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M. Imran Bhat, N. Hosseinzadeh and Ahmad M. Alghamdi

SOME PROPERTIES OF ZERO-DIVISOR GRAPH OF ANNIHILATORS OF A COMMUTATIVE RING

Abstract. In this paper, for a commutative ring $\mathcal{R}$ with non-zero zero-divisor set $\mathcal{Z}^*(\mathcal{R})$, the zero-divisor graph of annihilators of $\mathcal{R}$ is a simple graph, denoted by $\Gamma_{\mathcal{A}}(\mathcal{R})$ with vertex set $V(\mathcal{Z}_{\mathcal{A}}(\mathcal{R})) = \{\bar{a} = ann(a),\ a \in \mathcal{Z}^*(\mathcal{R})\}$ and there is an edge $\bar{a} \to \bar{b}$ between two vertices if and only if $ann(a) \subset ann(b)$ and there does not exist $\bar{c}$ with $ann(a) \subset ann(c) \subset ann(b)$. The structure of zero-divisor graph of annihilators of $\mathbb{Z}_n$ is described. We give a combinatorial formula to find various parameters of this graph. Further, we provide a partition of the vertex set of $\Gamma_{\mathcal{A}}(\mathbb{Z}_n)$. We further study the lattice graph $\Gamma_{\mathcal{L}}(\mathbb{G})$ of a group $\mathbb{G}$ defined as a graph whose vertex set is the set of proper subgroups of a group $\mathbb{G}$ and the two vertices H_1 and H_2 in $\Gamma_{\mathcal{L}}(\mathbb{G})$ are adjacent if and only if $H_1 \cap H_2 \neq \{e\}$. We study isomorphism and several structural properties of the lattice graph of groups.

1. Introduction and Preliminaries

Throughout, $\mathcal{R}$ will denote a commutative ring with identity denoted by 1. The set $\mathcal{Z}^*(\mathcal{R}) = \mathcal{Z}(\mathcal{R}) \setminus \{0\}$ will denote the set of non-zero zero-divisors of $\mathcal{R}$. The annihilator of an element a of a ring $\mathcal{R}$ is the set $ann(a) = \{r \in \mathcal{R} \mid ar = 0\}$. By $\mathbb{Z}_n$, we denote the ring of integers modulo n. A ring is said to be local ring if it has a unique maximal ideal. The zero-divisor graph $\Gamma(\mathcal{R})$ of $\mathcal{R}$, is a simple undirected graph whose vertex set is $\mathcal{Z}^*(\mathcal{R})$ and the two vertices $u, v \in \mathcal{Z}^*(\mathcal{R})$ are adjacent if and only if $uv = vu = 0$. We define vertex set of a zero-divisor graph of annihilators $\Gamma_{\mathcal{A}}(\mathcal{R})$ as $V(\mathcal{Z}_{\mathcal{A}}(\mathcal{R})) = \{\bar{a} = ann(a), a \in \mathcal{Z}^*(\mathcal{R})\}$ and there is an edge $\bar{a} \to \bar{b}$ between the two vertices in $\Gamma_{\mathcal{A}}(\mathcal{R})$ if and only if $ann(a) \subset ann(b)$ and there does not exist $\bar{c}$ with $ann(a) \subset ann(c) \subset ann(b)$.

The concept of the zero-divisor graph of a commutative ring $\mathcal{R}$ was first introduced by Beck [5], where he was mainly interested in colorings. In his work, he let all the elements of ring as vertices of the graph. This investigation of colorings of a commutative ring was then continued by Anderson and Naseer in [3]. A different approach of associating a graph to a commutative ring $\mathcal{R}$ was given by Anderson and Livingston in [2], where the graph $\Gamma(\mathcal{R})$ has its vertices as elements of $\mathcal{Z}^*(\mathcal{R})$ and two vertices $u, v \in \mathcal{Z}^*(\mathcal{R})$ are adjacent if and only if $uv = 0$. The authors believed that this definition better illustrates the zero-divisor structure of the ring. The zero-divisor graph translates the algebraic properties of a ring to graph theoretical tools, thus helps in exploring interesting results in both graph theory and abstract algebra. Associating a graph to a ring, a group, semigroup or a module has been studied extensively (see for example [2–5, 7, 14]).

There are many papers which interlink graph theory with annihilators and lattice theory (see [1, 6, 9–12, 15]). These papers discuss the properties of graphs derived from partially ordered sets and lattices. Recall that a lattice is an algebra $(\mathcal{L}, \vee, \wedge)$ satisfying

the following conditions: for all $a, b, c \in \mathcal{L}$,
1. $a \wedge a = a$, $a \vee a = a$,
2. $a \wedge b = b \wedge a$, $a \vee b = b \vee a$,
3. $(a \wedge b) \wedge c = a \wedge (b \wedge c)$, $(a \vee b) \vee c = a \vee (b \vee c)$, and
4. $a \vee (a \wedge b) = a \wedge (a \vee b) = a$.
There is an equivalent definition for a lattice. Let S be a non-empty ordered set, that is, we are given a relation $a \leq b$ on S which is reflexive and such that $a \leq b$ and $b \leq a$ together imply $a = b$. Then, a subset T of S is a chain if either $a \leq b$ or $b \leq a$ for every pair of elements a, b in T and let $\mathcal{L}$ be lattice, for any $a, b \in \mathcal{L}$, we set $a \leq b$ if and only if $a \wedge b = a$. Then $(\mathcal{L}, \leq)$ is an ordered set in which every pair of elements has a greatest lower bound (g.l.b.) and a least upper bound (l.u.b.).

A graph G with vertex set $V(G) \neq \phi$ and edge set $E(G)$ of unordered pairs of distinct vertices is called a *simple graph*. The cardinality of $V(G)$ is called the order of G and the cardinality of $E(G)$ is called its size. A graph G is *connected* if and only if there exists a path between every pair of vertices u and v. A graph on n vertices such that any pair of distinct vertices is joined by an edge is called a *complete graph*, denoted by K_n. A graph is said to be a *bipartite graph* if its vertex set can be partitioned into two disjoint sets V_1 and V_2 with $V(G) = V_1 \cup V_2$ such that uv is an edge of G if $u \in V_1$ and $v \in V_2$. The number of edges incident on a vertex is called its *degree* and a vertex of degree 1 is called a *pendent vertex*. In a connected graph G, the distance between two vertices u and v is the length of the shortest path between u and v. The diameter of a graph G is defined as $diam(G) = sup\{(d(u,v)|u, v \in V(G))\}$, where $d(u,v)$ denotes the distance between vertices u and v of G. The length of a smallest cycle in a graph G is called as *girth* and is denoted by $gr(G)$. The eccentricity of a vertex v of a connected graph G is the distance between v and a vertex u farthest from v. The minimum eccentricity among the vertices of G is called the radius of G, denoted by $rad(G)$. Two graphs G and G' are said to be isomorphic to each other if there is a one-one correspondence between their vertices and between their edges such that the incidence relationship is preserved. For basic definitions in graph theory and lattice theory, we refer to [8, 13].

2. Structure of $\Gamma_{\mathcal{A}}(\mathbb{Z}_{p_1 p_2 \dots p_r})$

THEOREM 1. *Let $n = p_1 p_2 \dots p_r$ be the product of r distinct primes of n. Then*

(i) $\left|V\left(\mathcal{Z}_{\mathcal{A}}(\mathbb{Z}_{p_1 p_2 \dots p_r})\right)\right| = 2^r - 2.$

(ii) $\left|E\left(\mathcal{Z}_{\mathcal{A}}(\mathbb{Z}_{p_1 p_2 \dots p_r})\right)\right| = \binom{r}{1}\binom{r-1}{1} + \binom{r}{2}\binom{r-2}{1} \cdots + \binom{r}{r-2}\binom{2}{1} = \sum_{k=1}^{r-2} (r-k)\binom{r}{k}.$

(iii) For any $v_1 \in V_1$, $d_{V_1}(v_1) = r-1$; for any $v_{r-1} \in V_{r-1}$, $d_{V_{r-1}}(v_{r-1}) = r-1$; for any $v_k \in V_k$ $d_{V_k}(v_k) = r$, where $k = 2, 3, \dots, r-2$, where $d_{V_i}(v_i)$ denotes the degree of a vertex v_i in some $V_i \subset V$

Proof. Let the vertex set V of $\Gamma_{\mathcal{A}}(\mathbb{Z}_{p_1 p_2 \dots p_r})$ be divided into disjoint subsets

$V_1, V_2, ..., V_{r-1}$, where

$$V_1 = \{\overline{p_i}, 1 \le i \le r\}$$
$$V_2 = \{\overline{p_i p_j}, 1 < i, j \le r\}$$
$$\cdots$$
$$V_{r-1} = \{\overline{p_i p_j p_k \ldots p_s}, 1 < i, j, k, \ldots, s \le r\}$$

(i) By partitioning a vertex set of $\Gamma_{\mathcal{A}}(\mathbb{Z}_{p_1 p_2 \ldots p_r})$, it is not difficult to see that $\left|V(\Gamma_{\mathcal{A}}(\mathbb{Z}_{p_1 p_2 \ldots p_r}))\right| = \binom{r}{1} + \binom{r}{2} + \binom{r}{3} \cdots + \binom{r}{r-1} = 2^r - 2$

(ii) By definition of zero-divisor graph of annihilators of a ring, we see that $ann(p_i) \subsetneq ann(p_i p_j)$, implies that $\overline{p_i}$ is adjacent to $\overline{p_i p_j}$ and therefore each vertex in V_1 is adjacent to $r-1$ vertices in V_2. Using the same argument, we see that $ann(p_i p_j) \subsetneq ann(p_i p_j p_k)$, $1 < i \neq j \neq k \le r$, implies that there is an edge from $\overline{p_i p_j}$ to $\overline{p_i p_j p_k}$ and each vertex in V_2 is adjacent to $r-2$ vertices in V_3. On proceeding, we see that, $ann(p_i p_j \ldots p_k) \subsetneq ann(p_i p_j \ldots p_s)$, $1 < i, j, k \le r$ and $1 < i, j, \ldots, s \le r$ and therefore each vertex in V_{r-2} is adjacent to 2 vertices in V_{r-1}. This, completes the second part.

(iii) To find the degree of a vertex in V_i, $1 \le i \le r-1$, we see that each vertex in V_1 is adjacent $r-1$ vertices. So that, for any $v_1 \in V_1$ $d_{V_1}(v_1) = r-1$. The degree of a vertex in V_2 equals to the sum of the number of incoming edges from V_1 and the number of outgoing edges to V_3, which sums up $2 + r - 2 = r$, that is, for any $v_2 \in V_2$, $d_{V_2}(v_2) = r$. Proceeding in this way, we see that $d_{V_k}(v_k) = r$ for any $v_k \in V_k$, $2 \le k \le r-2$ and for any $v_{r-1} \in V_{r-1}$, $d_{V_{r-1}}(v_{r-1}) = r-1$.

We notice that the graph so obtained is the chain of bipartite graphs, clearly contains no odd cycle. So, $gr(\Gamma_{\mathcal{A}}(\mathbb{Z}_{p_1 p_2 \ldots p_r})) = 4$.
Also, $diam(\Gamma_{\mathcal{A}}(\mathbb{Z}_{p_1 p_2 \ldots p_r})) = r-1$. □

EXAMPLE 1. Let $\mathbb{Z}_n$ be the ring of integers modulo n. Then for four distinct prime integers p, q, r and s

(i) $|\Gamma_{\mathcal{A}}(\mathbb{Z}_{pqrs})| = 14$.

(ii) $|E(\Gamma_{\mathcal{A}}(\mathbb{Z}_{pqrs}))| = 24$.

(iii) for any $v_1 \in V_1$, $d_{V_1}(v_1) = \binom{3}{1}$, for any $v_2 \in V_2$, $d_{V_2}(v_2) = 2.\binom{2}{1}$, $diam(\Gamma_{\mathcal{A}}(\mathbb{Z}_{pqrs})) \le 3$, $rad(\Gamma_{\mathcal{A}}(\mathbb{Z}_{pqrs})) \le 2$ and $gr(\Gamma_{\mathcal{A}}(\mathbb{Z}_{pqrs})) = 4$

Proof. Let the vertex set V of $\Gamma_{\mathcal{A}}(\mathbb{Z}_{pqrs})$ be partitioned into disjoint sets given by $V_1 = \{\bar{p}, \bar{q}, \bar{r}, \bar{s}\}$, $V_2 = \{\overline{pq}, \overline{pr}, \overline{ps}, \overline{qr}, \overline{qs}, \overline{rs}\}$, $V_3 = \{\overline{pqr}, \overline{pqs}, \overline{prs}, \overline{qrs}\}$.
We see, the set V_1 contain four distinct primes, the set V_2 contain elements by choosing two primes at a time and the set V_3 contain elements by choosing three primes at a time. Therefore, it is clear, $|V| = \binom{4}{1} + \binom{4}{2} + \binom{4}{3} = 14$.

Now, after determining the annihilator of each element of V_1, we see that annihilator of each prime in V_1 is contained in the annihilator of the elements of V_2 containing

that prime and therefore there are three outgoing edges from each prime of V_1 into the set V_2. Similarly, by the same argument, the annihilator of each element of V_2 is contained in the annihilator of the elements of V_3 containing the corresponding element and therefore are adjacent to each other. Moreover, there is no outgoing edge from the elements of V_3 to the elements of V_1. For if, the elements of V_3 are adjacent to the elements of V_2, but then, the annihilator of an element of V_1 is contained in annihilator of an element of V_2, which contradicts our definition of zero-divisor graph of annihilators. This is because $ann(x) \subsetneq ann(z)$, but then $ann(y) \subsetneq ann(z)$ for $x \in V_1$, $y \in V_2$, $z \in V_3$, implies x is not adjacent to z. Moreover, we can see that $ann(p) \subsetneq ann(pq) \subsetneq ann(pqr)$, which implies p is not adjacent to pqr. Thus, $|E(\Gamma_{\mathcal{A}}(\mathbb{Z}_{pqrs}))| = 3.\ \binom{4}{1}+2.\ \binom{4}{2}\ +0.\ \binom{4}{3}$.
(iii) Note that for any $v_1 \in V_1$, $d_{V_1}(v_1) = \binom{4-1}{1} = 3$, since it is adjacent to vertices in V_2 containing v_1. In the above case, p is adjacent to pq, pr, ps. Also, for any $v_2 \in V_2$, $d_{V_2}(v_2) = \binom{2}{1} + \binom{4-2}{1}$. Further, for any $v_3 \in V_3$, $d_{V_3}(v_3) = 3$, since the vertices in V_2 are adjacent to the elements in V_3 and there is no any other outgoing edge from V_3. Thus it follows that the diameter is 3 and the radius is 2. □

THEOREM 2. *Let $\mathcal{R} \cong \mathcal{R}_1 \times \mathcal{R}_2$ be a finite commutative local ring with unity. Then $\Gamma_{\mathcal{A}}(\mathcal{R})$ is connected.*

(i) If both $\mathcal{R}_1$ and $\mathcal{R}_2$ are local rings

(ii) If $\mathcal{R}_1$ (or $\mathcal{R}_2$) is a field

(iii) If and only if both $\mathcal{R}_1$ and $\mathcal{R}_2$ are fields.

Proof. (i) Let $\mathcal{R} \cong \mathcal{R}_1 \times \mathcal{R}_2 = \{(r_1, r_2) \mid r_1 \in \mathcal{R}_1, r_2 \in \mathcal{R}_2\}$ be a finite commutative local ring with unity. The vertex set of $\Gamma_{\mathcal{A}}(\mathcal{R})$ is the set of annihilators given by:
$\{ann(0,1), ann(1,0), ann(a_i,0), ann(0,b_j), ann(a_i,1), ann(1,b_j), ann(a_i,b_j)\}$. Now,

$$\begin{aligned}
ann(0,1) &= \{(x,0)\ \forall\, 0 \neq x \in \mathcal{R}_1\}\\
ann(1,0) &= \{(0,y)\ \forall\, y \in \mathcal{R}_2\}\\
ann(a_i,0) &= \{(x_i,y) : a_i x_i = 0, x_i \in \mathcal{Z}^*(\mathcal{R}_1), y \in \mathcal{R}_2,\},\\
ann(a_i,1) &= \{(x_i,0) : a_i x_i = 0, x_i \in \mathcal{Z}^*(\mathcal{R}_1)\},\\
ann(0,b_j) &= \{(x,y_j) : b_j y_j = 0, x \in \mathcal{R}_1, y_j \in \mathcal{Z}^*(\mathcal{R}_2)\},\\
ann(1,b_j) &= \{(0,y_j) : b_j y_j = 0, y_j \in \mathcal{Z}^*(\mathcal{R}_2)\},\\
ann(a_i,b_j) &= \{(x_i,0),(0,y_j),(x_i,y_j) : a_i x_i = 0 = b_j y_j, x_i \in \mathcal{Z}^*(\mathcal{R}_1), y_j \in \mathcal{Z}^*(\mathcal{R}_2)\}.
\end{aligned}$$

We note that, $ann(a_i,1) \subset ann(0,1)$ and $ann(a_i,0) \subset ann(0,1)$. Thus, $\Gamma_{\mathcal{A}}(\mathcal{R})$ is connected.

(ii) Let $\mathcal{R}_2$ be a field, say $\mathbb{F}$. That is, $\mathcal{R} \cong \mathcal{R}_1 \times \mathbb{F}$. The possible annihilators of the elements in $\mathcal{R}_1 \times \mathbb{F}$ are $ann(0,1), ann(1,0), ann(a_i,0), ann(a_i,1)$, where a_i is the nonzero zero-divisor in $\mathcal{R}_1$ and $1 \in \mathbb{F}$.

Now,

$$ann(0,1) = \{(x,0), \text{ for all } 0 \neq x \in R_1\},$$
$$ann(1,0) = \{(0,y) \text{ for all } 0 \neq y \in \mathcal{R}_2\},$$
$$ann(a_i,0) = \{(x_i,y) : a_i x_i = 0, x_i \in \mathcal{Z}^*(\mathcal{R}_1)\}$$
$$ann(a_i,1) = \{(x_i,0) : a_i x_i = 0, x_i \in \mathcal{Z}^*(\mathcal{R}_1)\}$$

It is now easy to see that $ann(a_i,1) \subset ann(0,1)$ and $ann(a_i,0) \subset ann(1,0)$. Thus, $\Gamma_{\mathcal{A}}(\mathcal{R})$ is connected.

(iii) Let $\mathcal{R} \cong \mathbb{F}_1 \times \mathbb{F}_2$. Then the annihilator set of each element of $\mathbb{F}_1 \times \{0\}$ is same, that is, $ann(a_i,0) = \{(0,b_j) : b_j \in \mathbb{F}_2\}$, where $a_i \in \mathbb{F}_1$. Similarly, $ann(0,b_j) = \{(a_i,0) : a_i \in \mathbb{F}_1\}$. Clearly, $ann(a_i,0) \subsetneq ann(0,b_j)$ and vice versa $ann(0,b_j) \subsetneq ann(a_i,0)$. Thus, $\Gamma_{\mathcal{A}}(\mathcal{R})$ is connected. □

3. Lattice graph of subgroups of a group

We know that the set of subgroups of a group $\mathbb{G}$ forms a lattice $\mathcal{L} = (H, \wedge, \vee)$, where for any two subgroups H_1 and H_2, $H_1 \vee H_2 = H_1 \cup H_2$ and $H_1 \cap H_2 = H_1 \wedge H_2$. The lattice graph $\Gamma_{\mathcal{L}}(\mathbb{G})$ of a group is defined as a graph whose vertex set is the set of proper subgroups of $\mathbb{G}$ and the two vertices H_1 and H_2 in $\Gamma_{\mathcal{L}}(\mathbb{G})$ are adjacent if and only if $H_1 \cap H_2 \neq \{e\}$. Let $(\mathcal{L}, \vee, \wedge)$ be a lattice with least element 0. Then $a \in \mathcal{L}$ is called an *atom* if there is no element $b \in \mathcal{L}$ such that $0 < b < a$. We gather atoms of $\mathcal{L}$ in the set $A(\mathcal{L})$. Let $(\mathcal{L}, \vee, \wedge)$ be a lattice with the maximal element 1. Then $d \in DA(\mathcal{L})$ is called a *dual atom*, if $0 < d < 1$.

THEOREM 3. *Let $\mathcal{L} = (H, \vee, \wedge)$ be the lattice of subgroups of the group $\mathbb{Z}_{p^n}$, then $diam(\Gamma_{\mathcal{L}}(\mathbb{Z}_{p^n})) = 1$ and $|A(\mathcal{L})| = 1$.*

Proof. We know that the proper subgroups of a group $\mathbb{Z}_{p^n}$ are generated by the divisors of p^n and are $\langle p \rangle, \langle p^2 \rangle, \langle p^3 \rangle, \ldots, \langle p^{n-1} \rangle$. Clearly, $\langle p^i \rangle \wedge \langle p^j \rangle$ is a non-trivial subgroup for all $i \neq j$. Thus, $\Gamma_{\mathcal{L}}(\mathbb{Z}_{p^n})$ is a complete graph K_{n-1}. Therefore, $diam(\Gamma_{\mathcal{L}}(\mathbb{Z}_{p^n})) = 1$. Since, $\Gamma_{\mathcal{L}}(\mathbb{Z}_{p^n}) \cong K_{n-1}$, then any two vertices H_1 and H_2 different from $H = \langle p^{n-1} \rangle$ are adjacent in $\Gamma_{\mathcal{L}}(\mathbb{Z}_{p^n})$. So, $H \leq H_1 \wedge H_2$, implies $H \leq H_i$ for all $H_i \in \mathcal{L}$, implies that H is a solitary atom, that is, $|A(\mathcal{L})| = 1$. □

EXAMPLE 2. Consider a group $\mathbb{Z}_{p^4}$. The set of proper subgroups are:
$H_1 = \langle p \rangle = \{0, p, 2p, \ldots, (p-1)p, p^2, \ldots, (p^3-1)p\}$,
$H_2 = \langle p^2 \rangle = \{0, p^2, 2p^2, \ldots, (p^2-1)p^2\}$ and
$H_3 = \langle p^3 \rangle = \{0, p^3, 2p^3, \ldots, (p-1)p^3\}$.
Clearly, $H_i \wedge H_j \neq \{0\}$. Thus, $\Gamma_{\mathcal{L}}(\mathbb{Z}_{p^4}) \cong K_3$.

THEOREM 4. *Let $\mathbb{G}$ be a group such that $\mathbb{G} \cong \mathbb{Z}_{p_1 p_2 \ldots p_n}$, where every p_i $(1 \leq i \leq n)$ is a prime integer and $p_1 < p_2 < \cdots < p_n$. Then, $\Gamma_{\mathcal{L}}(\mathbb{Z}_{p_1 p_2 \ldots p_n})$ is isomorphic to K_{2^n-2}.*

Proof. The proper subgroups of $\mathbb{G} \cong \mathbb{Z}_{p_1 p_2 \dots p_n}$ are generated by divisors of $p_1 p_2 p_3 \dots p_n$ and are $2^n - 2$ in number. Therefore, it is clear that the element $p_1 p_2$ belongs to all the proper subgroups of $\mathbb{G}$. Thus, there is an edge between every pair of subgroups of $\mathbb{G}$, that is $\mathbb{G} \cong K_{2^n-2}$. □

THEOREM 5. *The lattice graph of a group $\mathbb{G} = \mathbb{Z}_p \times \mathbb{Z}_p$ is disconnected.*

Proof. We know that $\mathbb{G} = \mathbb{Z}_p \times \mathbb{Z}_p$ contains $p+1$ subgroups of order p given by:
$H_1 = \{00, 01, 02, \dots, 0(p-1)\}$, note that here $a_i a_j = (a_i, a_j)$.
$H_2 = \{00, 10, 20, \dots, (p-1)0\}$,
$H_3 = \{00, 11, 22, \dots, (p-1)(p-1)\}, \dots$
$H_{p+1} = \{00, 1(p-1), 2(p-2), \dots, (p-1)(p-1)\}$.
Clearly, the intersection of no two subgroups is non-trivial. Thus, $\Gamma_{\mathcal{L}}(\mathbb{Z}_p \times \mathbb{Z}_p)$ is disconnected. □

Let $\mathcal{L}$ and $\mathcal{L}'$ be two lattices, a lattice isomorphism is one to one mapping $f : \mathcal{L} \longrightarrow \mathcal{L}^\star$ such that $f(a \vee b) = f(a) \vee f(b)$ and $f(a \wedge b) = f(a) \wedge f(b)$.

THEOREM 6. *Let $\mathbb{G}$ and $\mathbb{G}^\star$ be two isomorphic groups. Then $\Gamma_{\mathcal{L}}(\mathbb{G})$ and $\Gamma_{\mathcal{L}}(\mathbb{G}^\star)$ are isomorphic, where $\mathcal{L}$ and $\mathcal{L}^\star$ are lattice of subgroups of $\mathbb{G}$ and $\mathbb{G}^\star$.*

Proof. Let $f : \mathbb{G} \longrightarrow \mathbb{G}^\star$ be an isomorphism. Then for a subgroup $H^\star \leq \mathbb{G}^\star$, there exists a unique subgroup $H \in \mathbb{G}$ such that $f(H) = H^\star$. Let $V(\Gamma_{\mathcal{L}}(\mathbb{G})) = \{H_1, H_2, \dots, H_m\}$ and $V(\Gamma_{\mathcal{L}}(\mathbb{G}^\star)) = \{H_1', H_2', \dots, H_n'\}$. Define a mapping $\phi : \mathcal{L} \longrightarrow \mathcal{L}'$ by $\phi(H) = f(H)$. It is easy to see that ϕ is a bijection. To show that ϕ is a graph isomorphism, we show for any H_i and H_j in $\Gamma_{\mathcal{L}}(\mathbb{G})$ are adjacent if and only if $\phi(H_i)$ and $\phi(H_j)$ are adjacent in $\Gamma_{\mathcal{L}}(\mathbb{G}^\star)$. Let H_i and H_j be two adjacent vertices in $\Gamma_{\mathcal{L}}(\mathbb{G})$, that is,

$$\begin{aligned} & H_1 \wedge H_2 \neq \{e\} \\ \Leftrightarrow \quad & f(H_1 \wedge H_2) \neq \{e^\star\} \\ \Leftrightarrow \quad & f(H_1) \wedge f(H_2) \neq \{e^\star\} \\ \Leftrightarrow \quad & \phi(H_1) \wedge \phi(H_2) \neq \{e^\star\} \\ \Leftrightarrow \quad & \phi(H_1) \text{ is adjacent to } \phi(H_2). \end{aligned}$$

□

We observe that there exists a family of non-isomorphic groups whose lattice graphs are also non-isomorphic. Say, for example, Dihedral group $D_n = \{x^i y^j | i = 0, 1, j = 0, 1, \cdots, n-1, x^2 = e = y^n, xy = yx^{-1}\}$, in particular octic group $D_8 = \{R_0, R_{90}, R_{180}, R_{270}, H, H', D, D'\}$ and $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$.

Note that there exist non-isomorphic groups, but their lattice graph is identical. For illustration, consider Klien's 4-group and $\mathbb{Z}_4$.

THEOREM 7. *Let $\Gamma_{\mathcal{L}}(\mathbb{G})$ be the lattice graph with no cycle of length* 3*, associated to a group $\mathbb{G} = \mathbb{G}^\star \times \mathbb{Z}_2$ and if $H \leq \mathbb{G}$ such that $H \in A(\mathcal{L})$, then H is a pendent vertex in $\Gamma_{\mathcal{L}}(\mathbb{G})$.*

Proof. Consider a group $\mathbb{G} = \mathbb{G}^{\star} \times \mathbb{Z}_2$. We note that the subgroup $H = \{(0,0),(0,1)\}$ is a minimal element, that is, $H \in A(\mathcal{L})$ of $\mathbb{G}$. We claim that $deg(H) = 1$. If $deg(H) \geq 2$, then H is adjacent to atleast two subgroups, say H_j and H_k. So, that $H \wedge H_j \neq \{e\}$ and $H \wedge H_k \neq \{e\}$. But then, $H \wedge H_j = H = H \wedge H_k$, implies that $H \leq H_j$ and $H \leq H_k$, implies that $H \leq H_j \wedge H_k$, implies that H_j and H_k are adjacent, which is a contradiction to the hypothesis that $\Gamma_{\mathcal{L}}(\mathbb{G})$ has no cycle of length 3. Thus, $deg(H) = 1$. □

COROLLARY 1. *If* $H \in A(\mathcal{L})$ *and* $deg(H) = 1$, *then* $H \leq H_k$, *where* $H_k \in DA(\mathcal{L})$.

Proof. Let $H_j \in DA(\mathcal{L})$, then $H \wedge H_j \neq \{e\}$, implies $H \wedge H_j = H$, implies $H \leq H_j$. But H_k is the maximal element, implies $H \leq H_j \leq H_k$, implies $H \leq H_k$, implies $H \wedge H_k \neq \{e\}$, that is H and H_k are adjacent, a contradiction. Thus, $H \leq H_k$. □

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M. Imran Bhat, N. Hosseinzadeh and Ahmad M. Alghamdi
Department of Mathematics, Central University of Kashmir, Ganderbal
191201, Tulmulla, INDIA
Department of Mathematics, Dezful Branch, Islamic Azad University, Dezful, IRAN
Umm Al-qura University, Faculty of applied Sciences, Department of Mathematical Sciences
P.O. Box 14035, Makkah 21955, SAUDI ARABIA
e-mail: imran@cukashmir.ac.in, narges.hosseinzadeh@gmail.com, amghamdi@uqu.edu.sa

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