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WEAK SOLUTIONS FOR A QUASILINEAR ELLIPTIC EQUATION INVOLVING THE $P(X)$-LAPLACIAN

Abstract. In this paper, we prove the existence of weak solutions, in the Sobolev spaces with variable exponent $W_0^{1,p(x)}(\Omega)$, for the quasilinear Dirichlet boundary value problem involving the $p(x)$-Laplacian operator

$$-\Delta_{p(x)}u = \lambda u + f(x,u,\nabla u)$$

where Ω is an open bonded subset of $\mathbb{R}^N$, the operator $\Delta_{p(x)}$ is the $p(x)-$Laplacian, $\lambda \in \mathbb{R}$, the exponent $p(\cdot)$ is such that $2 \leq p(x) < \infty$, and $f(x,s,\xi)$ is a Carathéodory function verifying only some growth condition. Our technical approach is based on the recent Berkovits topological degree for a class of bounded and demicontinuous operators of (S_+) type.

1. Introduction

Equations involving the $p(x)$-Laplacian operator and variable exponent growth conditions have constituted a real interest in the study of the partial differential equations in the last few decades (see [8, 10, 11]). This kind of problems can be used to model dynamical phenomena which arise from the study of electrorheological fluids (smart fluids) or elastic mechanics, as well as in robotics and space technology. Problems with variable exponent growth conditions also appear in the modelling of stationary thermo-rheological viscous flows of non-Newtonian fluids and in the mathematical description of the processes filtration of an ideal barotropic gas through a porous medium (for more details see, for examples [2, 13]).

In this paper, we prove the existence of weak solutions for the Dirichlet problem

$$(1) \qquad \begin{cases} -\Delta_{p(x)}u = \lambda u + f(x,u,\nabla u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $\Delta_{p(x)}u := div(|\nabla u|^{p(x)-2}\nabla u)$ (the so called the $p(x)-$Laplacian), $\lambda \in \mathbb{R}$, $2 \leq p(x)$, $p(x) \in C(\overline{\Omega})$ and $f(x,s,\xi)$ be a Carathéodory function verifying some growth condition by using the topological degree theory for a class of bounded and demicontinuous operators of (S_+) type.

For $\lambda = 0$ and f independent of ∇u, Fan and Zhang [8] presents several sufficient conditions for the existence of solutions for the problem (1). The same problem

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is studied after by P.S. Iliaş [11] who gives sufficient conditions which allow to use variational and topological methods to prove the existence of weak solutions.

For $\lambda = 0$ and f dependent of ∇u, Ait Hammou et al. [1] prove the existence of at least one solution for this problem by using the topological degree theory for a class of demicontinuous operators of generalized $(S+)$ type.

With the same approach, we study the problem (1) with an additional term λu. We then generalise the last cited works. Note that with this term, the problem (1) can be seen as a nonlinear eigenvalue problem of the form

$$Au = \lambda u \tag{2}$$

where $Au := -\Delta_{p(x)} u - f(x,u,\nabla u)$. When (2) admits a non-zero weak solution u, λ is an eigenvalue of (2) and u is an associated eigenfunction. So, proving that (2) admits a weak solution, we prove at the same time that each real λ can be chosen as a eigenvalue of the problem (2).

The plan of the paper is as follows: after introduction in section 1, we present in section 2 some some basic properties of variable Lebesgue and Sobolev spaces. In section 3 deals with an outline of Berkovits degree defined for some classes of operators. Finaly, in section 4, we give some auxiliary results and the main result and its proof.

2. Variable Lebesgue and Sobolev Spaces

We introduce the setting of our problem with some auxiliary results of the variable exponent Lebesgue and Sobolev spaces $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$. For convenience, we only recall some basic facts with will be used later, we refer to [9, 12] for more details. Let Ω be an open bounded subset of $I\!R^N$, $N \geq 2$, with a Lipschitz boundary denoted by $\partial\Omega$. Denote

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) |\inf_{x\in\overline{\Omega}} h(x) > 1\}.$$

For any $h \in C_+(\overline{\Omega})$, we define

$$h^+ := max\{h(x), x \in \overline{\Omega}\}, h^- := min\{h(x), x \in \overline{\Omega}\}.$$

For any $p \in C_+(\overline{\Omega})$ we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \{u : \Omega \to I\!R \text{ measurable} ; \int_\Omega |u(x)|^{p(x)}\, dx < \infty\}$$

endowed with *Luxemburg norm*

$$|u|_{p(x)} = \inf\{\lambda > 0 / \rho_{p(x)}(\frac{u}{\lambda}) \leq 1\}.$$

where

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)}\, dx, \quad \forall u \in L^{p(x)}(\Omega).$$

$(L^{p(x)}(\Omega), |\,.\,|_{p(x)})$ is a Banach space [12, Theorem 2.5], separable and reflexive [12, Corollary 2.7]. Its conjugate space is $L^{p'(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for all $x \in \Omega$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, Hölder inequality holds [12, Theorem 2.1]

$$\begin{aligned} \left|\int_\Omega uv\,dx\right| &\leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)|u|_{p(x)}|v|_{p'(x)} \\ &\leq 2|u|_{p(x)}|v|_{p'(x)}. \end{aligned} \tag{3}$$

Notice that if (u_n) and $u \in L^{p(.)}(\Omega)$ then the following relations hold true (see [9])

$$|u|_{p(x)} < 1 (= 1; > 1) \quad \Leftrightarrow \quad \rho_{p(x)}(u) < 1 (= 1; > 1),$$

$$|u|_{p(x)} > 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+}, \tag{4}$$

$$|u|_{p(x)} < 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-}, \tag{5}$$

From (4) and (5), we can deduce the inequalitie

$$\rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-} + |u|_{p(x)}^{p^+}. \tag{6}$$

If $p, q \in C_+(\overline{\Omega}), p(x) \leq q(x)$ for any $x \in \overline{\Omega}$, then the embedding $L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ is continuous.

Next, we define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ as

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) / |\nabla u| \in L^{p(x)}(\Omega)\}.$$

It is a Banach space under the norm

$$||u|| = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

We also define $W_0^{1,p(.)}(\Omega)$ as the subspace of $W^{1,p(.)}(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ with respect to the norm $||\,.\,||$. If the exponent $p(.)$ satisfies the log-Hölder continuity condition, i.e. there is a constant $\alpha > 0$ such that for every $x, y \in \Omega, x \neq y$ with $|x - y| \leq \frac{1}{2}$ one has

$$|p(x) - p(y)| \leq \frac{\alpha}{-\log|x - y|} \quad , \tag{7}$$

then we have the Poincaré inequality (see [14]), i.e. the exists a constant $C > 0$ depending only on Ω and the function p such that

$$|u|_{p(x)} \leq C |\nabla u|_{p(x)}, \forall u \in W_0^{1,p(.)}(\Omega). \tag{8}$$

In particular, the space $W_0^{1,p(.)}(\Omega)$ has a norm $|\cdot|_{1,p(x)}$ given by

$$|u|_{1,p(x)} = |\nabla u|_{p(.)} \text{ for all } u \in W_0^{1,p(x)}(\Omega),$$

which is equivalent to $||\ .\ ||$. In addition, we have the compact embedding $W_0^{1,p(.)}(\Omega) \hookrightarrow L^{p(.)}(\Omega)$(see [12]). The space $(W_0^{1,p(x)}(\Omega), |\ .\ |_{1,p(x)})$ is a Banach space, separable and reflexive (see [9, 12]). The dual space of $W_0^{1,p(x)}(\Omega)$, denoted $W^{-1,p'(x)}(\Omega)$, is equipped with the norm

$$|v|_{-1,p'(x)} = \inf\{|v_0|_{p'(x)} + \sum_{i=1}^{N} |v_i|_{p'(x)}\},$$

where the infinimum is taken on all possible decompositions $v = v_0 - divF$ with $v_0 \in L^{p'(x)}(\Omega)$ and $F = (v_1, ..., v_N) \in (L^{p'(x)}(\Omega))^N$.

3. Berkovits degree

We start by an outline of Berkovits degree theory. For more details we can see [3–6].

Let X be a real separable reflexive Banach space with dual X^* and with continuous pairing $\langle .\ ,\ .\rangle$, Ω be a nonempty subset of X and Y be a real Banach space.

We recall that a mapping $F : \Omega \subset X \to Y$ is *bounded,* denote $F \in BD$, if it takes any bounded set into a bounded set. F is said to be *demicontinuous,* denote $F \in DC$, if for any $(u_n) \subset \Omega$, $u_n \to u$ implies $F(u_n) \rightharpoonup F(u)$. F is said to be *compact* if it is continuous and the image of any bounded set is relatively compact.

A mapping $F : \Omega \subset X \to X^*$ is said to be *of class* (S_+), denote $F \in (S_+)$, if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$ and $limsup\langle Fu_n, u_n - u\rangle \leq 0$, it follows that $u_n \to u$. F is said to be *quasimonotone,* if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, it follows that $limsup\langle Fu_n, u_n - u\rangle \geq 0$.

For any operator $F : \Omega \subset X \to X$ and any bounded operator $T : \Omega_1 \subset X \to X^*$ such that $\Omega \subset \Omega_1$, we say that F satisfies *condition* $(S_+)_T$, denote $F \in (S_+)_T$, if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$ and $limsup\langle Fu_n, y_n - y\rangle \leq 0$, we have $u_n \to u$.

Let $\mathscr{O}$ be the collection of all bounded open set in X. For any $\Omega \subset X$, we consider the following classes of operators:

$$\begin{aligned}
\mathscr{F}_1(\Omega) &:= \{F : \Omega \to X^* \mid F \in BD \cap DC \cap (S_+)\},\\
\mathscr{F}_{T,B}(\Omega) &:= \{F : \Omega \to X \mid F \in BD \cap DC \cap (S_+)_T\},\\
\mathscr{F}_T(\Omega) &:= \{F : \Omega \to X \mid F \in DC \cap (S_+)_T\},\\
\mathscr{F}_B(X) &:= \{F \in \mathscr{F}_{T,B}(\overline{G}) \mid G \in \mathscr{O}, \mathrm{T} \in \mathscr{F}_1(\overline{\mathrm{G}})\}.
\end{aligned}$$

Here, $T \in \mathscr{F}_1(\overline{G})$ is called an *essential inner map* to F.

DEFINITION 1. *Let G be a bounded open subset of a real reflexive Banach space X, $T \in \mathscr{F}_1(\overline{G})$ be continuous and let $F, S \in \mathscr{F}_T(\overline{G})$. The affine homotopy H :*

$[0,1]\times\overline{G}\to X$ defined by

$$H(t,u):=(1-t)Fu+tSu \text{ for } (t,u)\in[0,1]\times\overline{G}$$

is called an admissible affine homotopy with the common continuous essential inner map T.

REMARK 1. [4] The above affine homotopy satisfies condition $(S_+)_T$.

We introduce the topological degree for the class $\mathscr{F}_B(X)$ due to Berkovits [4].

THEOREM 1. *Let*

$$\mathscr{M}=\{(F,G,h)|G\in\mathscr{O},T\in\mathscr{F}_1(\overline{G}),F\in\mathscr{F}_{T,B}(\overline{G}),h\notin F(\partial G)\}.$$

There exists a unique degree function

$$d:\mathscr{M}\to\mathbb{Z}$$

that satisfies the following properties

1. *(Existence) if $d(F,G,h)\neq 0$, then the equation $Fu=h$ has a solution in G.*
2. *(Additivity) Let $F\in\mathscr{F}_{T,B}(\overline{G})$. If G_1 and G_2 are two disjoint open subset of G such that $h\notin F(\overline{G}\setminus(G_1\cup G_2))$, then we have*
$$d(F,G,h)=d(F,G_1,h)+d(F,G_2,h).$$
3. *(Homotopy invariance) If $H:[0,1]\times\overline{G}\to X$ is a bounded admissible affine homotopy with a common continuous essential inner map and $h:[0,1]\to X$ is a continuous path in X such that $h(t)\notin H(t,\partial G)$ for all $t\in[0,1]$,then the value of $d(H(t,.),G,h(t))$ is constant for all $t\in[0,1]$.*
4. *(Normalization) For any $h\in G$, we have $d(I,G,h)=1$.*

4. Assumptions and main results

In this section, we study the Dirichlet boundary value problem (1) based on the degree theory in Section 3, where $\Omega\subset\mathbb{R}^N$, $N\geq 2$, is a bounded domain with a Lipschitz boundary $\partial\Omega$, $p\in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition (7) and

$$2\leq p^-\leq p(x)\leq p^+<\infty.$$

4.1. Basic assumptions and technical Lemmas

Consider the operator
$L:W_0^{1,p(x)}(\Omega)\to W^{-1,p'(x)}(\Omega)$, defined by

$$\langle Lu,v\rangle=\int_\Omega|\nabla u|^{p(x)-2}\nabla u\nabla v dx, \text{ for all } u,v\in W_0^{1,p(x)}(\Omega).$$

LEMMA 1. *[7, Theorem 3.1] The operator L is a continuous homeomorphism, bounded, strictly monotone operator and of class* (S_+).

Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a real-valued function such that:

(A_1) f satisfies the Carathéodory condition, i.e. $f(.,s,\xi)$ is measurable on Ω for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and $f(x,.,.)$ is continuous on $\mathbb{R} \times \mathbb{R}^N$ for a.e. $x \in \Omega$.

(A_2) f verifies the flowing growth condition

$$|f(x,s,\xi)| \leq c(k(x) + |s|^{q(x)-1} + |\xi|^{q(x)-1})$$

for a.e. $x \in \Omega$ and all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, where c is a positive constant, $k \in L^{p'(x)}(\Omega)$ and $q \in C_+(\overline{\Omega})$ with $q^+ < p^-$.

LEMMA 2. *[1] Under assumptions* (A_1) *and* (A_2)*, the operator* $S_1 : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ *defined as*

$$\langle S_1 u, v\rangle = -\int_\Omega f(x,u,\nabla u)v\,dx, \quad \forall u,v \in W_0^{1,p(x)}(\Omega)$$

is compact.

LEMMA 3. *Under assumptions* (A_1) *and* (A_2)*, the operator* $S : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ *defined as*

$$\langle Su, v\rangle = -\int_\Omega (\lambda u + f(x,u,\nabla u))v dx, \quad \forall u,v \in W_0^{1,p(x)}(\Omega)$$

is compact.

Proof. Let $S_2 : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ be the operator given by

$$\langle S_2 u, v\rangle = -\int_\Omega \lambda uv\,dx \text{ for } u,v \in W_0^{1,p(x)}(\Omega).$$

Since the embedding $I : W_0^{1,p(x)}(\Omega) \to L^{p(x)}(\Omega)$ is compact, it is known that the adjoint operator $I^* : L^{p'(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ is also compact. Since $p(x) \geq 2$, we have $p'(x) \leq 2 \leq p(x)$, then the embedding $i : L^{p(x)} \hookrightarrow L^{p'(x)}$ is continuous. Therefor, $S_2 = -\lambda I^* \circ i \circ I$ is compact.
We conclude that $S = S_1 + S_2$ is compact. □

LEMMA 4. *[4, Lemma 2.2 and 2.4] Suppose that* $T \in \mathscr{F}_1(\overline{G})$ *is continuous and* $S : D_S \subset X^* \to X$ *is demicontinuous such that* $T(\overline{G}) \subset D_s$*, where G is a bounded open set in a real reflexive Banach space X. Then the following statement are true:*

- *If S is quasimonotone, then* $I + S \circ T \in \mathscr{F}_T(\overline{G})$*, where I denotes the identity operator.*
- *If S is of class* (S_+)*, then* $S \circ T \in \mathscr{F}_T(\overline{G})$.

4.2. Existence result

DEFINITION 2. *We call that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of (1) if*

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx = \int_\Omega (\lambda u + f(x,u,\nabla u))v dx,$$

for all $v \in W_0^{1,p(x)}(\Omega)$.

THEOREM 2. *Under assumptions (A_1) and (A_2), the problem* (1) *has a weak solution u in $W_0^{1,p(x)}(\Omega)$.*

Proof. Let L and $S : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ be as in subsection 4.1. Then $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of (1) if and only if

$$Lu = -Su \tag{9}$$

Thanks to the properties of the operator L seen in Lemma 1 and in view of Minty-Browder Theorem (see [15], Theorem 26A), the inverse operator $T := L^{-1} : W^{-1,p'(x)}(\Omega) \to W_0^{1,p(x)}(\Omega)$ is bounded, continuous and satisfies condition (S_+). Moreover, note by Lemma 3 that the operator S is bounded, continuous and quasimonotone.

Consequently, equation (9) is equivalent to

$$v + S \circ Tv = 0. \tag{10}$$

where $u = Tv$.

To solve equation (10), we will apply the degree theory introduced in section 3. To do this, we first claim that the set

$$B := \{v \in W^{-1,p'(x)}(\Omega) | v + tS \circ Tv = 0 \text{ for some } t \in [0,1]\}$$

is bounded. Indeed, let $v \in B$. Set $u := Tv$, then $|Tv|_{1,p(x)} = |\nabla u|_{p(x)}$.

If $|\nabla u|_{p(x)} \leq 1$, then $|Tv|_{1,p(x)}$ is bounded.

If $|\nabla u|_{p(x)} > 1$, then we get by the implication (4), the growth condition (A_2), the

Hölder inequality (3), the inequality (6)and the Young inequality the estimate

$$\begin{aligned} |Tv|_{1,p(x)}^{p^-} &= |\nabla u|_{p(x)}^{p-} \\ &\leq \rho_{p(x)}(\nabla u) \\ &= \langle Lu, u\rangle \\ &= \langle v, Tv\rangle \\ &= -t\langle S\circ Tv, Tv\rangle \\ &= t\int_\Omega (\lambda u + f(x,u,\nabla u))u dx \\ &\leq const(\int_\Omega \lambda |u(x)|^2 dx + \int_\Omega |k(x)u(x)|dx + \rho_{q(x)}(u) + \int_\Omega |\nabla u|^{q(x)-1}|u|dx) \\ &\leq const(\lambda |u|_{L^2}^2 + |k|_{p'(x)}|u|_{p(x)} + |u|_{q(x)}^{q^+} + |u|_{q(x)}^{q^-} + \frac{1}{q'^-}\rho_{q(x)}(\nabla u) + \frac{1}{q^-}\rho_{q(x)}(u)) \\ &\leq const(|u|_{L^2}^2 + |u|_{p(x)} + |u|_{q(x)}^{q^+} + |u|_{q(x)}^{q^-} + |\nabla u|_{q(x)}^{q^+}). \end{aligned}$$

From the Poincaré inequality (8) and the continuous embedding $L^{p(x)} \hookrightarrow L^2$ and $L^{p(x)} \hookrightarrow L^{q(x)}$, we can deduct the estimate

$$|Tv|_{1,p(x)}^{p^-} \leq const(|Tv|_{1,p(x)}^2 + |Tv|_{1,p(x)} + |Tv|_{1,p(x)}^{q^+}).$$

It follows that $\{Tv | v \in B\}$ is bounded.
Since the operator S is bounded, it follows from (10) that the set B is bounded in $W^{-1,p'(x)}(\Omega)$. Consequently, there exists $R > 0$ such that

$$|v|_{-1,p'(x)} < R \text{ for all } v \in B.$$

This says that

$$v + tS\circ Tv \neq 0 \text{ for all } v \in \partial B_R(0) \text{ and all } t \in [0,1].$$

From Lemma (4) it follows that

$$I + S\circ T \in \mathscr{F}_T(\overline{B_R(0)}) \text{ and } I = L\circ T \in \mathscr{F}_T(\overline{B_R(0)}).$$

Since the operators I, S and T are bounded, $I + S\circ T$ is also bounded. We conclude that

$$I + S\circ T \in \mathscr{F}_{T,B}(\overline{B_R(0)}) \text{ and } I \in \mathscr{F}_{T,B}(\overline{B_R(0)}).$$

Consider a homotopy
$H : [0,1]\times \overline{B_R(0)} \to W^{-1,p'(x)}(\Omega)$ given by

$$H(t,v) := v + tS\circ Tv \text{ for } (t,v) \in [0,1]\times \overline{B_R(0)}.$$

Applying the homotopy invariance and normalization property of the degree d stated in Theorem(1), we get

$$d(I + S\circ T, B_R(0), 0) = d(I, B_R(0), 0) = 1,$$

and hence there exists a point $v \in B_R(0)$ such that

$$v + S \circ T v = 0.$$

We conclude that $u = Tv$ is a weak solution of (1). □

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