, consists of 150 volumes and about 5.000 offprints. According to the study of such collection [58, pp. 45-73], we can define exactly which sources Fano had in his availability in 1938, and which were his readings before flying Italy, when he was forced to separate from his library and miscellany. As far as Godeaux's legacy is concerned, for example, the copy of his 1926 work, widely annotated by Fano, shows the close connection between Godeaux's

[18] In particular, when Fano approached the issue, it was already known that, for a general Enriques surface, Castelnuovo-Enriques invariant ω is equal to 1, Zeuthen-Segre invariant I is 8 and Severi base number ρ is 10.

contributions on Enriques surfaces and Fano's research on Fano-Enriques threefolds. Significant is the fact that in Fano's handwritten notes, references to three-dimensional varieties also appeared, even if Godeaux's work was focused on surfaces only.[19]

The choice of the subject for Fano's 1938's paper is equally indicative of the placing of its author into a teamwork, a scholarly community, a mathematical School. Fano felt belonging (and declared such identity on various occasions) to the Italian School of algebraic geometry, that is a group of scholars who shared a research program and a peculiar style of geometric investigation characterized by some epistemological and linguistic patterns.

The research agenda, defined by L. Cremona in the 1860s, then extended by Segre, Castelnuovo, Enriques and Severi in the late XIX and early XX century, was characterised by some distinctive features that can be synthetized in the following terms: transformational approach, i.e. a widespread use of the so-called Cremonian transformations (birational maps); in-depth culture in projective geometry, with the exploitation of its properties to enlighten the birational properties of algebraic varieties; geometrization of algebraic language; interpretation of the elements of an n-dimensional linear space as geometrical entities (and not as simple attributes of analytical objects) [4, pp. 186-187 and 199], [15, pp. 5-7].

As far as the Italian geometric style is concerned, Italian geometers adopted a sort of heuristic approach, relying on intuition and projective methods. The interpretation of projective elementary operations (such as sections and projections) from a birational point of view was one of its distinctive features. Hyperspatial projective geometry – giving a concrete model to n-dimensional linear algebra – was the main research tool, to be employed to discover new issues and intrinsic properties of varieties. Stressing the creative power of geometrical intuition, the Italian School steadily turned algebraic results into synthetic geometric language. Moreover, conceiving geometry as an experimental science, this teamwork – the Masters Segre, Castelnuovo, Veronese, first, followed by their pupils – started from the analysis of several particular cases, extending by analogy the results they progressively obtained to higher spaces and to the general case [9, pp. 194, 200, 201], [75, pp. 211-212], [14, p. 152], [76, pp. 9-10], [49, pp. 174-175], [3, pp. 194-195], [66, pp. 2805-2807].

Within such epistemological frame, starting from the systematization of geometry over an algebraic curve by Segre around 1890, Castelnuovo, Enriques, Severi, B. Levi, G. Bagnera and M. De Franchis extended the field to surfaces, reaching in 1914 their complete classification according to some projective invariants and thus laying the groundwork for further developments by B. Segre, A. Comessatti and F. Conforto. In a second step, with Severi, Fano, G. Albanese and U. Morin, attention turned to varieties of higher dimension, trying to generalize methods and previous results, but mostly neglecting the need for a sounder foundation of algebraic geometry.

In light of this collective research program, it is possible to identify at least two reasons

[19] See, for instance, the offprint 286 I, 3, Fano miscellany, Special Mathematical Library, University of Turin, where Fano wrote down, next to the equation of the surface "anallagmantique" F of the 6^{th} order: *sistema* ∞^6 *rappresentazione di* V_3^{10}*?*. See also 286 I, 13: *onde* $(7,3)$ *particolare*; and 286 III, 16: *le rette congiungenti le* ∞^2 *coppie di punti coniugati formano una congruenza contenuta in un complesso lineare: certo le* $(3,3)$ *riferibili a* F^6 *di* S_4 *di genere* 1.

that lead Fano to deal with Fano-Enriques threefolds. In the first place, since some decades, there was a strong interest by the Italian geometers in K3 surfaces, an interest that was also related to the analysis of linear systems of these surfaces over Fano threefolds (see, among the main works, [18], [8], [21], [73]). The investigations into these varieties had occupied Fano himself for many years: in an ongoing effort to solve Lüroth problem in dimension three, he had undertaken the study of these threefolds in [25], presenting the first systematic results at the International Congress of Mathematicians in Bologna in 1928 [29]. In two works published in 1936 (see [30], [31]) he had then started to outline and feature the classification of such varieties, by tackling various special cases and by analysing the nature of their hyperplane sections. Finally, in [32], just a year before the publication of the memoir *Sulle varietá algebriche a tre dimensioni* ..., Fano had given a limitation on their degree.
Secondly, this sort of investigation was motivated by the attention that the Italian geometers paid to automorphisms groups (see, among the main works, [22], [20], [24], [74], [27]). The automorphisms groups of Enriques or K3 surfaces are specifically discrete and trivial for a general K3. As early as in 1906, Enriques [20] had proved that every surface having a non-finite, discrete group of automorphisms is either elliptic (of which Enriques surfaces are a particular case) or a K3 surface. In the same year, Fano had constructed the first example of a non-elliptic K3 surface of this type. But the main result, highly correlated with Fano-Enriques threefolds, had concerned the automorphisms groups of Enriques surfaces. While this group is not finite for a general Enriques surface, Fano had established in [26] that it could become finite for special elements in the family of all these surfaces. The example given by Fano had been a hyper-special case of Reye congruence. Bearing in mind that this is a particular Enriques surface embedded in the Grassmannian $Gr(1,3)$, its Plücker embedding had represented an example of Fano model of an Enriques surface.[20]
The fact that Fano-Enriques threefolds are an exquisitely Italian research subject, which found its suitable and coherent position within the tradition of the Italian geometric School, justifies the very limited – even inexistent – reception of the 1938 work. Only two geometers, the English Roth and the Belgian Godeaux dealt with the Fano-Enriques threefolds at the same time as Fano and using the same mathematical tools and language. The circumstance is not casual. As a matter of fact, Roth's scientific activity "was devoted almost entirely to the study of algebraic geometry, following the methods of the Italian School" [69, p. 194]. After completing his training in Cambridge, he started doing research under the mentorship of H.F. Baker. Awarded a Rockefeller fellowship, in 1930-31 Roth moved to Rome, where he got in touch with the great Italian mathematicians of the time: Castelnuovo, Enriques, Severi and T. Levi-Civita [78], [69]. The legacy of the Italian experience is all but marginal: from the Thirties onwards, Roth's papers include a large pattern of concrete geometric constructions of varieties. His main contributions are related to the group varieties and, specifically, to problems of unirationality and birationality of algebraic manifolds. This last point would explain the reason why the unirationality of three-dimensional

[20]This example was recently rediscovered by I. Dolgachev [17, p. 28] who did not hide, on this occasion, the difficulties experienced in reading Fano's original works.

varieties was at the origin of an interesting exchange between Roth and Fano, less than a year before the publication of the memoir *Sulle varietá algebriche a tre dimensioni* .... In a letter dated 18 February 1937, Roth confronted Fano about the topic in these terms:

> I could not study properly Your really wonderful works so quickly, but I must say how satisfying is to know that the series of the V_3^{2p-2} ends for $p = 37$ and if $p > 10$ they are rational. For now, I am content merely to answer some of Your questions. As regards the unirationality of the general V_3^8, I used the method You outlined in the first 1907 Note, i.e. I proved that the V_3^8 does not contain a homaloid system of surfaces for the sake of argument. You remember that a part of the proof consists in proving that the intersection of the V_3^8 and a form of order n cannot have a multiple point of order $> 2n$. Well, this fact does no longer subsist for the V_3^{10} and, thus, there is also hope for determining the unirationality of this latter, and let alone that of the V_3^{12} that contains just complete intersections is rational; but this study is full of surprises. It is interesting to know that this property maybe does not extend to the V_3 having $p > 8$; from my part, I don't have anything definitive to tell You about the actual existence of the V_3 containing just complete intersections. And now I would like to add another remark: in a recent note – not yet published – I determined that a quartic form of S_4 cannot have more than 45 isolated nodes and, after all, it is well known that this limit is reached, since a rational V_3^4 of this kind exists (see Todd, Quarterly Journal, Oxford 1936). Perhaps we might use this result to show, via subsequent projections, that the V_3^{2p-2} of the first species do not exist for $p > 23$. My work on the V_3^{2p-2} should be published around next June, and I will be delighted to send an offprint to You. [Fano Archive, Special Mathematical Library, University of Turin, letter no. 23] [21]

As far as Godeaux is concerned, after graduating in mathematics from the University

[21] In questo breve tempo non ho potuto studiare per bene i Suoi risultati davvero meravigliosi, ma devo dire quanto é soddisfacente sapere che la serie delle V_3^{2p-2} termina per $p = 37$ e che per $p > 10$ esse sono razionali. Per ora mi contenteró di rispondere ad alcune delle Sue domande. Per quanto riguarda l'unirazionalitá della V_3^8 generale, ho adoperato il metodo da Lei esposto nella prima Nota del '07, cioé ragionando per assurdo ho dimostrato che la V_3^8 non contiene un sistema omaloidico di superficie. Ella si ricorderá che una parte della dimostrazione consiste nel provare che l'intersezione della V_3^8 con una forma di ordine n non puó avere un punto multiplo di ordine $> 2n$. Ebbene questo fatto non sussiste piú per la V_3^{10} e cosí c'é pure speranza di stabilire l'unirazionalitá di quest'ultima, e tanto meno quella della V_3^{12} di Segre, la quale contiene superficie che non sono intersezioni complete. In proposito, sembra strano che la V_3^{12} che contiene soltanto intersezioni complete sia razionale; ma questo studio é pieno di sorprese. É interessante sapere che tale proprietá forse non si estende alle V_3 aventi $p > 8$; da parte mia io non so dirle nulla intorno all'esistenza effettiva delle V_3 contenenti sole intersezioni complete. Ed ora vorrei aggiungere un'altra osservazione: in una Nota recente –non ancora pubblicata– ho stabilito che una forma quartica di S_4 non puó aver piú di 45 nodi isolati, e del resto si sa che tale limite é raggiunto, perché esiste una V_3^4 razionale di questa natura (ved. Todd, Quarterly Journal, Oxford 1936). Forse si potrebbe usare questo risultato per dimostrare, mediante proiezioni successive, che le V_3^{2p-2} della prima specie non esistono per $p > 23$. Il mio lavoro sulle V_3^{2p-2} dovrá uscire verso giugno prossimo, e sará per me un gran piacere mandare un estratto a Lei.

of Liége (1911), he spent the years 1912-14 in Bologna, having Enriques as main advisor [6,39,40,68]. From that period, he concentrated on one of the Enriques' beloved subjects – the study of general type surfaces, according to the properties of the respective canonical or pluricanonical models – and in 1926 he dedicated to the topic a substantial series of essays (*Recherches sur les surfaces algébriques de genres zéro et de bigenre un*). Starting from the 1930s, Godeaux began to address three-dimensional varieties whose hyperplane sections are surfaces of genus zero and bigenus one, firstly concentrating on a special case of the $W_3^{16} \subset \mathbb{P}^9$ that is obtained by projectively relating to the hyperplanes of $\mathbb{P}^6$ the surfaces F going through the edges of the referent tetrahedron [44, p. 922]. Godeaux would have resume these investigations from 1962 onward, for example showing that the multiple points of W_3^{16} are birationally equivalent to some rational surfaces and that this threefold is rational, as Fano had stated in 1938. In one of his last works, Godeaux succeeded in building a new example of Fano-Enriques threefold for $p = 9$ following the path of Fano's 1938 work, to which he referred explicitly at several points (see, for instance, [46, p. 751], [47, p. 1252], [48, p. 439]). Truly speaking, such Godeaux's 'loyalty' to the Italian research classical venue seems rather bizarre, since another approach had progressively established from the 1930s onwards, whose slogan could be summarized as: 'to plant the harpoon of topology in the whale body of algebraic geometry' [53, p. 854]. To witness a rediscovery of the research on Fano-Enriques threefolds one should wait until the 1980s, and with a singular coincidence such critical re-reading will take place again in the Turin context.

8. A synthesis of the best of the Italian geometric culture and one of the signs of its decline.

The memory of 1938 represents a good synthesis of the best of the Italian geometric culture as well as a testimony of its style. In particular Fano attempted to extend to varieties of higher dimension the classical methods, which had proved to be successful for curves and surfaces. Such approach was based on the correspondence between curve-systems and hyperplane sections of a certain surface: through the linear systems of planar curves of given genus p it was possible to suitably study those surfaces having hyperplane sections of genus p. Interpreted in this way, the problems relating to linear systems of curves were translated into issues concerning projective surfaces. As the 1938 essay makes clear, Fano tried to apply this kind of reasoning to tree-dimensional varieties too: he analysed the surfaces obtained as hyperplane sections of the starting threefolds to deduce some properties about these three-dimensional varieties. To study these surfaces, in turn, Fano drew attention to the systems of curves over them. This continuous transfer between threefolds, surfaces and curves was not so effective: without due caution, a concrete risk of losing some relevant information about the varieties considered impended, and Fano fell into error in some parts of his 1938 essay, for example when he explored the two possibilities for the hyperplane section-surfaces of W_3^{2p-2} going through the pair of points of the involution I_2. Here, Fano came to exclude both the cases by analysing the system of hyperplane sections of W_3^{2p-2} and the systems of curves over these last surfaces [33, pp. 60-62].

The way of working by means of intuitive synthesis revealed its limits in the case of three-dimensional varieties. Fano made use of processes which he himself described as 'empirical': starting from the careful analysis of individual cases, he sought to include single results within a general theory, to be deductively re-arranged *a posteriori*. However, this last step was not easy in the case of threefolds and Fano failed in several parts of the memoir of 1938.

Eventually, this work displays another characteristic feature of the Italian approach in hyperspatial geometry. Indeed, like the great Italian Masters, Fano treated points, lines, and planes of $\mathbb{P}^n$ as real geometrical entities, not as mere attributes of analytic entities. According to a chronology consolidated by historical research, the Italian geometric School blossomed in the years 1860-1880, reached its apogee around 1908, began its decline in the Thirties and was liquidated by racial persecution, since it was an ethnically homogeneous group. Whereas literature agrees on many aspects of this reconstruction, much less is the case as far as the factors of decline of this teamwork are concerned (see [4, pp. 186-188 and 200-201], [3, pp. 203-204], [38, pp. 144-146]) either internal, or external. Fano's 1938 memoir give interesting piece of information about the last issue.

In the first place, adopting again a material history lens, the analysis of Fano's patrimony revealed that he did not have an appropriate knowledge of new international literature in algebraic geometry. The presence of writings by S. Lefschetz, G. de Rham, E. Cartan and many others, sometimes annotated, does not suffice to dispel the feeling of being in front of an old-fashioned library, strongly polarized towards the past and deprived of those texts [11, 52, 79–81] that in 1938 were seminal references for any active algebraic geometer.

To invoke the policy of cultural autarky, promoted by the fascist regime, which substantially prevented Italian mathematicians from keeping abreast of English and American production [62], it would be unproper because until 1936 there were no restrictions on the import of books and periodicals from abroad or on the circulation of individuals. Fano, however, travelled to the UK in 1923 and to the USA in 1934, but progressively lost the contacts with their emerging geometric groups. This is evident in the English case. Fano maintained regular exchanges only with G. Chisholm, W. Young, H.F. Baker, L. Roth, and with the Cambridge group (W.D. Hodge, J. Todd), that not surprisingly up to 1939 promoted a geometric research in which "almost everything was argued in words, which was really a negation of mathematics" [51, p. 89].

The fact that Fano never used and neither mentioned the new techniques developed abroad gives evidence of the Italian School's difficulty in addressing and solving some classical problems without the tools of abstract algebra, and of its basic mistrust of the structural and topological methods. As Fano wrote in his special lecture on Italian geometry at the University College of Wales in Aberystwyth (1923):

> Reason and intuition both have their tasks, both have their fields of work; and particularly in the science which is still in the course of construction, the part played by intuition must be a greater one. Intuition is the true helpmeet of invention; it is an indispensable attribute to the inventor and of those who wish to understand him. Intuition is the pioneer of progress

> and points the way to the logical development. [28, pp. 14-15]

However, as successful as they may have been in the late XIX and early XX centuries, the intuitive methods enhanced by the Italian tradition were steadily reaching the limits of their creative power in face of the increasing complexity of the problems to be addressed. One example will suffice: Roth's proof of the unirationality of W_3^6. Although taking part to the Italian geometrical tradition, Roth corrected Fano's statement, based only on a hyperspatial projective geometrical reasoning, using some algebraic tools, such as the notion of divisor thanks to which he could show that the Enriques threefold had divisor $\sigma = 2$.

9. A specimen of the Italian way of writing geometry.

The memoir *Sulle varietá algebriche ...* is also emblematic of a very peculiar kind of prose. Italian geometers thought themselves as explorers of a new land, a "dark forest" in which they were able to orientate only thanks to the "faint flame"[22] of intuition. Accordingly, they used to present their results in a prose suggestive of their coming into being.

Moreover Fano had always been the most geometric of Segre's protégés, to the point that as early as in 1893, Segre had suggested him to spend a period in Göttingen, in the hope that at the Klein's School his tendency would have been, if not corrected, at least attenuated:

> [Fano] is provided of a large amount of memory and he is characterized by a sharp intellect. His inclinations are essentially directed toward geometry, especially toward pure geometry. But, even if I repeatedly encouraged him to cultivate analysis too, and during my courses I showed him not only synthetic methods, but also the analytical ones, he remained exclusively a geometer so far. I really encouraged him to go to Göttingen, at Your school, because I hope You are able to expand the field of his ideas and studies. Perhaps, considering his strong attitudes toward geometry, it will not be the case of steer him to the pure analysis, to that sector of analysis which scruples to avoid using geometrical supports; I believe that he could be strengthen as geometer if it were possible to let him obtain permanently the analytical instruments, and that also in mathematical analysis he could produce useful stuff based on his geometrical notions and leanings. Therefore, I entrust my pupil to You, with the hope that under such a valuable Master, he pushes himself further than he already made under my guidance. (Segre to Klein, Turin 4.10.1893, in [56, pp. 164-165]) [23]

[22] [9, p. 201]: Ma rinunziare all'intuizione geometrica, la sola che abbia permesso sinora di orientarsi in questo territorio intricato, vorrebbe dire spegnere la tenue fiammella che puó guidarci nell'oscura foresta.

[23] é dotato di molta memoria ed ha un ingegno vivace. Ma le sue tendenze sono essenzialmente geometriche, per la pura geometria. E quantunque io l'abbia eccitato ripetutamente a coltivare anche l'analisi, e nei miei corsi gli abbia fatto vedere non solo i metodi sintetici ma anche quelli analitici, egli finora é rimasto troppo esclusivamente geometra. Io l'ho incoraggiato molto a recarsi a Gottinga, alla Sua scuola, perché

Fano's prose had always been nebulous (his same colleagues often criticized it!), but in the works on classical algebraic varieties of dimension three this peculiar feature of his writing reached perhaps its acme. It should, however, be noticed that Fano was not the only Italian geometer in the 1930s to think and write in this way. Apodictic statements such as those that occurred in Fano's 1938 memoir made up a large part of the contemporary output by Severi and his protégés [12, pp. 15-16]. For all these scholars we might speak of a sort of self-referentiality that affected the Italian School of algebraic geometry, but also of a sort of jargon, a family lexicon constructed and shared by the members, in which a great deal of content was downplayed, because of a wide patrimony of oral geometric culture transmitted from Masters to pupils and circulating among them.

10. Concluding remarks.

The historical reading of Fano's work on Fano-Enriques threefolds documents why the realization of a complete scientific biography of Gino Fano appears as a mandatory task and, at the same time, it brought elements of novelty to the narrative of Italian algebraic geometry between the two World Wars.
Besides representing the first systematic attempt to a birational classification of Fano-Enriques threefolds, it perfectly gives the state of the art of Italian achievements in classical projective geometry according to the synthetic address of study ("the amounts to what people call geometrical intuition" [60, p. viii]).
The historical reappraisal of Fano's 1938 paper, combined with the reconstruction of the material sources (articles, books, letters, etc.) that Fano mobilized in writing his last paper before the exile in Switzerland led to assess the heritage of geometric culture, especially in the field of projective geometry of hyperspaces, that the Masters of the Italian School (Cremona, Bertini, Segre, Castelnuovo) shared with the first generation of their students (Fano, Enriques, Severi) and which expressed in a really peculiar research and expository style, quite clear for the members of the School but rather obscure outside the group.
The analysis of this paper also mirrors the delay accumulated by Italian geometers in the 1930s in the fields of topology and abstract algebra, compared to foreign colleagues, a delay which cannot be merely ascribed to external causes (lack of language competences, political-ideological blueprint, autarky, etc.) and which is even more bizarre because for many years Fano had entertained scientific relationships with Coolidge, Baker, Hodge, Snyder, and various others, not to mention the Germans. As A. Andreotti, Fano's successor to the chair of geometry in Turin in 1950, was used to say: "All Italian mathematicians were a bit ignorant, but especially the researchers of alge-

spero che Ella riesca ad allargare molto la cerchia delle sue idee e dei suoi studi. Forse, date le attitudini spiccatissime che egli ha per la geometria, non sará il caso di farlo passare alla pura analisi, a quell'analisi che si fa uno scrupolo di evitare sussidi geometrici; ma invece credo che si possa rinforzarlo di molto come geometra se si riesce a fargli acquistare pienamente gli strumenti analitici, e che anche nell'analisi gli si possan far produrre cose utili basate sulle sue cognizioni e tendenze geometriche. Insomma io affido a Lei il mio discepolo con la speranza che sotto un tale Maestro egli si spinga molto piú in lá di quel che non abbia ancor fatto sotto di me.

braic geometry" [50, p. 281].
Finally, the memoir *Sulle varietá algebriche* ... shows Fano's deep insight in opening new paths of geometric investigation. Although some hasty conclusions and incomplete proofs, Fano's contributions on Fano-Enriques threefolds have a pioneering character, so much so that one can validly state:

> The clarification occurred later (frequently after some time) almost always proved that the major protagonists moved – instinctively I'd say – within the limits dictated by their intuition and experience, even if those bounds were not always explicitly stated [...]. Every time controversies and critics caused fruitful systemisations and specifications, that nearly always confirmed the validity of the hunches of the great scholars. [59, p. 5] [24]

Fano's memoir of 1938 had an extremely limited reception, even if it focused on a mathematical object that would later met a surprising vitality and even if its contents would provide source of inspiration for further research until recent years. Racial persecution can be invoked as mitigating factor for such a failure. However, *Sulle varietá algebriche* ... could reasonably appear, outside Italy, as a peripheral work on a fringe subject, definitely not interesting for an international readership research context that 'looked elsewhere'. Fano's 1938 paper, in this sense, represents at once the 'best' and the 'worst' that an Italian algebraic geometer could have written at that time.

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[24] Le precisazioni venute in seguito (spesso a distanza di tempo) hanno quasi sempre dimostrato che i grandi si muovevano – istintivamente direi – entro i limiti che la loro intuizione e la loro esperienza dettava, anche se tali limiti non venivano sempre esplicitamente enunciati [...]. Controversie e critiche hanno ogni volta provocato feconde sistematizzazioni e precisazioni, le quali hanno confermato quasi sempre la validitá delle intuizioni dei grandi cultori.

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