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S. Ssekajja*

A NOTE ON CONTACT CR-LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE SASAKIAN MANIFOLD

Abstract. In this paper, we prove that any totally contact umbilic contact proper CR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$, tangent to the structure vector field, is totally contact geodesic and minimally immersed in $\overline{M}$. For a Sasakian space form $\overline{M}(c)$, we show that the above result implies that $c = -3$.

1. Introduction

In [3, 4], A. Bejancu introduced the theory of CR-submanifolds, as a generalisation of invariant and anti-invariant submanifolds. More precisely, a submanifold M of a Kaehler manifold $(\overline{M}, \overline{J})$, where $\overline{J}$ is the Hermitian structure on $\overline{M}$, is called a CR-submanifold if there exists on M a differentiable $\overline{J}$-holomorphic distribution D such that its orthogonal complement $D^{\perp}$ is a totally real distribution, that is; $\overline{J}D_p^{\perp} \subset T_pM^{\perp}$. This notion was later introduced for submanifolds of almost contact metric manifolds, see instance [1, 14, 20]. CR-submanifolds of almost contact metric manifolds are often called contact CR-submanifolds. In the book [8], the authors introduced the notion of CR-lightlike submanifolds of an indefinite Kaehler manifold. This was later extended to contact CR-lightlike submanifolds of an indefinite Sasakian manifold in [9, Chapter 7]. This motivated a further extension of the theory to contact CR-lightlike submanifolds of other almost contact metric manifolds, like the Kenmotsu manifolds [10]. Among other contributions to the general theory of lightlike submanifolds, we quote the following; [2], [5], [6], [7], [10], [15], [16], [17], [18].

In this paper, we study the geometry of a lightlike submanifold of a Sasakian manifold $\overline{M}$ tangent to the structure vector field of $\overline{M}$. A characterisation of totally contact umbilic submanifold is given in Theorem 1. In fact, it is proved that any totally contact umbilic contact proper CR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$, tangent to the structure vector field, is totally contact geodesic and minimally immersed in $\overline{M}$. For a Sasakian space form $\overline{M}(c)$, we have shown that the above theorem implies that $c = -3$ (see Theorem 2). The rest of the paper is arranged as follows; In section 2, we give some basic facts about lightlike submanifolds and Sasakian geometry needed in the rest of the paper. In Section 3, we present the main results of this paper.

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2. Preliminaries

Consider an $(m+n)$-dimensional semi-Riemannian manifold $(\overline{M},\overline{g})$, of constant index q, $1 \leq q \leq m+n$. Let (M,g) be a submanifold of $\overline{M}$ of codimension n. We assume, through out this paper, that $m,n \geq 1$. For each point $p \in M$, define the orthogonal complement $T_pM^{\perp}$ of the tangent space T_pM by

$$T_pM^{\perp} = \{u \in T_pM : \overline{g}(u,v) = 0,\ \forall\, v \in T_pM\}.$$

Let us put $\mathrm{Rad}\,T_pM = \mathrm{Rad}\,T_pM^{\perp} = T_pM \cap T_pM^{\perp}$. The submanifold M of $\overline{M}$ is said to be r-lightlike submanifold (one supposes that the index of $\overline{M}$ is $p \geq r$), if the mapping

$$\mathrm{Rad}\,TM : p \in M \longrightarrow \mathrm{Rad}\,T_pM,$$

defines a smooth distribution on M of rank $r > 0$. We call $\mathrm{Rad}\,TM$ the radical distribution on M. Throughout this paper, an r-lightlike submanifold will simply be called a *lightlike submanifold* and g is *lightlike metric*.

Consider a screen distribution $S(TM)$ of M, which is a semi-Riemannian complementary distribution of $\mathrm{Rad}\,TM$ in TM, that is,

$$TM = \mathrm{Rad}\,TM \perp S(TM).$$

Choose a screen transversal bundle $S(TM^{\perp})$, which is semi-Riemannian and complementary to $\mathrm{Rad}\,TM$ in $TM^{\perp}$. It is well known [8, 9] that, for any local basis $\{\xi_i\}$, $1 \leq i \leq r$, of $\mathrm{Rad}\,TM$, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $S(TM)^{\perp}$ such that

$$g(\xi_i,N_j) = \delta_{ij}, \quad \overline{g}(N_i,N_j) = 0, \quad 1 \leq i,j \leq r.$$

It follows that there exists a lightlike transversal vector bundle $l\mathrm{tr}(TM)$ locally spanned by $\{N_i\}$ [8,9]. Let us denote by $\mathrm{tr}(TM)$ the complementary (but not orthogonal) vector bundle to TM in $T\overline{M}$. Then, we have the decompositions:

$$\begin{aligned} \mathrm{tr}(TM) &= l\mathrm{tr}(TM) \perp S(TM^{\perp}), \\ \text{and} \quad T\overline{M}_{|M} &= S(TM) \perp S(TM^{\perp}) \perp \{\mathrm{Rad}\,TM \oplus l\mathrm{tr}(TM)\}. \end{aligned}$$

We remark that the screen distribution $S(TM)$ is not unique, but is canonically isomorphic to the factor vector bundle $TM/\mathrm{Rad}\,TM$ [12]. A lightlike submanifold with a chosen screen distribution is denoted by $(M,g) = (M,g,S(TM),S(TM^{\perp}))$. Moreover, a lightlike submanifold (M,g) is called:

1. r-lightlike if $1 \leq r < \min\{m,n\}$;
2. co-isotropic if $1 \leq r = n < m, \quad S(TM^{\perp}) = \{0\}$;
3. isotropic if $1 \leq r = m < n, \quad S(TM) = \{0\}$;
4. totally lightlike if $r = n = m, \quad S(TM) = \{0\} = S(TM^{\perp})$.

Throughout this paper, we denote by $F(M)$ the algebra of smooth functions on M and $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E (same notation for any other vector bundle) over M. Next, the Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \ \forall X,Y \in \Gamma(TM), \tag{1}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^t_X V, \ \forall X \in \Gamma(TM), \ V \in \Gamma(\mathrm{tr}(TM)), \tag{2}$$

where $\{\nabla_X Y, A_V X\}$ and $\{h(X,Y), \nabla^t_X V\}$ belongs to $\Gamma(TM)$ and $\Gamma(\mathrm{tr}(TM))$ respectively. Further, ∇ and ∇^t are linear connections on M and $\mathrm{tr}(TM)$, respectively. The second fundamental form h is a symmetric $F(M)$-bilinear form on $\Gamma(TM)$ with values in $\Gamma(\mathrm{tr}(TM))$ and the shape operator A_V is a linear endomorphism of $\Gamma(TM)$. Moreover, (1) and (2) leads to (see [9, p. 196–198])

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \quad \forall X,Y \in \Gamma(TM), \tag{3}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N), \quad \forall N \in \Gamma(l\mathrm{tr}(TM)), \tag{4}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W), \quad \forall W \in \Gamma(S(TM^{\perp})), \tag{5}$$

where A_N and A_W are called the shape operators of M. Furthermore, we call h^l and h^s the lightlike second fundamental form and the screen second fundamental form, respectively. Furthermore, ∇^l and ∇^s are, respectively, linear connections on $l\mathrm{tr}(TM)$ and $S(TM^{\perp})$, called the lightlike connection and the scree Otsuki connections on $l\mathrm{tr}(TM)$ and $S(TM^{\perp})$, respectively. Denote the projection of TM on $S(TM)$ by P. Then, we have

$$\nabla_X PY = \nabla^*_X PY + h^*(X,PY), \quad \forall X,Y \in \Gamma(TM), \tag{6}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \quad \forall \xi \in \Gamma(\mathrm{Rad}\,TM), \tag{7}$$

where ∇^* and A^*_ξ are, respectively, the linear connection and shape operator of $S(TM)$. Furthermore, h^* and ∇^{*t} are the second fundamental form and a linear connection on $\mathrm{Rad}\,TM$, respectively. Furthermore, by using (1), (3)-(7), we obtain

$$\overline{g}(h^s(X,Y),W) + \overline{g}(Y, D^l(X,W)) = g(A_W X, Y), \tag{8}$$

$$\overline{g}(D^s(X,N),W) = \overline{g}(N, A_W X), \tag{9}$$

$$\overline{g}(h^l(X,PY),\xi) = g(A^*_\xi X, PY), \quad \overline{g}(h^l(X,\xi),\xi) = 0, \tag{10}$$

$$\overline{g}(h^*(X,PY),N) = g(A_N X, PY),$$

where $X,Y \in \Gamma(TM)$, $\xi \in \Gamma(\mathrm{Rad}\,TM)$ and $W \in \Gamma(S(TM^{\perp}))$. In general, the induced connection ∇ on M is not a metric connection. Since $\overline{\nabla}$ is a metric connection, by using (3) we get

$$(\nabla_X g)(Y,Z) = \overline{g}(h^l(X,Y),Z) + \overline{g}(h^l(X,Z),Y), \tag{11}$$

for all $X,Y,Z \in \Gamma(TM)$. It is important to note that ∇^* is a metric connection on $S(TM)$.

Denote by R the curvature tensor of M, then we have (see [9, p. 217] for more details)

$$\begin{aligned}\overline{R}(X,Y)Z &= R(X,Y)Z + A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y - A_{h^s(Y,Z)}X \\ &\quad + (\tilde{\nabla}_X h^l)(Y,Z) - (\tilde{\nabla}_Y h^l)(X,Z) + D^l(X, h^s(Y,Z)) - D^l(Y, h^s(X,Z)) \\ &\quad + (\tilde{\nabla}_X h^s)(Y,Z) - (\tilde{\nabla}_Y h^s)(X,Z) + D^s(X, h^l(Y,Z)) - D^s(Y, h^l(X,Z)),\end{aligned} \tag{12}$$

where $\tilde{\nabla}h^l$ and $\tilde{\nabla}h^s$ are given by

$$(\tilde{\nabla}_X h^l)(Y,Z) = \nabla^l_X h^l(Y,Z) - h^l(\nabla_X Y, Z) - h^l(Y, \nabla_X Z), \tag{13}$$

$$\text{and} \quad (\tilde{\nabla}_X h^s)(Y,Z) = \nabla^s_X h^s(Y,Z) - h^s(\nabla_X Y, Z) - h^s(Y, \nabla_X Z), \tag{14}$$

for all $X,Y,Z \in \Gamma(TM)$.

A lightlike submanifold (M,g) of a semi-Riemannian manifold $\overline{M}$ is said to be totally umbilic [9, Definition 5.3.1, p. 215] in $\overline{M}$ if there is a smooth transversal vector field $\alpha \in \Gamma(\mathrm{tr}(TM))$ on M, called the transversal curvature vector field of M, such that,

$$h(X,Y) = g(X,Y)\alpha, \tag{15}$$

for all $X,Y \in \Gamma(TM)$. We say that M is totally geodesic if $\alpha = 0$. The screen distribution $S(TM)$ is said to be totally umbilic in M [7, Definition 2, p. 62] if there is a smooth vector field $\beta \in \Gamma(\mathrm{Rad}\,TM)$ on M, such that

$$h^*(X,PY) = g(X,PY)\beta, \tag{16}$$

for all $X,Y \in \Gamma(TM)$. In case $\beta = 0$, we say $S(TM)$ is totally geodesic.

A $(2s+1)$-dimensional semi-Riemannian manifold $\overline{M} = (\overline{M}, \overline{g}, \overline{\phi}, \zeta, \eta)$ is said to be an indefinite Sasakian manifold if it admits an almost contact structure $(\overline{\phi}, \zeta, \eta)$, that is $\overline{\phi}$ is a tensor of type $(1,1)$ of rank $2s$, ζ is a unit spacelike vector field, also called the structure vector field, and η is a 1-form satisfying

$$\overline{\phi}^2 = -I + \eta \otimes \zeta, \quad \eta(\zeta) = 1, \quad \eta \circ \overline{\phi} = 0, \quad \overline{\phi}\zeta = 0, \tag{17}$$

$$\overline{g}(\overline{\phi}\overline{X}, \overline{\phi}\overline{Y}) = \overline{g}(\overline{X}, \overline{Y}) - \eta(\overline{X})\eta(\overline{Y}), \quad \eta(\overline{X}) = \overline{g}(\overline{X}, \zeta), \tag{18}$$

$$(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Y} = \overline{g}(\overline{X}, \overline{Y})\zeta - \eta(\overline{Y})\overline{X}, \quad \overline{\nabla}_{\overline{X}}\zeta = -\overline{\phi}\overline{X}, \tag{19}$$

for all $\overline{X}$ and $\overline{Y}$ tangent to $\overline{M}$. $\overline{\nabla}$ is the Levi-Civita connection for a semi-Riemannian metric $\overline{g}$.

A plane section π in $T_p\overline{M}$ of a Sasakian manifold $\overline{M}$ is called a $\overline{\phi}$-section if it is spanned by a unit vector $\overline{X}$ orthogonal to ζ and $\overline{\phi}\overline{X}$, where $\overline{X}$ is a non-null vector field on $\overline{M}$. The sectional curvature $K(\overline{X}, \overline{\phi}\overline{X})$ of a $\overline{\phi}$-section is called a $\overline{\phi}$-sectional curvature. If $\overline{M}$ has a $\overline{\phi}$-sectional curvature c which does not depend on the $\overline{\phi}$-section at each point, then, c is constant in $\overline{M}$ and $\overline{M}$ is called a Sasakian space form, denoted by $\overline{M}(c)$. Moreover, the curvature tensor $\overline{R}$ of $\overline{M}$ satisfies (see K. Ogiue [19, p. 229] for details):

$$\overline{R}(\overline{X},\overline{Y})\overline{Z} = \frac{c+3}{4}\{\overline{g}(\overline{Y},\overline{Z})\overline{X} - \overline{g}(\overline{X},\overline{Z})\overline{Y}\}$$
$$+\frac{c-1}{4}\{\eta(\overline{X})\eta(\overline{Z})\overline{Y} - \eta(\overline{Y})\eta(\overline{Z})\overline{X} + \overline{g}(\overline{X},\overline{Z})\eta(\overline{Y})\zeta$$
$$-\overline{g}(\overline{Y},\overline{Z})\eta(\overline{X})\zeta + \overline{g}(\overline{\phi}\overline{Y},\overline{Z})\overline{\phi}\overline{X} - \overline{g}(\overline{\phi}\overline{X},\overline{Z})\overline{\phi}\overline{Y} - 2\overline{g}(\overline{\phi}\overline{X},\overline{Y})\overline{\phi}\overline{Z}\}, \tag{20}$$

for any $\overline{X}$, $\overline{Y}$ and $\overline{Z}$ tangent to $\overline{M}$.

LEMMA 1. *On any lightlike submanifold (M,g) of an indefinite Sasakian space form $\overline{M}(c)$, the following holds:*

$$\overline{g}((\tilde{\nabla}_X h^l)(Y,Z) - (\tilde{\nabla}_Y h^l)(X,Z),\xi)$$
$$= \frac{c-1}{4}\{\overline{g}(Y,\overline{\phi}Z)\overline{g}(X,\overline{\phi}\xi) + \overline{g}(\overline{\phi}X,Z)\overline{g}(Y,\overline{\phi}\xi) + 2\overline{g}(\overline{\phi}X,Y)\overline{g}(Z,\overline{\phi}\xi)\}$$
$$+\overline{g}(h^s(X,Z),h^s(Y,\xi)) - \overline{g}(h^s(Y,Z),h^s(X,\xi)), \tag{21}$$
$$\overline{g}((\tilde{\nabla}_X h^s)(Y,Z) - (\tilde{\nabla}_Y h^s)(X,Z),W)$$
$$= \frac{c-1}{4}\{\overline{g}(Y,\overline{\phi}Z)\overline{g}(X,\overline{\phi}W) + \overline{g}(\overline{\phi}X,Z)\overline{g}(Y,\overline{\phi}W) + 2\overline{g}(\overline{\phi}X,Y)\overline{g}(Z,\overline{\phi}W)\}$$
$$+\overline{g}(h^l(X,Z),A_W Y) - \overline{g}(h^l(Y,Z),A_W X), \tag{22}$$

for all X, Y and Z tangent M.

Proof. Follows directly while using (20), (8), (9) and (12). □

Let (M,g) be a lightlike submanifold of an almost contact metric manifold $\overline{M}$, with ζ tangent to M. We say that M is totally contact umbilic [9, Definition 7.4.7, p. 324] if there exist a smooth vector field $\mu \in \Gamma(\mathrm{tr}(TM))$, such that

$$h(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}\mu + \eta(X)h(Y,\zeta) + \eta(Y)h(X,\zeta), \tag{23}$$

for all $X,Y \in \Gamma(TM)$. In case $\mu = 0$, we say that M is totally contact geodesic. Moreover, for any totally contact umbilic lightlike submanifold, (23) leads to

$$h^l(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}\mu_l + \eta(X)h^l(Y,\zeta) + \eta(Y)h^l(X,\zeta), \tag{24}$$
$$h^s(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}\mu_s + \eta(X)h^s(Y,\zeta) + \eta(Y)h^s(X,\zeta), \tag{25}$$

for all $X,Y \in \Gamma(TM)$, $\mu_l \in \Gamma(l\mathrm{tr}(TM))$ and $\mu_s \in \Gamma(S(TM^\perp))$. For more details on lightlike submanifolds we refer the reader to the books [8] and [9].

3. Contact CR-lightlike submanifolds

In the book [9, Chapter 7], K. L. Duggal and B. Sahin initiated the study of contact CR-lightlike submanifolds (M,g) of an indefinite Sasakian manifold $\overline{M} = (\overline{M},\overline{g},\overline{\phi},\zeta,\eta)$.

The theory and constructions on such a submanifold is based on the assumption that the structure vector field ζ is tangent to M, and in particular, belonging to the screen distribution $S(TM)$ of M. C. Calin, in his thesis [6], has considered similar assumptions. We shall follow the same assumptions in this paper, and prove, in Theorem 1 and Corollary 3, that:

A totally contact umbilic contact proper CR-lightlike submanifold (M,g) of an indefinite Sasakian manifold $\overline{M}$, tangent to the structure vector field, is totally contact geodesic, and minimally immersed in $\overline{M}$.

This result comes in as an improvement of Theorem 7.4.9 of [9, p. 325]. Moreover, in case $\overline{M}$ is a Sasakian space form, we prove, in Theorem 2, that such a totally contact umbilic contact CR-lightlike submanifold is only embeddable in $\overline{M}$ if $\overline{M}$ is a space of constant holomorphic sectional curvature -3.

DEFINITION 1 (Duggal-Sahin [9]). Let $(M,g,S(TM),S(TM^{\perp})$ be a lightlike submanifold, tangent to the structure vector field ζ, immersed in an indefinite Sasakian manifold $(\overline{M},\overline{g})$. We say that M is a contact CR-lightlike submanifold of $\overline{M}$ if the following conditions are satisfied:

1. $\mathrm{Rad}\,TM$ is a distribution on M such that $\mathrm{Rad}\,TM \cap \overline{\phi}\mathrm{Rad}\,TM = \{0\}$.
2. There exist vector bundles D_0 and D' over M such that

$$S(TM) = \{\overline{\phi}\mathrm{Rad}\,TM \oplus D'\} \perp D_0 \perp \langle\zeta\rangle,$$
$$\overline{\phi}D_0 = D_0, \quad \overline{\phi}D' = \mathscr{L} \perp l\mathrm{tr}(TM),$$

where D_0 is nondegenerate, and $\mathscr{L}$ is a subbundle of $S(TM^{\perp})$.

In view of the above definition, we have the decomposition

$$TM = \{D \oplus D'\} \perp \langle\zeta\rangle, \quad D = \mathrm{Rad}\,TM \perp \overline{\phi}\mathrm{Rad}\,TM \perp D_0. \tag{26}$$

A contact CR-lightlike submanifold is *proper* if $D_0 \neq \{0\}$ and $\mathscr{L} \neq \{0\}$.

A basic example of a contact CR-lightlike submanifold is a lightlike hypersurface as described in Example 1 of [9, p. 321]. For an example of proper contact CR-lightlike submanifold, we need the following structure: Let $(\mathbb{R}^{2n+1}_q, \overline{\phi}_0, \zeta, \eta, \overline{g})$ denote the manifold $\mathbb{R}^{2n+1}_q$ with its usual Sasakian structure given by

$$\eta = \frac{1}{2}\left(dz - \sum_{i=1}^{n} y^i dx^i\right), \qquad \zeta = 2\partial z,$$
$$\overline{g} = \eta \otimes \eta + \frac{1}{4}\left(-\sum_{i=q+1}^{q/2} dx^i \otimes dx^i + dy^i \otimes dy^i + \sum_{i=1}^{q/2} dx^i \otimes dx^i + dy^i \otimes dy^i\right),$$
$$\overline{\phi}_0\left(\sum_{i}^{n}(X_i\partial x^i + Y_i\partial y^i) + Z\partial z\right) = \sum_{i=1}^{m}(Y_i\partial x^i - X_i\partial y^i) + \sum_{i=1}^{m} Y_i y^i \partial z,$$

where $(x^i; y^i; z)$ are the Cartesian coordinates. With such a structure, K. L. Duggal and B. Sahin in [9, p. 322] constructed the following proper contact CR-lightlike submanifold tangent to ζ:

EXAMPLE 1 ([9]). Let $\overline{M} = (\mathbb{R}^9_2, \overline{g})$ be a semi-Euclidean space, where $\overline{g}$ is of signature $(-,+,+,+,-,+,+,+,+)$ with respect to the canonical basis

$$\left\{\partial x^1, \partial x^2, \partial x^3, \partial x^4, \partial y^1, \partial y^2, \partial y^3, \partial y^4, \partial z\right\}.$$

Suppose M is a submanifold of $\mathbb{R}^9_2$ defined by

$$x^1 = y^4, \quad x^2 = \sqrt{1-(y^2)^2} : \ y \neq \pm 1.$$

By a simple calculation, we notice that a local frame of TM is given by

$$Z_1 = 2\left(\partial x^1 + \partial y^4 + y^1 \partial z\right), \quad Z_2 = 2\left(\partial x^4 - \partial y^1 + y^4 \partial z\right)$$
$$Z_3 = \partial x^3 + y^3 \partial z, \quad Z_4 = \partial y^3, \quad Z_5 = -\frac{y^2}{x^2}\partial x^2 + \partial y^2 - \frac{(y^2)^2}{x^2}\partial z,$$
$$Z_6 = \partial x^4 + \partial y^1 + y^4 \partial z, \quad Z_7 = \zeta = 2\partial z.$$

It follows that $\mathrm{Rad}\,TM = \mathrm{Span}\{Z_1\}$, $\overline{\phi}_0 \mathrm{Rad}\,TM = \mathrm{Span}\{Z_2\}$ and $\mathrm{Rad}\,TM \cap \overline{\phi}_0 \mathrm{Rad}\,TM = \{0\}$. Thus, (1) holds. Moreover, $\overline{\phi}_0 Z_3 = -Z_4$, and hence $D_0 = \mathrm{Span}\{Z_3, Z_4\}$ is invariant with respect to $\overline{\phi}_0$. By direct calculations, we have $S(TM^\perp) = \mathrm{Span}\{W = \partial x^2 + \frac{y^2}{x^2}\partial y^2 + y^2 \partial z\}$, such that $\overline{\phi}_0 W = -Z_5$, and $l\mathrm{tr}(TM) = \mathrm{Span}\{N = -\partial x^1 + \partial y^4 - y^1 \partial z\}$ such that $\overline{\phi}_0 N = Z_6$. It follows that (2) is satisfied and M is a contact CR-lightlike submanifold.

For a contact CR-lightlike submanifold (M, g) of an indefinite Sasakian manifold $\overline{M}$, K. L. Duggal and B. Sahin, in [9, Propositions 7.4.2, 7.4.3, 7.4.4 and 7.4.5], have studied the geometry of the distributions D, D', $D \oplus D'$ and $D \perp \langle\zeta\rangle$. More precisely, it has been proved that D and $D \oplus D'$ are never integrable.

Next, the following lemma is fundamental to this paper.

LEMMA 2. *On a contact CR-lightlike submanifold of an indefinite Sasakian manifold, the following holds:*

$$\nabla_X \zeta = -\overline{\phi} X, \quad h^l(X, \zeta) = h^s(X, \zeta) = 0, \quad \forall X \in \Gamma(D \perp \langle\zeta\rangle), \tag{27}$$

$$\nabla_X \zeta = 0, \quad h^l(X, \zeta) + h^s(X, \zeta) = -\overline{\phi} X, \quad \forall X \in \Gamma(D'). \tag{28}$$

Proof. Using the second relation in (19), with $\overline{X}$ replaced with X, which is tangent to M, and relation (3), we have $\nabla_X \zeta + h^l(X, \zeta) + h^s(X, \zeta) = -\overline{\phi} X$. Now, if X belongs to the invariant distribution $D \perp \langle\zeta\rangle$, the last relation leads to (27). Finally, when X belongs to the anti-invariant distribution D' we get (28), which completes the proof. □

PROPOSITION 1. *A contact CR-lightlike submanifold of an indefinite Sasakian manifold is neither totally umbilic nor totally screen umbilic.*

Proof. If M is totally umbilic, the second relation in (27) and (15) lead to $g(X,\zeta)\alpha = 0$, for any X tangent to $D \perp \langle\zeta\rangle$. As ζ is a spacelike vector field (see the second relation in (17)), we take $X = \zeta$ in the previous relation and $g(\zeta,\zeta)\alpha = \alpha = 0$. That is, M is totally geodesic. Thus, the second relation in (28) leads to $\overline{\phi}X = 0$, for any X tangent to D'. This is clearly a contradiction. In fact, if we take $X = \overline{\phi}N$, where $N \in \Gamma(l\mathrm{tr}(TM))$, we get $\overline{\phi}^2 N = -N + \eta(N)\zeta = 0$, that is; $N = 0$, which is a contradiction. On the other hand, when $S(TM)$ is totally umbilic, (16), the first relation in (27) and relation (6) leads to $-\overline{\phi}X = \nabla^*_X\zeta + h^*(X,\zeta) = \nabla^*_X\zeta + g(X,\zeta)\beta$, for any X tangent to $D \perp \langle\zeta\rangle$. In particular, taking $X = \overline{\phi}\xi$ in the above relation, we have $\nabla^*_{\overline{\phi}\xi}\zeta = -\overline{\phi}^2\xi = \xi$, which gives the obvious contradiction $\xi = 0$. □

The following is an immediate consequence Proposition 1.

COROLLARY 1. *There exist no contact CR-lightlike submanifold such that:*

1. *$\overline{\nabla}_X N$ belongs to* $\mathrm{tr}(TM)$*, for any X tangent to M.*
2. *$\nabla_X PY$ belongs to $S(TM)$, for any X and Y tangent to M.*
3. *∇ is a metric connection on M.*

Proof. If $\overline{\nabla}_X N$ belongs to $\mathrm{tr}(TM)$, then (4) leads to $A_N X = 0$, for any X tangent to M. It follows that $S(TM)$ is totally geodesic, which leads to a contradiction as seen in Corollary 1. On the other hand, if $\nabla_X PY$ belongs to $S(TM)$, (6) gives $h^*(X,PY) = 0$, for any X and Y tangent to M, which shows that $S(TM)$ is totally geodesic. Again, by Corollary 1, this is an impossible case. Finally, assume that ∇ is a metric connection, then $\nabla g = 0$. It follows from (11) that $\overline{g}(h^l(X,Y),Z) + \overline{g}(h^l(X,Z),Y) = 0$, for any X, Y and Z tangent to M. Taking $Z = \xi$, we get $h^l(X,PY) = 0$. Thus, if $X \in \Gamma(D')$ and $PY = \zeta$, the last relation together with (28) leads to $\overline{\phi}X = 0$, for all $X \in \Gamma(D')$. This is, clearly, a contradiction. □

Let $\mathscr{L}^\perp$ denote the orthogonal complementary subbundle to $\mathscr{L}$ in $S(TM^\perp)$. Then, for a contact CR-lightlike submanifold, we put

$$\overline{\phi}X = fX + \omega X, \tag{29}$$

for any X tangent to M, where fX belongs to D and ωX belongs to $\mathscr{L} \perp l\mathrm{tr}(TM)$. We also have

$$\overline{\phi}W = BW + CW, \tag{30}$$

for any W tangent to $S(TM^\perp)$, where BW belongs to $\overline{\phi}\mathscr{L}$ and CW belongs to $\mathscr{L}^\perp$. Let us define the covariant derivatives of f and ω by

$$(\nabla_X f)(Y) = \nabla_X fY - f(\nabla_X Y) \quad \text{and} \quad (\nabla^t_X \omega)(Y) = \nabla^t_X \omega Y - \omega(\nabla_X Y),$$

for all X and Y tangent to M. Then, in view of (2), (3), (19), (29) and (30), we have

$$(\nabla_X f)(Y) = A_{\omega Y}X + Bh^s(X,Y) + \overline{\phi}h^l(X,Y) + \overline{g}(X,Y)\zeta - \eta(Y)X, \tag{31}$$
$$\text{and} \quad (\nabla^t_X \omega)(Y) = Ch^s(X,Y) - h^l(X,fY) - h^s(X,fY),$$

for any X and Y tangent to M.

In what follows, we characterise a totally contact umbilic proper contact CR-lightlike submanifold. To do that, we first quote the following remark about what is so far known on such a submanifold:

REMARK 1. Suppose that (M,g) is a totally contact umbilic proper contact CR-lightlike submanifold of an indefinite Sasakian manifold. Then:

1. $\mu_l = 0$ (see Lemma 7.4.8 in [9, p. 325]).
2. $\mu_s \in \Gamma(\mathscr{L})$ (see the proof of Theorem 7.4.9 in [9, p. 325]).

In view of Remark 1, we prove the following important result.

THEOREM 1. *A totally contact umbilic proper contact CR-lightlike submanifold of an indefinite Sasakian manifold is totally contact geodesic.*

Proof. First, by Remark 1, we know that $\mu_l = 0$ and $\mu_s \in \Gamma(\mathscr{L})$. It is left to prove that $\mu_s = 0$. In that line, replacing X with $\overline{\phi}N$ and Y with $\overline{\phi}\xi$ in (31), while noting that $fX = 0$, $\omega X = -N$, $fY = -\xi$ and $\omega Y = 0$, we get

$$-\nabla_{\overline{\phi}N}\xi - f(\nabla_{\overline{\phi}N}\overline{\phi}\xi) = \overline{\phi}h^l(\overline{\phi}N,\overline{\phi}\xi) + Bh^s(\overline{\phi}N,\overline{\phi}\xi) + \overline{g}(\overline{\phi}\xi,\overline{\phi}N)\zeta. \tag{32}$$

From (24), (25) and the fact $\mu_l = 0$ (see the first part of Remark 1), we derive

$$h^l(\overline{\phi}N,\xi) = h^s(\overline{\phi}N,\xi) = 0, \quad h^l(\overline{\phi}N,\overline{\phi}\xi) = \overline{g}(\overline{\phi}N,\overline{\phi}\xi)\mu_l = 0. \tag{33}$$

On the other hand, we have

$$h^s(\overline{\phi}N,\overline{\phi}\xi) = \overline{g}(\overline{\phi}N,\overline{\phi}\xi)\mu_s. \tag{34}$$

Considering (33) and (34) in (32), noting that μ_s is tangent to $\mathscr{L}$ (see the second part of Remark 1), we have

$$-\nabla_{\overline{\phi}N}\xi - f(\nabla_{\overline{\phi}N}\overline{\phi}\xi) = \overline{g}(\overline{\phi}N,\overline{\phi}\xi)B\mu_s + \overline{g}(\overline{\phi}\xi,\overline{\phi}N)\zeta. \tag{35}$$

Taking the inner product of (35) with $\overline{\phi}W$, where $W \in \mathscr{L}$, we get

$$-g(\nabla_{\overline{\phi}N}\xi,\overline{\phi}W) = \overline{g}(\overline{\phi}N,\overline{\phi}\xi)\overline{g}(B\mu_s,\overline{\phi}W). \tag{36}$$

Using (7), the first relation in (10) and (24), we derive

$$-g(\nabla_{\overline{\phi}N}\xi,\overline{\phi}W) = g(A^*_\xi\overline{\phi}N,\overline{\phi}W) = \overline{g}(h^l(\overline{\phi}N,\overline{\phi}W),\xi) = 0. \tag{37}$$

It follows from (36), (37) and the first relation in (18), that

$$\overline{g}(\xi,N)\overline{g}(\mu_s,W)=0. \tag{38}$$

As $\overline{g}(\xi,N)\neq 0$ we deduce, from (38) and the fact $\mu_s\in\Gamma(\mathscr{L})$, that $\overline{g}(\mu_s,W)=0$, which implies that $\mu_s=0$. That is, M is totally contact geodesic. □

COROLLARY 2. *The distribution $D\perp\langle\zeta\rangle$ is integrable.*

REMARK 2. Theorem 1 improves Theorem 7.4.9 of [9], by giving the exact nature of totally contact umbilic proper contact CR-lightlike submanifold of an indefinite Sasakian manifold. Furthermore, we note that h vanishes on $D\oplus D'$ for any totally contact umbilic proper contact CR-lightlike submanifold.

DEFINITION 2 (Duggal-Sahin [9]). A lightlike submanifold $(M,g,S(TM))$ isometrically immersed in a semi-Riemannian manifold $(\overline{M},\overline{g})$ is minimal if

1. $h^s=0$ on $\mathrm{Rad}\,TM$;
2. $\mathrm{trace}\,h=0$, where trace is written with respect to g restricted to $S(TM)$.

COROLLARY 3. *A totally contact umbilic proper contact CR-lightlike submanifold of an indefinite Sasakian manifold is minimal.*

Proof. Note that $\mu_l=\mu_s=0$, which shows that h vanishes on $D\oplus D'$. Clearly, $h^s=0$ on $\mathrm{Rad}\,TM$ since it is a subbundle of D (see relation (26) above). Also, note that $h(\zeta,\zeta)=0$, and hence $\mathrm{trace}\,h=0$, where trace is taken with respect to the screen distribution $S(TM)$ of M. □

THEOREM 2. *Let (M,g) be a totally contact umbilic proper contact CR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}(c)$. Then, $c=-3$.*

Proof. Using (13) and (24), with $X\in\Gamma(D)$, $Y=\overline{\phi}X$ and $Z\in\Gamma(D')$, we have

$$\begin{aligned}(\tilde{\nabla}_X h^l)(\overline{\phi}X,Z)&=\nabla^l_X h^l(\overline{\phi}X,Z)-h^l(\nabla_X\overline{\phi}X,Z)-h^l(\overline{\phi}X,\nabla_X Z)\\&=-\eta(\nabla_X\overline{\phi}X)h^l(Z,\zeta)-\eta(\nabla_X Z)h^l(\overline{\phi}X,\zeta).\end{aligned} \tag{39}$$

In view of Lemma 2, we see that $h^l(Z,\zeta)=-\overline{\phi}Z$ and $h^l(\overline{\phi}X,\zeta)=0$. Moreover, considering (11), we have

$$\eta(\nabla_X\overline{\phi}X)=g(\nabla_X\overline{\phi}X,\zeta)=-g(\overline{\phi}X,\nabla_X\zeta)=g(\overline{\phi}X,\overline{\phi}X). \tag{40}$$

It follows from (39) and (40) that

$$(\tilde{\nabla}_X h^l)(\overline{\phi}X,Z)=g(X,X)\overline{\phi}Z. \tag{41}$$

In a similar way, we derive

$$(\tilde{\nabla}_{\overline{\phi}X} h^l)(X,Z)=-g(X,X)\overline{\phi}Z. \tag{42}$$

Then, replacing (41) and (42) in (21) leads to

$$2g(X,X)\overline{g}(\overline{\phi}Z,\xi)=\frac{c-1}{2}g(X,X)g(Z,\overline{\phi}\xi),$$

from which, we get

$$(c+3)g(X,X)g(Z,\overline{\phi}\xi)=0. \tag{43}$$

As $D_0\neq\{0\}$ and $\overline{g}(Z,\overline{\phi}\xi)\neq 0$ for any $Z\in\Gamma(D')$, it follows from (43) that $c=-3$. $\square$

REMARK 3. In the proof of Theorem 1, one could have used relation (14) and (22) instead of (21). Again, in this case, we get $c=-3$.

COROLLARY 4. *There does not exist any totally contact umbilic proper contact CR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}(c)$ such that $c\neq -3$.*

PROPOSITION 2. *On a totally contact umbilic proper contact CR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}(c)$, R satisfies:*

$$g(R(X,\xi)\xi,X)=-3\overline{g}(\overline{\phi}X,\xi)\overline{g}(\xi,\overline{\phi}X),$$

for all X tangent to $S(TM)$.

Proof. First, we note from (24), (25) and Lemma 2 that

$$h^l(X,\xi)=\eta(X)h^l(\xi,\zeta)=0,\quad h^s(X,\xi)=\eta(X)h^s(\xi,\zeta)=0, \tag{44}$$

for any $X\in\Gamma(TM)$. Now, taking the inner product of (12) with PX and applying (44), we get

$$g(R(X,Y)\xi,PX)=\overline{g}(\overline{R}(X,Y)\xi,PX). \tag{45}$$

Then, applying (20) to (45), and remembering that $c=-3$, we derive

$$g(R(X,Y)\xi,PX)=\overline{g}(Y,\overline{\phi}\xi)\overline{g}(\overline{\phi}X,PX)+\overline{g}(\overline{\phi}X,\xi)\overline{g}(\overline{\phi}Y,PX)+2\overline{g}(\overline{\phi}X,Y)\overline{g}(\overline{\phi}\xi,PX),$$

for any X, Y tangent to M, from which our claims follows. $\square$

COROLLARY 5. *There does not exist a totally contact umbilic proper contact CR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}(c)$, with constant curvature.*

Proof. Suppose that M has constant curvature, then $R(X,\xi)\xi=0$. Proposition 2 gives $-\overline{g}(X,\overline{\phi}\xi)=0$, for any X tangent to $S(TM)$. Taking $X=\overline{\phi}N$, where $N\in\Gamma(l\mathrm{tr}(TM))$, in this relation, we get $\overline{g}(\overline{\phi}N,\overline{\phi}\xi)=\overline{g}(N,\xi)=0$, which is a contradiction. $\square$

Concluding remarks

Let (M,g) be an r-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$, tangent to ζ. Then, using the second relation in (19), relation (3) and the fact that $\overline{\phi}\zeta = 0$ (see (17)), we derive $\nabla_{\zeta}\zeta = 0$ and $h^l(\zeta,\zeta) + h^s(\zeta,\zeta) = 0$. It follows from the second relation that $h^l(\zeta,\zeta) = h^s(\zeta,\zeta) = 0$. Since ζ is a unit spacelike vector field, it follows from these relations together with (15) that any totally umbilic lightlike submanifold of an indefinite Sasakian manifold is totally totally geodesic, and should be invariant since $\overline{\phi}X = -\overline{\nabla}_X\zeta = -\nabla_X\zeta \in \Gamma(TM)$, for any X tangent to M. It is on this basic fact that K. L. Duggal and B. Sahin based on to introduce the concept of totally contact umbilic lightlike submanifolds in [9, Definition 7.4.7, p. 324]. Our findings, in Theorem 1 and Corollary 3, have demonstrated that totally contact umbilic *proper* contact CR-lightlike submanifolds of indefinite Sasakian manifold are totally contact geodesic and minimal. This leads to the following open problem:

PROBLEM 1. *Characterise a totally contact umbilic non-proper contact CR-lightlike submanifold (M,g) of an indefinite almost contact metric manifold $\overline{M}$.*

By a non-proper contact CR-lightlike submanifold, we mean a contact CR-lightlike submanifold in which $\dim D_0 \geq 0$ and $\dim \mathscr{L} \geq 0$. Clearly, such submanifolds will contain the proper contact CR-lightlike submanifolds. Note that lightlike hypersurfaces as described in [9, p. 313] (you may also see [13]) are basic examples of non-proper contact CR-lightlike submanifold in which $\dim \mathrm{Rad}\,TM = 1$, $D_0 \geq 0$ and $S(TM^{\perp}) = \{0\}$ implying $\mathscr{L} = \{0\}$. It is easy to see that a totally contact umbilic lightlike hypersurface is not necessarily contact geodesic, and in fact when $\overline{M}$ is a Sasakian space form, then $c = -3$, and the vector field μ of (23) satisfy the relations

$$\nabla^l_{\xi}\mu = \overline{g}(\xi,\mu)\mu \quad \text{and} \quad \nabla^l_{PX}\mu = 0, \tag{46}$$

for any X tangent to M, in which we have used (22). Of course, the equations above can be reduced to the well-known differential equations as shown in [13, Theorem 2.4, p. 1562]. In fact, we may write $\mu = \lambda N$, where λ is some smooth function on $\mathscr{U} \subset M$ and N the lightlike transversal vector field spanning $l\mathrm{tr}(TM)$. Moreover, we may write $\nabla^l_X N = \tau(X)N$, where τ is a differential 1-form on M, given by $\tau(X) = \overline{g}(\nabla^l_X N,\xi)$, for any X tangent to M. By a direct calculation, we have

$$\nabla^l_X\mu_l = \nabla^l_X(\lambda N) = (X\lambda)N + \lambda\nabla^l_X N = \{X\lambda + \lambda\tau(X)\}N, \tag{47}$$

It then follows from (47) and the equations in (46) that

$$\xi\lambda + \lambda\tau(\xi) - \lambda^2 = 0 \quad \text{and} \quad PX\lambda + \lambda\tau(PX) = 0,$$

for any X tangent to M.

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Samuel SSEKAJJA,
School of Mathematics, University of the Witwatersrand
Private Bag 3, Wits 2050, Johannesburg, South Africa
e-mail: `samuel.ssekajja@wits.ac.za, ssekajja.samuel.buwaga@aims-senegal.org`

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