Rendiconti Sem. Mat. Univ. Pol. Torino
Vol. 80, 2022 (2022), 29 – 40

Alessandro Fazzari

A SURVEY ON A WEIGHTED ONE-LEVEL DENSITY OF FAMILIES OF *L*-FUNCTIONS

Abstract. This paper is an extended abstract of my talk "A weighted one-level density of families of *L*-functions", for the conference 5^{th} *Number theory meeting*, Torino 2021.

1. Introduction

Linear statistics for low-lying zeros of families of L-functions are of major interest in modern analytic number theory and there is an extensive literature on this topic. A fundamental contribution is due to Katz and Sarnak [19], who studied the one-level density of a variety of families of L-functions. In particular, they conjectured that the distribution of the (normalized) zeros close to the central point $s = \frac{1}{2}$ can be predicted just by a few random matrix ensembles, being unitary, symplectic or orthogonal. The group of matrices which encodes the distribution of the low-lying zeros of the L-functions in the family is then called "symmetry type" of the family itself.

Namely, given $\mathscr{F}$ a "natural" family of L-functions in the Selberg class, ordered by log-conductor $c(L)$, Katz and Sarnak looked at the quantity

$$\mathscr{D}^{\mathscr{F}}(f) := \frac{1}{\#\mathscr{F}} \sum_{L\in\mathscr{F}} \sum_{\gamma_L} f(c(L)\gamma_L) \tag{1}$$

where f is a Schwartz test function and $\frac{1}{2} + \gamma_L$ are the non-trivial zeros of L. Note that, since f is Schwartz, then only those zeros which are close to the central point contribute significantly to the sum. Their work suggests that the above quantity can be described by a function which only depends on the symmetry type G of the family; more specifically, their *density conjecture* predicts that

$$\mathscr{D}^{\mathscr{F}}(f) \overset{\#\mathscr{F}\to\infty}{\longrightarrow} \int_{\mathbb{R}} f(x)W_G(x)dx. \tag{2}$$

In particular, $W_G(x)$ is (conjecturally) the kernel appearing in the analogous computation in the corresponding random matrix theory setting.

There is a vast literature proving (2) for numerous specific families, the most famous probably being Iwaniec, Luo and Sarnak's work [18] (see e.g. [23], and the paper therein cited, for a more comprehensive overview). In all the results towards the density conjecture, however, it is necessary to place some conditions on the test function f; in particular, in order to handle the quantity (1), one needs to restrict the support of the Fourier transform of f^*. Enlarging the support of $\hat{f}$ is then a crucial

*Note that the nature of this restriction is the same as in Montgomery's work [24].

(and very hard indeed) problem, which has been studied intensively in the past years; we refer to [11] for a more detailed account of the problem. Typically the assumption of GRH allows to extend the support of $\hat{f}$ (see e.g. [15,18]) and even more can be done by assuming the asymptotic formulae suggested by the recipe [7] for the moments of ratios of L-functions (see e.g. [10]).

In this paper we are interested in a weighted version of the quantity (1), tilting the external average over the family by the central value of the L-function; more precisely, for any $k \in \mathbb{N}$, we define

$$\mathscr{D}_k^{\mathscr{F}}(f) := \frac{1}{\sum_{L\in\mathscr{F}} V(L(1/2))^k} \sum_{L\in\mathscr{F}} \sum_{\gamma_L} f(c(L)\gamma_L) V(L(1/2))^k, \tag{3}$$

where $V(z) = |z|^2$ if the family $\mathscr{F}$ is unitary, while $V(z) = z$ in symplectic and orthogonal cases.

Similarly to [5,12,13], the philosophy of this approach is to give more relevance to the L-functions in the family which are large at central point. Of course, these L-functions have less zeros close to $s = \frac{1}{2}$, therefore we expect a different low-lying zeros distribution with respect to the weighted average in (3).

Moreover, the weighted one-level density $\mathscr{D}_k^{\mathscr{F}}(f)$ gives a connection between moments and one-level density. Indeed, since the quantity $\sum_{\gamma_L} f(c(L)\gamma_L)$ does not grow too fast, then in (3) only the L-functions in the family $\mathscr{F}$ which are responsible for the moments are expected to contribute significantly. For instance, for a unitary family, assuming the Riemann Hypothesis we know from Soundararajan's work [26] that the main term of the $2k$-th moment comes from the L-functions in the family which are approximately of size $(\log X)^k$ at the central point. In addition, the classical n-th level density [25] implies that $\sum_{\gamma_L} f(c(L)\gamma_L) \ll c(L)^{\varepsilon}$ for almost all the L-funcitions in $\mathscr{F}$; as a consequence, also in (3), only the L-functions such that $|L(1/2)| \asymp (\log X)^{k\pm\varepsilon}$ should give a significant contribution to the main term of the sum. Thus, in a unitary case, $\mathscr{D}_k^{\mathscr{F}}(f)$ can be seen as a one level density for the thin subset $\{L \in \mathscr{F} : (\log X)^{k-\varepsilon} \ll |L(1/2)| \ll (\log X)^{k+\varepsilon}|\}$, i.e. those L-functions which are responsible for the $2k$-th moment. Analogous considerations can be done also for the other symmetry types.

In [14], we speculate that the weighted one-level density defined in (3) has the same structure as the classical one; more specifically we conjecture that the quantity $\mathscr{D}_k^{\mathscr{F}}(f)$ only depends on the symmetry type on the family $\mathscr{F}$. In addition, we give explicit conjectural formulae for the weighted kernels, as described in the following conjecture.

CONJECTURE 1.1. *(from [14, Conjecture 2.1]).* Let us consider a test function f, holomorphic in the strip $|\Im(z)| < 2$, even, real on the real line and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$, then for any $k \in \mathbb{N}$. Given a family $\mathscr{F}$ of L-functions with symmetry type $G \in \{U, USp, SO^+\}$, we have

$$\mathscr{D}_k^{\mathscr{F}}(f) = \int_{-\infty}^{+\infty} f(x) W_G^k(x)\,dx + O\left(\frac{1}{\log X}\right) \tag{4}$$

as $X \to \infty$, where the weighted one-level density function W_G^k depends on k and G only. In addition the following relations hold

$$W_{SO^+}^k(x) = W_{USp}^{k-1}(x) \quad \text{and} \quad W_U^k(x) = \frac{W_{USp}^k(x) + W_{SO^+}^k(x)}{2} \tag{5}$$

for any $k \in \mathbb{Z}_+$ and $k \in \mathbb{N}$ respectively. Moreover, for every $k \in \mathbb{Z}_+$, in the symplectic case (the others can be recovered by the above relations), we have that

$$\hat{W}_{USp}^k(y) = \delta_0(y) + P_{USp}^k(|y|)\chi_{[-1,1]}(y) \tag{6}$$

where P_{USp}^k is a polynomial of degree $2k-1$, given by

$$P_{USp}^k(y) = -\frac{2k+1}{2} - k(k+1)\sum_{j=1}^{k}(-1)^j c_{j,k}\frac{y^{2j-1}}{2j-1}, \tag{7}$$

with

$$c_{j,k} = \frac{1}{j}\binom{k-1}{j-1}\binom{k+j}{j-1}.$$

Other than the general structure of the weighted one-level density (4), which is in accordance to (2), the above conjecture brings out three main aspects.

First of all, we notice that the relations (5) put in light a connection between the kernels of different symmetry types. In particular, the orthogonal kernel with weight k equals the symplectic kernel with weight $k-1$; moreover the unitary kernel is the arithmetic mean of the other two. It is interesting to note that the leading order moment coefficients satisfy analogous relations linking them with each other (see [20, Equations (6.10) and (6.11)]). Beside this, the relations (5) allow us to focus only on one symmetry type in the following (say symplectic), as the others can be then deduced from it.

The second aspect we want to look at is the explicit formula for the weighted kernel W_{USp}^k; as shown by (6)-(7), its Fourier transform is essentially given by a completely explicit polynomial of degree $2k-1$, for $k > 1$. Therefore, by Fourier inversion we can obtain a conjectural formula also for the kernel itself, being

$$W_{USp}^k(x) = 1 - (2k+1)\frac{\sin(2\pi x)}{2\pi x} + \sum_{j=1}^{k}\frac{k(k+1)}{2^{2j-2}\pi^{2j-1}}\frac{c_{j,k}}{2j-1}\frac{d^{2j-1}}{dx^{2j-1}}\left[\frac{1-\cos(2\pi x)}{2\pi x}\right].$$

The above expression underlines that the weighted kernel is basically a linear combination of derivatives of the function $\frac{1-\cos(2\pi x)}{2\pi x}$.

Finally we look at the behaviour at $x = 0$ of the weighted kernel $W_{USp}^k(x)$; since as k increases the quantity (3) gives more and more relevance to the L-functions in the family which are large at the central point, for which it is less likely to have zeros near to $\frac{1}{2}$, it is natural to expect that the order of vanishing of $W_{USp}^k(x)$ at $x = 0$ increases with k. This phenomenon is formally described in the following conjecture.

CONJECTURE 1.2. *(from [14, Conjecture 2.2]).* For $G \in \{U, USp, SO^+\}$, $k \in \mathbb{N}$, the weighted kernels W_G^k defined in Conjecture 1.1 satisfy the following asymptotic relations as $x \to 0$:

$$W_U^k(x) \sim \frac{\pi^{2k} x^{2k}}{(2k-1)!!(2k+1)!!}$$
$$W_{USp}^k(x) \sim \frac{2\pi^{2(k+1)} x^{2(k+1)}}{(2k+1)!!(2k+3)!!}$$
$$W_{SO^+}^k(x) \sim \frac{2\pi^{2k} x^{2k}}{(2k-1)!!(2k+1)!!}.$$

These estimates highlight some relevant information about the asymptotic weighted distribution of the low-lying zeros; in particular, for example in the unitary case, Conjectures 1.1 and 1.2 suggest that, on weighted average over the considered family, the number of normalized zeros which are less than ε away from the central point is typically $\asymp_k \varepsilon^{2k+1}$.

Note that Conjecture 1.1 actually implies Conjecture 1.2 in a stronger form, with the complete asymptotic expansion at 0 of the weighted kernel (see [14, Theorem 2.7]).

We conclude this section by mentioning another context where this weighted one-level density appears. As already pointed out in [14], Kowalski, Saha and Tsimerman [22] consider the spinor L-functions of a given Siegel modular form F of genus 2 and they study the one-level density tilted by a weight ω^F, which is essentially the modulus square of the first Fourier coefficient of F. Even though the family is expected to show orthogonal behaviour, with the weight ω^F they get a symplectic distribution for the low-lying zeros. They explain this discrepancy by Böcherer's conjecture [4] (now proved by Furusawa and Morimoto [16]), that claims that ω^F is proportional to the central value $L(\frac{1}{2}, F)$, which carries a deep arithmetical meaning that influences the symmetry of the family. In our notation, the quantity they studied is therefore the weighted one-level density $\mathscr{D}_1^{\mathscr{F}}(f)$ for a specific orthogonal family and the transition from orthogonal to symplectic can be then explained by (5) ([21, 27] are other examples where this phenomenon of change of symmetry type is observed).

2. Weighted one-level density via ratio conjecture

This section is devoted to an investigation on the quantity $\mathscr{D}_k^{\mathscr{F}}(f)$ defined in (3), under the assumption of the ratio conjecture. This conjecture comes from an application of the recipe [7] and gives an explicit formula with all the main terms for the moments of ratios of L-functions, for a variety of families (see [8, 9]). The method we will use to study the weighted one-level density was first introduced introduced by Conrey and Snaith [10] to compute the classical one-level density, complete with lower order terms.

All the results mentioned throughout this section already appeared in [14] and are currently submitted for publication

We now describe roughly the method; let's consider a family $\mathscr{F}$ and assume the Riemann Hypothesis for all the L-functions in $\mathscr{F}$; we denote by $\frac{1}{2}+i\gamma_L$ a generic non-trivial zero of L ($\gamma_L \in \mathbb{R}$). Let also k be a non-negative integer and f be a test function, holomorphic in the strip $|\Im(z)| < 2$, even, real on the real line and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$. Finally let $\alpha = (\alpha_1, \dots, \alpha_k)$ be a set of shifts of size $1/c(L)$ (we recall that $c(L)$ denotes the log-conductor of L). We are interested in the quantity

$$\mathscr{I}(\alpha) := \sum_{L\in\mathscr{F}} \sum_{\gamma_L} f(c(L)\gamma_L) \prod_{i=1}^{k} L(\tfrac{1}{2}+\alpha_i).$$

By the residue theorem and the assumption of RH, $\mathscr{I}(\alpha)$ can be expressed as an integral around the critical line; namely, denoting $c = \frac{1}{2}+\delta$ for some positive δ of size $1/c(L)$, we write

$$\mathscr{I}(\alpha) = \sum_{L\in\mathscr{F}} \frac{1}{2\pi i} \left(\int_{(c)} - \int_{(1-c)} \right) \frac{L'}{L}(s) \prod_{i=1}^{k} L(\tfrac{1}{2}+\alpha_i) f\left(\frac{c(L)}{i}(s-\tfrac{1}{2}) \right) ds.$$

We start with the integral over the c-line

$$\begin{aligned} \mathscr{I}_{(c)}(\alpha) &= \sum_{L\in\mathscr{F}} \frac{1}{2\pi i} \int_{(c)} \frac{L'}{L}(s) \prod_{i=1}^{k} L(\tfrac{1}{2}+\alpha_i) f\left(\frac{c(L)}{i}(s-\tfrac{1}{2}) \right) ds \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} f\left(\frac{c(L)}{i}(\delta+ix) \right) \frac{d}{d\gamma} \left[\sum_{L\in\mathscr{F}} \frac{L(\frac{1}{2}+\gamma+ix)}{L(\frac{1}{2}+\delta+ix)} \prod_{i=1}^{k} L(\tfrac{1}{2}+\alpha_i) \right]_{\gamma=\delta} dx. \end{aligned}$$

After a truncation of the integral over x to guarantee the convergences, at this point we can use the ratio conjecture in order to evaluate the inner sum. Differentiating the formula that we have for moment of the ratio of L-functions, we get a nice expression for $I_{(c)}(\alpha)$.

As for the integral over the $(1-c)$-line, we apply the functional equation for $\frac{L'}{L}$, being

$$\frac{L'}{L}(1-s) = \frac{X'}{X}(s) - \frac{L'}{L}(s),$$

to write $I_{(1-c)}(\alpha)$ as a sum of two terms; the one coming from the $\frac{X'}{X}$ will be completely explicit, while the one from $\frac{L'}{L}$ will be exactly the same as $I_{(c)}(\alpha)$, being f even.

By proceeding in this way, for small values of k, in [14] we investigated three specific families of L-functions, one for each symmetry type. For the unitary case, we study the continuous family of the Riemann zeta function, parametrized by a vertical shift, i.e. $\zeta := \{\zeta(s+ia) : a \in \mathbb{R}\}$; assuming the relevant ratio conjecture (see [9, Conjecture 5.1]), we define

$$\mathscr{D}_k^{\zeta}(f) = \frac{1}{\int_T^{2T} |\zeta(1/2+it)|^{2k} dt} \int_T^{2T} \sum_{\gamma} f\left(\frac{\log T}{2\pi}(\gamma - t) \right) |\zeta(1/2+it)|^{2k} dt$$

and we prove the following result.

THEOREM 2.1. *(from [14, Proposition 3.1-3.2]).* Let us assume the Ratio Conjecture and the Riemann Hypothesis for the Riemann zeta function. We consider a test function $f(z)$ which is holomorphic throughout the strip $|\Im(z)| < 2$, real on the real line, even and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$. Then, for $k \leq 2$, we have

$$\mathscr{D}_k^{\zeta}(f) = \int_{-\infty}^{+\infty} f(x) W_U^k(x) dx + O\Big(\frac{1}{\log T}\Big)$$

with

$$W_U^1(x) = 1 - \operatorname{sinc}^2(x) = 1 - \frac{\sin^2(\pi x)}{(\pi x)^2}$$

and

$$W_U^2(x) = 1 - \frac{2+\cos(2\pi x)}{(\pi x)^2} + \frac{3\sin(2\pi x)}{(\pi x)^3} + \frac{3(\cos(2\pi x)-1)}{2(\pi x)^4}.$$

As a symplectic example, we consider the symplectic family L_χ of quadratic Dirichlet L-function and denote

$$\mathscr{D}_k^{L_\chi}(f) = \frac{1}{\sum_{d \leq X} L(1/2,\chi_d)^k} \sum_{d \leq X} \sum_{\gamma_d} f\left(\frac{\log X}{2\pi}\gamma_d\right) L(1/2,\chi_d)^k.$$

Under the ratio conjecture for these L-functions (see [9, Conjecture 5.2]) we compute the weighted one-level density for this family in the following theorem.

THEOREM 2.2. *(from [14, Proposition 4.1-4.4]).* Assume GRH and the Ratio Conjecture for quadratic Dirichlet L-functions. For any test function f as in Theorem 2.1 and $k \leq 4$, we have

$$\mathscr{D}_k^{L_\chi}(f) = \int_{-\infty}^{+\infty} f(x) W_{USp}^k(x) dx + O\Big(\frac{1}{\log X}\Big)$$

as $X \to \infty$, where

$$\begin{aligned}
W_{USp}^1(x) &= 1 + \frac{\sin(2\pi x)}{2\pi x} - \frac{2\sin^2(\pi x)}{(\pi x)^2},\\
W_{USp}^2(x) &= 1 - \frac{\sin(2\pi x)}{2\pi x} - \frac{24(1-\sin^2(\pi x))}{(2\pi x)^2} + \frac{48\sin(2\pi x)}{(2\pi x)^3} - \frac{96\sin^2(\pi x)}{(2\pi x)^4},\\
W_{USp}^3(x) &= 1 + \frac{\sin(2\pi x)}{2\pi x} - \frac{12\sin^2(\pi x)}{(\pi x)^2} - \frac{240\sin(2\pi x)}{(2\pi x)^3}\\
&\quad - \frac{15(6-10\sin^2(\pi x))}{(\pi x)^4} + \frac{2880\sin(2\pi x)}{(2\pi x)^5} - \frac{90\sin^2(\pi x)}{(\pi x)^6},\\
W_{USp}^4(x) &= 1 - \frac{\sin(2\pi x)}{2\pi x} - \frac{10(1+\cos(2\pi x))}{(\pi x)^2} + \frac{90\sin(2\pi x)}{(\pi x)^3}\\
&\quad - \frac{15(3-31\cos(2\pi x))}{(\pi x)^4} - \frac{1470\sin(2\pi x)}{(\pi x)^5}\\
&\quad - \frac{315(1+9\cos(2\pi x))}{(\pi x)^6} + \frac{3150\sin(2\pi x)}{(\pi x)^7} - \frac{1575(1-\cos(2\pi x))}{(\pi x)^8}.
\end{aligned}$$

Finally, we look at the orthogonal family $L_{\Delta,\chi}$ of the quadratic twists of the L-function associated with the discriminant modular form Δ. We denote

$$\mathscr{D}_k^{L_{\Delta,\chi}}(f) = \frac{1}{\sum_{d\le X} L_\Delta(1/2,\chi_d)^k} \sum_{d\le X} \sum_{\gamma_d} f\left(\frac{\log X}{2\pi}\gamma_d\right) L_\Delta(1/2,\chi_d)^k$$

and, under the relevant ratio conjecture (see [9, Conjecture 5.3]) we prove the following result.

THEOREM 2.3. *(from [14, Proposition 5.1-5.4]).* Assume GRH and the Ratio Conjecture for the quadratic twists of L-function associated with the discriminant modular form Δ. For any test function f as in Theorem 2.1 and $k \le 4$, we have

$$\mathscr{D}_k^{L_{\Delta,\chi}}(f) = \int_{-\infty}^{+\infty} f(x) W_{SO^+}^k(x)dx + O\Big(\frac{1}{\log X}\Big)$$

as $X \to \infty$, where

$$\begin{aligned}
W_{SO^+}^1(x) &= 1 - \frac{\sin(2\pi x)}{2\pi x},\\
W_{SO^+}^2(x) &= 1 + \frac{\sin(2\pi x)}{\pi x} - \frac{2\sin^2(\pi x)}{(\pi x)^2},\\
W_{SO^+}^3(x) &= 1 - \frac{\sin(2\pi x)}{2\pi x} - \frac{24(1-\sin^2(\pi x))}{(2\pi x)^2} + \frac{48\sin(2\pi x)}{(2\pi x)^3} - \frac{96\sin^2(\pi x)}{(2\pi x)^4},\\
W_{SO^+}^4(x) &= 1 + \frac{\sin(2\pi x)}{2\pi x} - \frac{12\sin^2(\pi x)}{(\pi x)^2} - \frac{240\sin(2\pi x)}{(2\pi x)^3}\\
&\quad - \frac{15(6-10\sin^2(\pi x))}{(\pi x)^4} + \frac{2880\sin(2\pi x)}{(2\pi x)^5} - \frac{90\sin^2(\pi x)}{(\pi x)^6}.
\end{aligned}$$

3. An unconditional approach

In this section we present an unconditional method to attack the problem of estimating $\mathscr{D}_k^{\mathscr{F}}(f)$; the results occurring throughout the section already appeared in [3], a joint work with Sandro Bettin which is currently submitted for publication. This approach, which is quite classical and it was used in the context of the one-level density for the Riemann zeta function e.g. by Hughes and Rudnick [17], does not require the assumption of neither the ratio conjecture nor the Riemann Hypothesis; however, it is less flexible and in the following we will only deal with a specific case. Let us consider the unitary (continuous) family of the Riemann zeta function, parametrized by a vertical shift and let us fix $k = 1$. We denote

$$N_f(t) := \sum_\gamma f\left(\frac{\log T}{2\pi}(\gamma - t)\right)$$

where $\frac{1}{2} + i\gamma$ denotes the generic non-trivial zero of zeta ($\gamma \in \mathbb{C}$, as we are not assuming the Riemann Hypothesis). We first consider a smooth average over t; namely, given a non-negative smooth weight function ϕ supported in $[1/2,3]$, for any Riemann-integrable function $g:\mathbb{R}\to\mathbb{C}$, we define

$$\langle g\rangle_{|\zeta|^2,\phi} := \frac{1}{T\log T}\int_{-\infty}^{+\infty} g(t)|\zeta(1/2+it)|^2\phi\left(\frac{t}{T}\right)dt.$$

For any even and smooth test function f such that its Fourier transform is compactly supported, the subject of our study is $\left\langle N_f\right\rangle_{|\zeta|^2,\phi}$.

THEOREM 3.1. *(from [3, Theorem 1.2]).* For any smooth, even and real-valued function f with $\hat{f}$ smooth and compactly supported, we have

$$\langle N_f\rangle_{|\zeta|^2,\phi} = \hat{\phi}(0)\int_{-\infty}^{+\infty} f(x)\left(1-\frac{\sin^2(\pi x)}{(\pi x)^2}\right)dx + O_\phi\left(\frac{1}{\log T}\right)$$

as $T\to\infty$.

With some analytical considerations, at the cost of assuming the Riemann Hypothesis and of an error term of $o(1)$, we can also weaken the assumption that f is smooth and $\hat{f}$ is compactly supported to the hypothesis that f is continuous and decays faster than any power as $x\to\infty$. Moreover, we can deduce the sharp cut-off one-level density, as shown in the following corollary.

COROLLARY 3.2. *(from [3, Corollary 1.1]).* Assume the Riemann Hypothesis. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous, even and real-valued function such that $f(x)$ decays faster than any power as $x\to\infty$. Then, as $T\to\infty$, we have

$$\frac{1}{T\log T}\int_T^{2T} N_f(t)|\zeta(\tfrac{1}{2}+it)|^2dt = \int_{-\infty}^{+\infty} f(x)\left(1-\frac{\sin^2(\pi x)}{(\pi x)^2}\right)dx + o(1).$$

Very recently, Sugiyama and Suriajaya [28] proved the analogous of Corollary 3.2 in the case of Dirichlet L-functions, under the extra condition that $\mathrm{supp}(\hat{f})\subseteq(-\frac{1}{3},\frac{1}{3})$.

We now describe the method of the proof of Theorem 3.1 in broad strokes. The first step is a classical explicit formula originally due to Riemann; as an application of the residue theorem we write

$$N_f(t) = \frac{1}{2\pi i}\left(\int_{(2)}-\int_{(-1)}\right) f\left(-i\frac{\log T}{2\pi}(s-1/2)\right)\frac{\zeta'}{\zeta}(s+it)ds. \tag{8}$$

In the integral over the 2-line, we expand the logarithmic derivative of zeta into its Dirichlet series and we integrate term by term, getting

$$\begin{aligned} I_{(2)}(t) &= -\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\frac{1}{2\pi i}\int_{(2)} f\left(-i\frac{\log T}{2\pi}(s-1/2)\right)n^{-(s-\frac{1}{2}+it)}ds \\ &= -\frac{1}{\log T}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{1/2+it}}\hat{f}\left(\frac{\log n}{\log T}\right). \end{aligned} \tag{9}$$

For the integral over the line $\Re(s) = -1$, we use the functional equation for zeta in the form

$$\frac{\zeta'}{\zeta}(1-s) = \frac{X'}{X}(s) - \frac{\zeta'}{\zeta}(s),$$

and we easily get

$$\begin{aligned} I_{(-1)}(t) &= \frac{1}{2\pi i}\int_{(2)} \left(\frac{X'}{X}(s) - \frac{\zeta'}{\zeta}(s)\right) f\left(-i\frac{\log T}{2\pi}(s-1/2)\right) ds \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{X'}{X}(1/2 - it + ix) f\left(\frac{\log T}{2\pi}x\right) dx - I_{(2)}(-t) \end{aligned} \tag{10}$$

since f is even. Plugging (9) and (10) into (8), we finally get

$$\begin{aligned} N_f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{X'}{X}(1/2 - it + ix) f\left(\frac{\log T}{2\pi}x\right) dx \\ - \frac{1}{\log T}\sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}}(n^{it} + n^{-it})\hat{f}\left(\frac{\log n}{\log T}\right). \end{aligned} \tag{11}$$

We note that this explicit formula actually holds in a much wider context, for any L-function in the Selberg class, as explained by Conrey in [6].

The first term in (11) is completely explicit and one can show that

$$\left\langle \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{X'}{X}(1/2 - it + ix) f\left(\frac{\log T}{2\pi}x\right) dx \right\rangle_{|\zeta|^2,\phi} = \hat{f}(0) + O_\phi\left(\frac{1}{\log T}\right). \tag{12}$$

Therefore, our goal is now to estimate the weighted average over t of the second term in (11), being

(13)

$$\begin{aligned} &\left\langle -\frac{1}{\log T}\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1/2}}(n^{it} + n^{-it})\hat{f}\left(\frac{\log n}{\log T}\right) \right\rangle_{|\zeta|^2,\phi} \\ &\qquad = -\frac{1}{T(\log T)^2}\int_{-\infty}^{+\infty} \Big(P(1/2+it) + P(1/2-it)\Big)|\zeta(1/2+it)|^2 \phi\left(\frac{t}{T}\right) dt \end{aligned}$$

with

$$P(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}\hat{f}\left(\frac{\log n}{\log T}\right).$$

Being $\hat{f}$ compactly supported, $P(s)$ is a Dirichlet polynomial; the second moment of the Riemann zeta function twisted by the absolute value squared of a Dirichlet polynomial is known (see Balasubramanian, Conrey and Heath-Brown's work [1]), if the length of the polinomial is at most T^θ, for some $\theta < \frac{1}{2}$†. This would lead to Theorem 3.1 only for test function f such that $\hat{f}$ is supported in $[-a, a]$, with $0 < a < 1$. To extend the

†This was improved to $\theta < \frac{1}{2} + \frac{1}{66}$ by Bettin, Chandee and Radziwiłł [2].

support of the Fourier transform of f to any positive a, in [3] we prove a variation of the classical Balasubramanian, Conrey and Heath-Brown's result, providing a formula for the quantity

$$\int_{-\infty}^{+\infty} A(1/2+it)|\zeta(1/2+it)|^2\phi\left(\frac{t}{T}\right)dt$$

for Dirichlet polynomials $A(s)$ of any (fixed) length. Once we have this auxiliary result, we can evaluate explicitly (13), showing that

$$\left\langle -\frac{1}{\log T}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{1/2}}(n^{it}+n^{-it})\hat{f}\left(\frac{\log n}{\log T}\right)\right\rangle_{|\zeta|^2\cdot\phi} = -\int_{-\infty}^{+\infty} f(x)\frac{\sin^2(\pi x)}{(\pi x)^2} + O\left(\frac{1}{\log T}\right).$$

This proves Theorem 3.1, together with (11) and (12).

The deduction of Corollary 3.2 is instead analytical in nature. The proof is essentially based on the fact that a continuous function which decays faster than any powers can be *well-approximated* by a smooth function with compactly supported Fourier transform. The Riemann Hypothesis is needed to ensure that the test function f is a real function.

Acknowledgments. The author was supported by Czech Science Foundation, grant 21-00420M, and by the FRG grant DMS 1854398.

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1991 Mathematics Subject Classification. 11M06, 11M26.

Alessandro Fazzari,
Charles University, Faculty of Mathematics and Physics,
Department of Algebra, Sokolovska 83, 18600 Praha 8, Czech Republic e-mail:
fazzari@karlin.mff.cuni.cz

American Institute of Mathematics, 600 East Brokaw Road,
San Jose, CA 95112, US
e-mail: fazzari@aimath.org

Lavoro pervenuto in redazione il 07.03.2022.