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Emiliano Torti

ON CONGRUENCES BETWEEN LATTICES IN CRYSTALLINE REPRESENTATIONS OF GAL$(\overline{\mathbb{Q}}_P/\mathbb{Q}_P)$ OF DIMENSION TWO

Abstract. The aim of this article is to present a short survey on the main techniques of integral p-adic Hodge theory currently used to study (as esplicitly as possible) congruences modulo general prime powers between Galois stable lattices inside two-dimensional crystalline representations of the absolute Galois group of $\mathbb{Q}_p$. The focus will be on the general case of reductions modulo prime powers and at the same time we will recall some of the main results in the most studied semi-simple residual case.

1. Introduction

Crystalline representations arise in many contexts in number theory and they are crucial in the study of linear representations of the absolute Galois group of $\mathbb{Q}_p$ for some prime p. For example, if f is a classical modular form of weight $k \geq 2$ and p is a prime not dividing its level N, then the attached local Galois representation $V_p f$ is indeed crystalline. In this short article, we want to focus on two-dimensional irreducible crystalline representations. In particular, we want to study their Galois-stable lattices and their reductions modulo prime powers. We will start by setting the notations which will be fixed once and for all through the article.
Let p be an odd prime and let $\mathbb{E} \subseteq \overline{\mathbb{Q}}_p$ be a finite extension of $\mathbb{Q}_p$. We denote by $\mathscr{O}_\mathbb{E}$ the ring of integers of $\mathbb{E}$, by $\pi_\mathbb{E}$ a uniformizer, by $k_\mathbb{E}$ the residue field and denote by e the ramification index of $\mathbb{E}$ over $\mathbb{Q}_p$. Let $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ denote the absolute Galois group of $\mathbb{Q}_p$ (after we have fixed once and for all an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$). We will denote by v the normalized valuation on $\overline{\mathbb{Q}}_p$. Let $k \geq 2$ be a positive integer and a_p an element of positive valuation, or in other words an element in the maximal ideal of $\mathscr{O}_\mathbb{E}$. There is a unique two-dimensional irreducible crystalline representation denoted V_{k,a_p} of Hodge-Tate weights $\{0, k-1\}$ and such that the attached characteristic polynomial of the crystalline Frobenius is $x^2 - a_p x + p^{k-1}$. Moreover, up to twist of dimension one, they are the only ones. These implies essentially that one can study and treat these representations as a two-parameters family where the parameters are k and a_p. A lot of attention has been given in the literature to the study of their semi-simple residual reductions (see for example [BLZ04], [BL20], [Bha18]). To be precise, we have almost a complete classifications of the semi-simplified residual representation

$\overline{V}_{k,a_p} := (T \otimes_{\mathscr{O}_{\mathbb{E}}} k_{\mathbb{E}})^{\mathrm{ss}}$ where T denotes any Galois stable lattice inside V_{k,a_p}. Note that this is all well defined thanks to the Brauer-Nesbitt theorem. However, a full classification is still missing and turned out to be quite a challenging problem to solve. We are interested in studying reductions of lattices modulo prime powers and in such context no semi-simplification process is allowed. This implies that a certain care is required because congruence properties will a priori strictly depend on the chosen Galois-stable lattice. The advantage of working directly with a fixed choice of lattice is, as it will be clear later, that some results in the residual case can be strengthened by removing the semi-simplification restriction. This is the case of the two main results that we will present in this article. There are many techniques available in the literature for studying reductions of crystalline representations coming from integral p-adic Hodge theory or the p-adic and mod p Langland correspondence. We refer the interested reader to the work of Ghate et al. (see for example [Gha19] or [Bha18]) for understanding how the Langland correspondence can be used in this context. In this article we will focus on the integral p-adic Hodge theory side by showing how the theory of (φ, Γ)-modules can be used to control these reductions when we let p-adically vary separately the parameters k and a_p. These phenomena are usually addressed as local constancy for reductions of crystalline representations (see [Ber12], [Tor20], [Bha18]). The article is structured as follows. In the first section we will consider the case of letting the trace of the crystalline Frobenius a_p vary p-adically while in the second section we will focus on modifying the parameter k.

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2. Reductions of integral $G_{\mathbb{Q}_p}$-stable lattices in crystalline representations of dimension two

In what follows, we will show how the theory of (φ, Γ)-modules can be used to study reductions of crystalline representations but we want to point out that in integral p-adic Hodge theory also another tool can be used to study such reductions, namely the Breuil-Kisin modules (as an alternative to Wach modules which instead will be introduced later). This has been done in the semi-simple residual case by Bergdall-Levin (see [BL20]). It could be interesting to see wether such objects can give extra information also in the case of reductions modulo general prime powers.

2.1. Reductions via integral p-adic Hodge Theory I: Fontaine's theory of (φ,Γ)-modules

Let Γ be a group isomorphic to $\mathbb{Z}_p^\times$ via a map $\chi : \Gamma \to \mathbb{Z}_p^\times$. Fix once and for all a topological generator of Γ (which is procyclic as $p \neq 2$), say γ. For the sake of completeness, we will briefly recall the construction of some of the rings of Fontaine necessary for introducing the (φ,Γ)-modules and the theorem characterizing their relation with certain representations of $G_{\mathbb{Q}_p}$. Most of what we introduce here can be found in the original paper of Fontaine (see [Fon90]) and a very good summary has been written up by Berger in some lecture notes (see [**?**]).
Let $\{\varepsilon^{(n)}\}_{n\geq 1} \subset \bar{\mathbb{Q}}_p$ be a system of roots of unity such that:

$$\begin{aligned}&(1)\ \ \varepsilon^{(1)} \neq 1,\\ &(2)\ \ \varepsilon^{(n)} \in \mu_{p^n} \subset \overline{\mathbb{Q}}_p,\\ &(3)\ \ (\varepsilon^{(n+1)})^p = \varepsilon^{(n)}.\end{aligned}$$

One can think of $\varepsilon := (\varepsilon^{(1)}, \varepsilon^{(2)}, \dots)$ as an element of Fontaine's ring $\mathscr{E} = \varprojlim \mathbb{C}_p$ (with projective limit maps given by the Frobenius maps $z \mapsto z^p$) where $\mathbb{C}_p$ is the p-adic completion of $\overline{\mathbb{Q}}_p$. It is well known that $\mathscr{E}$ is an algebraically closed field of characteristic p. Now, consider the fields $\mathbb{Q}_p^{(n)} := \mathbb{Q}_p(\varepsilon^{(n)})$ and define $\mathbb{Q}_p^{(\infty)} = \cup_{n\geq 1}\mathbb{Q}_p^{(n)}$. Denote by $H_{\mathbb{Q}_p}$ the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^{(\infty)})$.
The p-adic cyclotomic character ω gives the exact sequence:

$$1 \longrightarrow H_{\mathbb{Q}_p} \longrightarrow G_{\mathbb{Q}_p} \xrightarrow{\omega} \Gamma \cong \mathbb{Z}_p^\times \longrightarrow 1.$$

Consider the field $\mathbb{F} = \mathbb{F}_p((\varepsilon - 1))$ inside $\mathscr{E}$. Let $\mathscr{A}_{\mathbb{Q}_p}$ be the p-adic completion of $\mathbb{Z}_p[[x]][\frac{1}{x}]$; it is a complete discrete valuation ring whose residue field can be identified with $\mathbb{F}$ (one can identify x with a suitable Teichmuller lift of $\varepsilon - 1$). Let $\mathscr{A}$ be the p-adic completion of the strict henselization $\mathscr{A}_{\mathbb{Q}_p}^{\mathrm{sh}}$ of $\mathscr{A}_{\mathbb{Q}_p}$ inside $\tilde{\mathscr{A}} := W(\mathscr{E})$. Note that $\mathscr{A}_{\mathbb{Q}_p}^{\mathrm{sh}}$ can be identified with the ring of integers of the maximal unramified extension of the field $\mathscr{A}_{\mathbb{Q}_p}[\frac{1}{p}]$ inside $\tilde{\mathscr{A}}[\frac{1}{p}]$.
The Galois group $G_{\mathbb{Q}_p}$ acts on $\mathscr{E}$ by acting on $\mathbb{C}_p$ and by functoriality on the projective limit. By functoriality of the Witt vectors, the group $G_{\mathbb{Q}_p}$ also acts on $\tilde{\mathscr{A}} = W(\mathscr{E})$ and we have that $\mathscr{A}$ is $G_{\mathbb{Q}_p}$-stable. It is also true that $\mathscr{A}^{H_{\mathbb{Q}_p}} = \mathscr{A}_{\mathbb{Q}_p}$.
Now, we define $\mathscr{A}_{\mathbb{E}}$ as $\mathscr{A}_{\mathbb{Q}_p} \otimes_{\mathbb{Z}_p} \mathscr{O}_{\mathbb{E}}$, by a result of Dee (see prop. 2.2.2 in [Dee01]) we have that $\mathscr{A}_{\mathbb{E}}$ is isomorphic to the $\pi_{\mathbb{E}}$-adic completion of $\mathscr{O}_{\mathbb{E}}[[x]][\frac{1}{x}]$. One can think of $\mathscr{A}_{\mathbb{E}}$ inside the ring $\mathscr{A}^{(\mathscr{O}_{\mathbb{E}})} := \mathscr{A} \otimes_{\mathbb{Z}_p} \mathscr{O}_{\mathbb{E}}$. The ring $\mathscr{A}_{\mathbb{E}}$ inherits a $G_{\mathbb{Q}_p}$-action via the natural action of $G_{\mathbb{Q}_p}$ on $\mathscr{A}$ and

trivial action on $\mathscr{O}_{\mathbb{E}}$. It follows that $(\mathscr{A}^{(\mathscr{O}_{\mathbb{E}})})^{H_{\mathbb{Q}_p}} = \mathscr{A}_{\mathbb{E}}$ and so $\mathscr{A}_{\mathbb{E}}$ has a structure of Γ-module.
Hence, the ring $\mathscr{A}_{\mathbb{E}}$ has a natural $\mathscr{O}_{\mathbb{E}}$-linear action of Γ and a $\mathscr{O}_{\mathbb{E}}$-linear Frobenius endomorphism φ given by the following expressions:

$$\varphi(f(x)) = f((1+x)^p - 1) \qquad \text{for all } f(x) \in \mathscr{A}_{\mathbb{E}},$$
$$\eta(f(x)) = f((1+x)^{\chi(\eta)} - 1) \quad \text{for all } f(x) \in \mathscr{A}_{\mathbb{E}}, \text{ for all } \eta \in \Gamma.$$

Finally, we can recall the following:

DEFINITION 1. *An étale (φ, Γ)-module D over $\mathscr{O}_{\mathbb{E}}$ is an $\mathscr{A}_{\mathbb{E}}$-module of finite type endowed with a semilinear Frobenius map φ such that $\varphi(D)$ generates D as $\mathscr{A}_{\mathbb{E}}$-module (this is the étale property) and a semilinear continuous action of Γ which commutes with φ.*
The category of étale (φ, Γ)-module over $\mathscr{O}_{\mathbb{E}}$ will be denoted by $\boldsymbol{Mod}^{ét}_{(\varphi,\Gamma)}(\mathscr{O}_{\mathbb{E}})$.

Denote by $\mathbf{Rep}_{\mathscr{O}_{\mathbb{E}}}(G_{\mathbb{Q}_p})$ the category of $\mathscr{O}_{\mathbb{E}}$-linear representations of $G_{\mathbb{Q}_p}$, i.e. the category of $\mathscr{O}_{\mathbb{E}}$-modules of finite type with a continuous $\mathscr{O}_{\mathbb{E}}$-linear action of $G_{\mathbb{Q}_p}$.
By a theorem of Fontaine (see A.3.4 in [Fon90]) and its generalization by Dee (see 2.2 in [Dee01]) we have the following:

THEOREM 1. *There exists a natural isomorphism*

$$\mathfrak{D} : \boldsymbol{Rep}_{\mathscr{O}_{\mathbb{E}}}(G_{\mathbb{Q}_p}) \to \boldsymbol{Mod}^{ét}_{(\varphi,\Gamma)}(\mathscr{O}_{\mathbb{E}})$$

given by $\mathfrak{D}(T) = (\mathscr{A}^{(\mathscr{O}_{\mathbb{E}})} \otimes_{\mathscr{O}_{\mathbb{E}}} T)^{H_{\mathbb{Q}_p}}$.
A quasi-inverse functor, which is a natural isomorphism as well, is given by:

$$\mathfrak{T} : \boldsymbol{Mod}^{ét}_{(\varphi,\Gamma)}(\mathscr{O}_{\mathbb{E}}) \to \boldsymbol{Rep}_{\mathscr{O}_{\mathbb{E}}}(G_{\mathbb{Q}_p})$$

given by $\mathfrak{T}(D) = (\mathscr{A}^{(\mathscr{O}_{\mathbb{E}})} \otimes_{\mathscr{A}_{\mathbb{E}}} D)^{\varphi=1}$.

REMARK 1. Note that the equivalence of categories given by the above theorem preserves the objects killed by a fixed power of a chosen uniformizer. This essentially follows from the exactness of the functor $\mathfrak{D}$ (see prop. 2.1.9 in [Dee01]) and so $(\pi_{\mathbb{E}}^n) \cdot \mathfrak{D}(T) = \mathfrak{D}((\pi_{\mathbb{E}}^n) \cdot T)$ in the category $\mathbf{Mod}^{\text{ét}}_{(\varphi,\Gamma)}(\mathscr{O}_{\mathbb{E}})$. Same goes for the quasi-inverse functor $\mathfrak{T}$ (see prop. 2.1.24 in [Dee01]).

The étale (φ, Γ)-modules corresponding to crystalline representations can be described more explicitly via the theory of Wach modules developed by Berger and Wach. We will briefly recall the definition of Wach modules and their main properties. First, we define $\mathscr{A}_{\mathbb{E}}^{+} = \mathscr{O}_{\mathbb{E}}[[x]]$ inside $\mathscr{A}_{\mathbb{E}}$. It inherits naturally the actions of φ and Γ by restriction from $\mathscr{A}_{\mathbb{E}}$. Note also that $\mathscr{A}_{\mathbb{E}}$ is obtained by taking the $\pi_{\mathbb{E}}$-adic completion of the localization $\mathscr{O}_{\mathbb{E}}[[x]][1/x]$ of $\mathscr{A}_{\mathbb{E}}^{+} = \mathscr{O}_{\mathbb{E}}[[x]]$ at the multiplicative set $\{1, x, x^2, \ldots\}$. Moreover, since $\mathscr{A}_{\mathbb{E}}$ is Noetherian, we have that $\mathscr{A}_{\mathbb{E}}$ is flat as $\mathscr{A}_{\mathbb{E}}^{+}$-module since localization and completion preserve such property. Following Berger (see [Ber12]), we have the following

DEFINITION 2. *A Wach module of height $h \geq 1$ is a free $\mathscr{A}_{\mathbb{E}}^{+}$-module N of finite rank endowed with commutative $\mathscr{A}_{\mathbb{E}}^{+}$-semilinear actions of a Frobenius map φ and of the group Γ such that:*

(1) $D(N) := \mathscr{A}_{\mathbb{E}} \otimes_{\mathscr{A}_{\mathbb{E}}^{+}} N \in \boldsymbol{Mod}^{ét}_{(\varphi,\Gamma)}(\mathscr{O}_{\mathbb{E}})$,

(2) Γ *acts trivially on* N/xN,

(3) $N/\varphi^*(N)$ *is killed by* Q^h,

where $\varphi^*(N)$ denotes the $\mathscr{A}_{\mathbb{E}}^{+}$-module generated by $\varphi(N)$, and $Q = \frac{(1+x)^p - 1}{x} \in \mathscr{A}_{\mathbb{E}}^{+}$.

We recall that the Wach modules are the right linear algebra objects to specialize Fontaine's equivalence to crystalline representations. Indeed, we have the following (see Prop. 1.1 in [Ber12]):

PROPOSITION 1. *Let N be a Wach module of height h. The $\mathbb{E}$-linear representation $\mathbb{E} \otimes_{\mathscr{O}_{\mathbb{E}}} \mathfrak{T}(\mathscr{A}_{\mathbb{E}} \otimes_{\mathscr{A}_{\mathbb{E}}^{+}} N)$ of $G_{\mathbb{Q}_p}$ is crystalline with Hodge-Tate weights in the interval $[-h; 0]$; and*

$$D_{cris}(\mathbb{E} \otimes_{\mathscr{O}_{\mathbb{E}}} \mathfrak{T}(\mathscr{A}_{\mathbb{E}} \otimes_{\mathscr{A}_{\mathbb{E}}^{+}} N)) \cong \mathbb{E} \otimes_{\mathscr{O}_{\mathbb{E}}} N/xN \quad \text{as } \varphi\text{-modules.}$$

Moreover, all crystalline representations with Hodge-Tate weights in $[-h; 0]$ arise in this way.

The above result clarifies that we are essentially able to control Galois-stable lattices in crystalline representations via simple linear algebra objects. A very interesting point is that such description comes in hand if one is interested in studying general reductions modulo prime powers. Indeed, let N be a Wach module, let P and G denote respectively the matrices representing the action of φ and of a fixed topological generator of Γ (say γ). Note the subtlety that when we say that the above matrices represent an action, such action is semi-linear and not linear. This implies, for example, that the actions of φ and Γ commute translates in terms of attached matrices as the equation $P\varphi(G) = G\gamma(P)$. It is possible, just via semi-linear algebra means, to deform p-adically the matrices P and G in another two matrices P' and G' which are arbitrary p-adically close respectively to P

and G and which correspond to a new Wach module N'. The corresponding Galois stable lattice will, under some conditions, live inside V_{k,a'_p} for some $a'_p \in m_{\mathbb{E}}$ which is p-adically close to a_p giving the desired congruence. This is the main idea which resulted in the theorem of Berger in the semi-simple case (see [Ber12]) and the author (see [Tor20]) to the following:

THEOREM 2. (Local constancy with respect to a_p)
Let $a_p, a'_p \in \mathfrak{m}_{\mathbb{E}}$ and $k \geq 2$ be an integer. Let $m \in \frac{1}{e}(\mathbb{Z}_{\geq 1})$ such that $v(a_p - a'_p) \geq 2 \cdot v(a_p) + \alpha(k-1) + m$, then for every $G_{\mathbb{Q}_p}$-stable lattice T_{k,a_p} inside V_{k,a_p} there exists a $G_{\mathbb{Q}_p}$-stable lattice T_{k,a'_p} inside V_{k,a'_p} such that

$$T_{k,a_p} \otimes_{\mathscr{O}_{\mathbb{E}}} \mathscr{O}_{\mathbb{E}}/(p^m) \cong T_{k,a'_p} \otimes_{\mathscr{O}_{\mathbb{E}}} \mathscr{O}_{\mathbb{E}}/(p^m) \text{ as } G_{\mathbb{Q}_p}\text{-modules;}$$

where $\alpha(k-1) = \sum_{n \geq 1} \lfloor \frac{k-1}{p^{n-1}(p-1)} \rfloor$.

By imposing $m = 1/e$ in the above theorem, one can recover Berger's theorem without the semi-simplification condition.

2.2. Reductions via p-adic Hodge Theory II: Kedlaya's theory of (φ, Γ)-modules over the Robba ring

In this section, we will recall another version of the theory of (φ, Γ)-modules developed by Kedlaya (see [Ked04]) which is equivalent to the one constructed by Fontaine but has several different features. We will briefly recall the necessary definitions and main properties and we will highlight what are the advantages in using such theory in the context of crystalline representations.
Let $\mathscr{R}_{\mathbb{E}}$ denote the Robba ring (over the finite extension $\mathbb{E}$ of $\mathbb{Q}_p$), i.e. the ring of Laurent power series of the form $f(x) = \sum_{n \in \mathbb{Z}} a_n x^n$ where $a_n \in \mathbb{E}$ and such that $f(x)$ converges on an annulus of the form $\rho \leq |x|_p < 1$ for some positive $\rho \in \mathbb{R}$.
As in the previous section, let Γ be a group isomorphic to $\mathbb{Z}_p^\times$ via a map $\chi : \Gamma \to \mathbb{Z}_p^\times$. The Robba ring $\mathscr{R}_{\mathbb{E}}$ can be endowed with linear actions of a Frobenius map φ and Γ (which we recall being a pro-cyclic group when $p \neq 2$). To be more precise, we can describe explicitly such actions. Let $f(x) \in \mathscr{R}_{\mathbb{E}}$, we define $\varphi(f)(x) := f((1+x)^p - 1)$ and if γ is a fixed choice for a topological generator of Γ, we define $(\gamma f)(x) := f((1+x)^{\chi(\gamma)} - 1)$. The actions of φ and Γ commute. We define the ring $\mathscr{O}^\dagger_{\mathscr{E}}$ as the ring of elements $f(x) \in \mathscr{R}_{\mathbb{E}}$ with $|a_n|_p \leq 1$ for all $n \in \mathbb{Z}$. We are now able to define a (φ, Γ)-module in the sense of Kedlaya (i.e. over the Robba ring):

DEFINITION 3. *A (φ, Γ)-module D over the Robba ring is a finite and free $\mathscr{R}_{\mathbb{E}}$-module of rank $d \in \mathbb{Z}_{\geq 0}$ endowed with a semilinear action of a Frobenius φ and a commuting semilinear action of Γ, such that once a*

basis is fixed the respectively attached matrices satisfy Mat$(\varphi) \in GL_d(\mathscr{R}_{\mathbb{E}})$ *and Mat*$(\gamma) \in Mat_{d\times d}(\mathscr{R}_{\mathbb{E}})$.
Moreover, we say that the (φ,Γ)*-module D is étale if there exists a basis such that Mat*$(\varphi) \in GL_d(\mathscr{O}^{\dagger}_{\mathscr{E}})$.

As in the Fontaine's case described in the previous section, the above definition is very important as (with a combination of results of Fontaine, Cherbonnier-Colmez and Kedlaya, see for example [Ked04]) it leads to the following crucial result:

THEOREM 3. *There is an explicit equivalence of categories between the category of étale* (φ,Γ)*-modules over the Robba ring* $\mathscr{R}_{\mathbb{E}}$ *and the category of* $\mathbb{E}$*-linear representations of Gal*$(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.

It is easy to describe explicitly the (φ,Γ)-modules of rank one. They are of the form $D = \mathscr{R}_{\mathbb{E}}(\delta)$, or in other words a free (φ,Γ)-module of rank one over $\mathscr{R}_{\mathbb{E}}$ such that φ and γ act on a basis (say e_δ) via the multiplicative character $\delta : \mathbb{Q}_p^\times \to \mathbb{E}^\times$ through the formulas $\varphi(e_\delta) = \delta(p)e_\delta$ and $\gamma(e_\delta) = \delta(\chi(\gamma))e_\delta$. The étale property corresponds to the explicit property $\mathrm{val}_p(\delta(p)) = 0$.
The modules of rank one are particularly useful to define a very large class of $\mathbb{E}$-linear representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ called trianguline representations. They are representations whose attached (φ,Γ)-modules can be realized as successive extension of (φ,Γ)-modules of rank one. To the interested reader, we refer to Berger (see [Ber11]) for a very nice survey on trianguline representations. For what concerns us, it is useful to note that the theory of (φ,Γ)-modules built up by Kedlaya has the disadvantage (contrary to the Fontaine's theory and Wach modules) that it cannot describe via integral structures the lattices inside the Galois representations (essentially because of properties of the Robba ring). However, every crystalline representation is in particular trianguline and those representations can be parametrized via a rigid analytic space described by Colmez and Chenevier. The trianguline representations arise naturally in rigid analytic families (and admit integral subfamilies) and by modifying the attached parameters (especially the one that allows to modify p-adically the Hodge-Tate weights) it is possible to produce congruences between some Galois stable lattices contained in crystalline representations of the form V_{k,a_p} and V_{k',a_p}. This was done by Berger (see [Ber12]) in the semi-simple residual case and by the author (see [Tor20]) for the general reductions modulo prime powers. The final main result is the following:

THEOREM 4. (Local constancy with respect to k)
Let $\mathbb{E}$ *be a finite extension of* $\mathbb{Q}_p$ *and denote by* $m_{\mathbb{E}}$ *the maximal ideal of its ring of integers* $\mathscr{O}_{\mathbb{E}}$. *Let* $a_p \in m_{\mathbb{E}}$ *such that* $a_p \neq 0$. *Let* $k \geq 2$ *and*

$m \in \frac{1}{e}(\mathbb{Z}_{\geq 1})$ be fixed. Assume that

$$k \geq (3v(a_p)+m)\cdot\left(1-\frac{p}{(p-1)^2}\right)^{-1}+1.$$

There exists an integer $r = r(k,a_p) \geq 1$ such that if $k'-k \in p^{r+m}(p-1)\mathbb{Z}_{\geq 0}$ then there exist $G_{\mathbb{Q}_p}$-stable lattices $T_{k,a_p} \subset V_{k,a_p}$ and $T_{k',a_p} \subset V_{k',a_p}$ such that

$$T_{k,a_p} \otimes_{\mathscr{O}_{\mathbb{E}}} \mathscr{O}_{\mathbb{E}}/(p^m) \cong T_{k',a_p} \otimes_{\mathscr{O}_{\mathbb{E}}} \mathscr{O}_{\mathbb{E}}/(p^m) \text{ as } G_{\mathbb{Q}_p}\text{-modules.}$$

As before, by imposing $m = 1/e$ in the above theorem, one can recover Berger's theorem without the semi-simplification condition. It is interesting to understand if is possible to produce an explicit radius for the local constancy with respect to the weight (i.e. an explicit description of r). A partial answer has been given in a special case by Bhattacharya (see [Bha18]).

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Emiliano Torti

University of Luxembourg
6 Av. de la Fonte,
4364 Esch-sur-Alzette,
Luxembourg

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