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A CHARACTERIZATION OF WEAKLY TRIPOTENT RINGS

Abstract. Recently, Breaz and Cîmpean defined and explored in Bull. Korean Math. Soc. (2018) the class of so-called *weakly tripotent rings* that are rings whose elements satisfy one of the equations $x^3 = x$ or $(1-x)^3 = 1-x$. However, they succeed to characterize these rings only in the commutative case. We will give here a complete characterization of these rings in the general (possibly noncommutative) case by using our recent results published in Commun. Korean Math. Soc. (2017).

1. Introduction and Background

Throughout the text of the current article, all rings R are assumed to be associative, possessing the identity element 1 which differs from the zero element 0 of R. Our standard terminology and notation are mainly in agreement with [10]. For instance, $U(R)$ denotes the set of all units in R, $Id(R)$ the set of all idempotents in R, $Nil(R)$ the set of all nilpotents in R and $J(R)$ the Jacobson radical of R. The more specific notions and notations will be given explicitly in the sequel.

An element b of a ring R is called *tripotent* if the equality $b^3 = b$ holds. If each element of R is equipped with this property, R is said to be *tripotent* as well. The complete description of such rings is given in [9]. Specifically, they are a subdirect product (and hence a subring of a direct product) of a family of copies of the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$.

On the similar vein, in [1] were recently explored the so-called *weakly tripotent* rings that are rings in which at least one of the elements b or $1-b$ is a tripotent. It is immediate that weakly tripotent rings of characteristic 3 are themselves tripotent as $(1-b)^3 = 1-b^3 = 1-b$ yields that $b^3 = b$. An interesting example of a weakly tripotent ring of characteristic 8 is the indecomposable ring $\mathbb{Z}_8$ whose elements are solutions of (one of) the equations $x^2 = 1$ or $(1-x)^2 = 1$ (thus x or $1-x$ is obviously a tripotent). In general, mainly in the non-commutative case, these rings have not a complete characterization yet.

On the other hand, in the recent paper [3] was defined the classes of *invo-clean* and *strongly invo-clean* rings. In fact, it was said there that a ring R is invo-clean if, for each $r \in R$, there exist $v \in U(R)$ with $v^2 = 1$ and $e \in Id(R)$ such that $r = v + e$. If, in addition, $ve = ev$, R is called strongly invo-clean.

Our motivation in writing up the present paper is to give a full description (up to an isomorphism) of the class of weakly tripotent rings by the usage of the aforementioned strongly invo-clean rings.

2. Main Results

Before stating and proving our chief characterizing theorems, we need the following preliminaries. Concretely, it was showed in [3, Corollary 2.17], accomplishing this corollary with [6, Theorem 4.3(4,5)], that *if a ring R is strongly invo-clean, then R is decomposable as $R_1 \times R_2$, where $R_1 = \{0\}$ or $R_1/J(R_1)$ is Boolean with nil $J(R_1)$ of index of nilpotence at most* 3, *and $R_2 = \{0\}$ or R_2 is a subdirect product of a family of copies of the field $\mathbb{Z}_3$*. Some further explicit applications to this isomorphic classification were given in [11].

In that aspect, the next relationship considerably strengthens [1, Corollary 4].

PROPOSITION 1. *Every weakly tripotent ring is strongly invo-clean.*

Proof. For such a ring R, we have $r^3 = r$ or $(1-r)^3 = 1-r$ whenever $r \in R$. In the first case, one writes that $r = (1-r^2) + (r^2+r-1)$. A direct manipulation shows that $1-r^2 \in Id(R)$ as $r^2 \in Id(R)$ and that $(r^2+r-1)^2 = 1$ as $r^3 = r$, observing elementarily that these two elements commute, as required.

Dealing with the other equality $(1-r)^3 = 1-r$, by replacing $r \to 1-r$, we can write by using the trick above that $1-r = f+w$ for some idempotent $f = 1-(1-r)^2 = 2r-r^2$ and an involution $w = 1-3r+r^2$. Therefore, $r = (1-f)+(-w) \in Id(R)+U(R)$, where $(-w)^2 = w^2 = 1$, as required. □

It is worthwhile noticing that the converse implication is untrue as it will be shown below in Example 1 and Remark 1.

We will be able now to improve somewhat [1, Corollary 9] in the following way.

PROPOSITION 2. *A ring R is weakly tripotent without non-trivial idempotents if, and only if, for each $r \in R$, one of the equations $r^2 = 1$ or $r^2 = 2r$ holds.*

Proof. "$\Rightarrow$". Given $r^3 = r$, it follows that $(r^2)^2 = r^2$ and so either $r^2 = 1$ or $r^2 = 0$. However, to be more precise, one also writes that $r = rbr$, where $b = 1+r-r^2$ has the property that $b^2 = 1$. Thus $rb = br$ is an idempotent implying that either $rb = 1$ or $rb = 0$ which gives the more exact equalities $r^2 = 1$ or $r = 0$. Notice that the second equality assures that $r^2 = 2r = 0$.

Similarly, $(1-r)^3 = 1-r$ will imply that either $(1-r)^2 = 1$ or $1-r = 0$, i.e., either $r^2 = 2r$ or $r = 1$. As the latter ensures that $r^2 = 1$, we are set.

"$\Leftarrow$". It is elementarily seen that $r^2 = 1$ forces $r^3 = r$ and that $r^2 = 2r$ forces $(1-r)^2 = 1$, thus yielding $(1-r)^3 = 1-r$, as required. □

As an immediate consequence to the last assertion, one can derive the following confirmation of the validity of [1, Corollary 9] by giving a more direct and transparent proof (notice that, under the given circumstances, we will have in this case that $2 \in Nil(R) \iff 3 \in U(R)$ as $U(R) = 1 + Nil(R)$).

COROLLARY 1. *Suppose that R is a ring with no non-trivial idempotents in which 2 is nilpotent. Then R is weakly tripotent if, and only if, $R/J(R) \cong \mathbb{Z}_2$ and the equation $x^2 = 1$ holds for any element of $U(R)$ if, and only if, $R/J(R) \cong \mathbb{Z}_2$ and the equation $x^2 = 2x = 0$ holds for any element of $J(R)$.*

Proof. "**Necessity.**" As $2 \in J(R)$ is a central nilpotent, by setting $\overline{R} := R/J(R)$, one observes that $\overline{R}$ is a weakly tripotent ring of characteristic 2 (i.e., $\overline{2} = \overline{0}$). Since $J(R)$ is nil, this surely makes up $\overline{R}$ a ring without non-trivial idempotents as well. In accordance with Proposition 2, the conditions $\overline{r}^2 = \overline{1}$ or $\overline{r}^2 = \overline{0}$ are true for any $\overline{r} \in \overline{R}$. Furthermore, in the first case, it must be that $(\overline{r} - \overline{1})^2 = \overline{0}$ whence $\overline{r} = \overline{1} + (\overline{r} - \overline{1})$, whereas in the second case $\overline{r} = \overline{0} + \overline{r}$. This means that, in both situations, $\overline{R}$ is a strongly nil-clean ring, so that [6] can be applied to get that $\overline{R}$ is Boolean, and hence it is isomorphic to $\mathbb{Z}_2$, as asked for.

The application of Proposition 2 insures that $r^2 = 1$ or $r^2 = 2r$ for every $r \in R$. Certainly, both equalities $r^2 = 1$ and $r^2 = 2r = 0$ cannot be true simultaneously, so that the relation $r^2 = 1$ will be fulfilled for all $r \in U(R)$ as the other one $r^2 = 2r \in J(R)$ (because $2 \in J(R)$) is impossible in $U(R)$ and, reciprocally, the relation $r^2 = 2r \in J(R)$ will be valid for all $r \in J(R)$ as the other one $r^2 = 1$ makes no sense in $J(R)$.

"**Sufficiency.**" As $U(R/J(R)) \cong U(R)/(1+J(R)) \cong U(\mathbb{Z}_2) = \{1\}$, it follows at once that $U(R) = 1 + J(R)$. Hence $Nil(R) \subseteq J(R)$ because $1 + Nil(R) \subseteq U(R)$. However, one sees that $Nil(R) = J(R)$ since $J(R)$ is nil; in fact, this follows automatically if $x^2 = 2x = 0$ is true in $J(R)$. However, if $x^2 = 1$ is true in $U(R)$, one has for any $z \in J(R)$ that $(1-z)^2 = 1$ and so $z^2 = 2z \in Nil(R)$ whence $z \in Nil(R)$, as pursued. But R being local implies that any element r of R is either nilpotent or unit. That is, $r^2 = 2r = 0$ or $r^2 = 1$. Therefore, $(1-r)^3 = 1 - 3r + 3r^2 - r^3 = 1 - 3r = 1 - r$ as $r^2 = 2r = 0$ or $r^3 = r$ as $r^2 = 1$, thus substantiating our claim. $\square$

One more critical remark, which clarifies all the things alluded to above, is that in [1, Corollary 9(2)] the condition posed on the ring R to be "*a local ring*" is superfluous as the quotient $R/J(R) \cong \mathbb{Z}_2$ is surely a field.

Recall that a ring is called *indecomposable* if it does *not* possess non-trivial central idempotents, that is, all its central idempotents are only 0 and 1. We observe that R may have a non-central nilpotent, however. E.g., in the ring $\mathbb{T}_2(\mathbb{Z}_2)$, the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is such an element. In fact, by consulting with [8, Theorem 1] every ring with central nilpotents is a ring with central idempotents, while the converse is generally untrue. Nevertheless, in von Neumann regular rings the reverse is true (see [8]).

We now have accumulated all the ingredients necessary to proceed by proving with the next achievement, which is a (significant) reminiscent of [1, Corollary 9] and which somewhat (considerably) extends the last assertion to the following one.

THEOREM 1. *Suppose R is a weakly tripotent ring. Then the following two points are true:*

(i) If R is without non-trivial (central) idempotent elements of $char(R) = 3$, then $R \cong \mathbb{Z}_3$.

(ii) If R is without non-trivial central nilpotent elements of even $char(R)$ not exceeding 8, then $char(R) = 2$ and $R/J(R)$ is Boolean.

Proof. Item (i) follows immediately with the aid of Proposition 1 in combination with [3] (see also [1]).

As for item (ii), as $2 \in Nil(R)$ is central, it must be that $2 = 0$. According to Proposition 1, for each $r \in R$, we have that $r = v + e = (1 + v) + (1 + e)$ for some $v \in R$ with $v^2 = 1$ and $e \in Id(R)$. But $(1 + v)^2 = 0$ and $(1 + e)^2 = 1 + e$, so that R is nil-clean (see, e.g., [6] or [7]). Moreover, we claim that the index of nilpotence of R is at most 2. In fact, taken an arbitrary $q \in Nil(R)$, we write as above that $q = v + e$. With [3, Corollary 2.6] at hand we arrive at $e = 1$. We, therefore, have that $q = v + 1$ and hence that $q^2 = 0$ as $2 = 0$. The claim sustained (compare also with [11, Lemma 2.1] where the proof is unnecessarily intricately stated). Another approach to extract this claim could be taken from Lemma 1 stated below.

Furthermore, since the homomorphic images of a weakly tripotent ring and of a nil-clean ring retain the same corresponding property, one deduces that $R/J(R)$ has to be simultaneously weakly tripotent and nil-clean. However, [11, Theorem 2.2] allows us to conclude that $R/J(R)$ is a subdirect product of a family of copies of $\mathbb{Z}_2$, i.e., this factor-ring is necessarily Boolean as asserted, bearing in mind that the matrix ring $\mathbb{M}_2(\mathbb{Z}_2)$ need not be weakly tripotent (see Example 2 (3) listed below).

An other approach to deduce that the quotient ring $R/J(R)$ is a subring of a direct product of the $\mathbb{Z}_2$'s is realizable by the usage of Proposition 1 and the comments before it. $\square$

By a slight modification of our arguments, accomplishing them with [1, Theorem 7], our conclusion in the last theorem can also be deduced successfully.

One can also note in accordance with Example 2(2) quoted below that the ring $\mathbb{T}_3(\mathbb{Z}_2)$ is surely *not* weakly tripotent and that there is an isomorphism

$$\mathbb{T}_3(\mathbb{Z}_2)/J(\mathbb{T}_3(\mathbb{Z}_2)) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,$$

as expected. In fact, this is an indecomposable ring having both a non-central nilpotent of order 2 and a non-central idempotent.

In view of the last two statements established above and concerning invo-clean rings, we can offer the following reminiscent of [3, Corollary 2.17].

PROPOSITION 3. *Suppose that R is a ring for which 2 is a nilpotent. Then R is strongly invo-clean if, and only if, $R/J(R)$ is Boolean and, for all elements $z \in J(R)$, the equation $z^2 = 2z$ holds.*

Proof. "$\Rightarrow$". Given z in $J(R)$, one writes that $z = v + e$ for some $v \in R$ with $v^2 = 1$ and $e \in Id(R)$. Hence $z - v = e \in Id(R) \cap U(R) = \{1\}$, so that $z = v + 1$. We now find by squaring that $z^2 = 2z$. Besides, as we commented above at the beginning of this section, the quotient ring $R/J(R)$ has to be Boolean, as stated.

"$\Leftarrow$". Apparently, $J(R) \subseteq Nil(R)$, because $z^2 = 2z \in Nil(R)$ amounts to $z \in Nil(R)$. Since $U(R)/(1+J(R)) \cong U(R/J(R)) = \{\bar{1}\}$, it follows that $U(R) = 1+J(R)$. Consequently, $U^2(R) = \{u^2 \mid u \in U(R)\} = \{1\}$ since $4z = 0$ is always fulfilled (cf. [3, Corollary 2.16] and its proof) and, therefore, R is allowed to be strongly invo-clean because it is strongly clean by an appeal to [6]. $\square$

The next characterization might be of some applicable purposes (compare with Proposition 1 quoted above).

LEMMA 1. *Suppose R is a ring of characteristic 2. Then the following three issues are equivalent:*

(1) R is weakly tripotent.

(2) All elements of R satisfy (one of) the equations $x^3 = x$ or $x^3 = x^2$.

(3) Each element r of R is presentable as $r = q+e$, where $q, e \in R$ commute with $q^2 = 0$ and $e^2 = e$ such that either $qe = 0$ or $q(1-e) = 0$.

Proof. First of all, let us assume that $\mathrm{char}(R) = 2$, i.e., $2 = 0$.

The equivalence (1) $\iff$ (2) follows by a direct routine check, because $(1-x)^3 = 1-x$ is tantamount to $x^3 = x^2$ as $2 = 0$.

Now, we handle the implication (3) $\Rightarrow$ (2). In fact, one derives that $r^2 = e$ and $r^3 = qe+e$. So, if $qe = 0$, we obtain that $r^3 = e$ and so $r^3 = r^2$. However, if $qe = q$, one gets that $r^3 = r$, as formulated.

What remains to show is the truthfulness of the converse implication (2) $\Rightarrow$ (3). Indeed, we claim foremost that the nilpotence index of R is exactly 2. To see that, given $q \in Nil(R)$, we have that $q^3 = q$ or $q^3 = q^2$. Thus $q(1-q^2) = 0$ or $q^2(1-q) = 0$. As both $1-q^2$ and $1-q$ invert in R, it must be in the two cases that $q^2 = 0$, as required. Furthermore, we directly can apply Proposition 1 to write that $r = v+e = (v+1)+(1+e)$, where $v+1$ is a nilpotent of order 2 and $1+e$ is an idempotent. However, a more precise analysis of this record is needed and we will establish it in what follows. Likewise, another useful approach might be as follows: If $u \in U(R)$, then either $u^3 = u$ or $u^3 = u^2$. Hence, in both cases $u^2 = 1$, which means that $(u-1)^2 = 0$. This, in turn, forces that $u \in 1+Nil(R)$, i.e., $U(R) = 1+Nil(R)$. As these rings have elements all of them being solutions of (one of) the equations $x^3 = x$ or $x^3 = x^2$, they are clearly exchange and the application of [6] shows that $r = q+e$ with $qe = eq$, $q^2 = 0$ and $e^2 = e$. Nevertheless, we need the more detailed relations between q and e stated above. To draw them, we foremost calculate that $r^2 = e$ and $r^3 = qe+e$ as $2 = 0$. Henceforth, $r^3 = r$ yields that $qe+e = q+e$, i.e., $q(1-e) = 0$; whereas $r^3 = r^2$ implies that $qe+e = e$, i.e., $qe = 0$, as promised. $\square$

Notice that the ring $\mathbb{Z}_2[X]/(X^2)$ is a non-elementary example of a weakly tripotent ring of characteristic 2.

A valuable criterion for rings of characteristic 2 can be extracted when they are strongly invo-clean only in terms of equations. Specifically, the following is valid.

PROPOSITION 4. *Let R be a ring in which $2 = 0$. Then R is strongly invo-clean if, and only if, all its elements are solutions of the equation $x^4 = x^2$.*

Proof. "$\Rightarrow$". Suppose $r \in R$. Given $r = v + e$ for some commuting elements $v, e \in R$ with $v^2 = 1$ and $e^2 = e$, one derives that $r^2 = 1 + e$ is an idempotent whence $r^4 = r^2$ will hold.

"$\Leftarrow$". Letting $r \in R$, we just write that $r = (1 + r + r^2) + (1 + r^2)$. Furthermore, one simple checks that the first term is an involution, i.e. $(1 + r + r^2)^2 = 1$, while the second term is an idempotent, i.e. $(1 + r^2)^2 = 1 + r^2$. This substantiates our claim after all. $\square$

It is worthy of noticing that any of the equations $x = x^3$ and $x^2 = x^3$ will imply that $x^2 = x^4$, but the validity of the reverse implication is wrong even in the commutative case. Indeed, by what we have shown so far in Lemma 1 and Proposition 4, the following holds:

EXAMPLE 1. There exists a strongly invo-clean ring of characteristic 2 which is *not* weakly tripotent, that is, there is a ring R of characteristic 2 in which $x^2 = x^4$, $\forall\, x \in R$ does *not* imply that $x^2 = x^3$ or $x = x^3$.

In fact, let us consider the group ring $R = BG$, where $B \ncong \mathbb{Z}_2$ is a Boolean ring and G is a group consisting only of elements of order at most 2. Clearly, the equality $x^4 = x^2$ is valid for each element $x \in R$ as $\mathrm{char}(R) = 2$. However, taking into account that B contains non-trivial idempotents, simple calculations show that both inequalities $x^3 \neq x^2$ and $x^3 \neq x$ are fulfilled, which facts we leave to the interested reader for a direct inspection. This concludes the example and substantiates our claim.

As for rings of characteristic 4, we may initiate the next discussion.

REMARK 1. If R is a ring such that $R/J(R)$ is a Boolean factor-ring with $J(R) = \{0, 2\}$ (whence $4 = 0$ and $z^2 = 2z = 0$ holds in $J(R)$), then one says with the aid of [5] that R is commutative and every element of R is the sum or the difference of two idempotents. Thus, an appeal to [1, Corollary 18], or by a pretty direct check, leads us to the conclusion that R is weakly tripotent. More generally, given a ring R of $\mathrm{char}(R) = 4$ such that $R/J(R)$ is a Boolean quotient and $J(R) = 2Id(R)$, one claims in view of [5] that R is a commutative ring whose elements are sums of three idempotents. This ring R is surely strongly invo-clean by virtue of the main results from [3]. However, such a ring R could *not* be weakly tripotent by the usage of the next arguments: Since $R/J(R)$ is Boolean, it follows that $r - r^2 \in J(R)$ and since $J(R)$ is obviously nil, it must be that $r - e \in J(R)$ for some $e \in Id(R)$. Hence $r = e + 2f$ for some $f \in Id(R)$, so that $r^2 = e$ (whence $r^4 = r^2$ is true) and $2r = 2r^2$ because $4 = 0$ by assumption. To be the ring R weakly tripotent, its elements r need to satisfy at least one of the equations $r^3 = r$ or $r^3 + r^2 + 2r = 0$, where the latter one is just equivalent to $r^3 = r^2$ because of the equality $2r = 2r^2$, as we have seen before. Therefore, $r^3 = r$ implies that $2ef = 2f$, whereas $r^3 = r^2$ implies that $2ef = 0$. With this in mind, the weak tripotentness of R depends on the structure of the idempotents in R. To be more precise, if we take $Id(R) = \{0, 1, e, 1-e\}$ and hence $J(R) = \{0, 2, 2e, 2(1-e)\}$, it will follow at once that our assertion is sustained, as we leave the direct verification to the interested reader who just needs to consider the four possibilities $f = 0$, $f = 1$, $f = e$ and $f = 1-e$, respectively.

Besides, as we already have seen above in Proposition 1, if R is a weakly tripotent ring, then R is strongly invo-clean, but we need a rather more precise description, however. This will be materialized in the next statement, which encompasses the previous Lemma 1.

THEOREM 2. *A ring R is weakly tripotent (of characteristic 2, 4 or 8) if, and only if, every element r of R is presentable as $r = q + e$, where $q, e \in R$ commute with $q^2 = 2q$ and $4q = 0$ (hence $q^2 = 0$ when $2 = 0$, $q^3 = 0$ when $4 = 0$ and even $q^3 = 0$ when $8 = 0$), and $e^2 = e$ such that either $qe = 0$ or $q(1-e) = 0$.*

Proof. "$\Rightarrow$". The case when $2 = 0$ was handled above in Lemma 1, so we shall assume hereafter in the proof that $2 \neq 0$.

So, let us now char$(R) = 4$. Then, for any $q \in Nil(R)$, we have that $q^3 = q$ or $(1-q)^3 = 1-q$. The first equality immediately ensures that $q = 0$ since $q(1-q^2) = 0$ and $1-q^2$ is invertible in R. The second one, however, implies that $(1-q)^2 = 1$ as $1-q$ inverts in R, so that we come to $q^2 = 2q$. Thus, $q^3 = 2q^2 = 4q = 0$ as $4 = 0$. Furthermore, with Proposition 1 at hand, we write that $r = q + e$, where $qe = eq$ with $e^2 = e$ and $q^3 = 0$. Consequently, $r = r^3 = q^3 + 3q^2e + 3qe + e = -q^2e - qe + e = q + e$ yields that $q^2e + qe = -q$. As $q^2 = 2q$, we derive that $3qe = -q$, i.e., $-qe = -q$, i.e., $q(1-e) = 0$, as desired. If now $(1-r)^3 = 1-r$, then as showed in Proposition 1 we may replace r by $1-r$ (and hence $q \to -q$ and $e \to 1-e$), deducing that $qe = 0$, as wanted.

Assuming now that char$(R) = 8$, as already observed above, $q^2 = 2q$ whence $q^4 = 4q^2 = 8q = 0$ as $8 = 0$. Furthermore, for $r^3 = r$ such that $r = q + e$ with $qe = eq$, one infers that $q^3 + 3q^2e + 3qe = q$. The last means that $q(1-q)(1+q) = 3qe(1+q)$. Since $1+q$ is invertible in R, this leads to $q(1-q) = 3qe$ and hence to $q(1-q)(1-e) = 0$. But $1-q$ also inverts in R, so that we obtain the pursued equality $q(1-e) = 0$ – another way of proving up this equality is to use as given above that $q^2 = 2q$. Replacing as above $q \to -q$ and $e \to 1-e$ in the case of $1-r = (1-r)^3$, one concludes that $qe = 0$, as asked for. Further, the element $0 \neq 2r$ does not satisfy the equation $x^3 = x$ as for otherwise $8r^3 = 2r = 0$. Hence one has that $(1-2r)^3 = 1-2r$ and thus $4r = 4r^2$. Substituting q in the last equality provided $2q \neq 0$, we therefore obtain that $4q = 4q^2 = 8q = 0$ as $q^2 = 2q$ and hence $2q^2 = 0$ which, finally, leads to $q^3 = 2q^2 = 0$, as wanted. If, however, $2q = 0$, we are directly over.

"$\Leftarrow$". Writing $r = q + e$ with q and e satisfying the conditions stated above, we can process like this: If $qe = q = eq$, then $r^3 = (q+e)^3 = q^3 + 3q^2e + 3qe + e =$

$4q+9qe+e=13q+e=q+e=r$ whenever $4=0$. If now $qe=0=eq$, then $1-r=-q+(1-e)$, so that $(-q)(1-e)=-q=(1-e)(-q)$ and thereby we may apply the same trick to obtain that $(1-r)^3=1-r$.

For $8=0$, as demonstrated above, one has that $r^3=r$ whenever $qe=eq=q$ or $(1-r)^3=1-r$ whenever $qe=eq=0$, because $4q=0$. □

Note that, even without $8=0$ at hand, the equalities $qe=0=eq$ (hence $q(1-e)=(1-e)q=q$), $q^2=2q$ and $e^2=e$ force whenever $r=q+e$, i.e. whenever $1-r=-q+(1-e)$, that $(1-r)^3=-q^3+3q^2(1-e)-3q(1-e)+(1-e)=-4q+3q(1-e)+(1-e)=-q+(1-e)=1-r$, as expected.

Likewise, as $4q^2=0$ when $8=0$, one may derive that $r^3=q^3-q^2e+3qe+e=4q+qe+e=5q+e=r+4q$, provided $qe=eq=q$. Thus $2r^3=2r$, and after the squaring of $r^3=r+4q$, we deduce that $r^6=r^2$, that is, r^2 is a tripotent.

The next (possibly non-commutative) examples shed some more light on the currently studied class of weakly tripotent rings:

EXAMPLE 2. (1) The (upper) triangular matrix ring $\mathbb{T}_2(\mathbb{Z}_2)$ is weakly tripotent by virtue of [1, Example 11].

(2) The (upper) triangular matrix ring $\mathbb{T}_3(\mathbb{Z}_2)$ is *not* weakly tripotent. Even more, it is *not* strongly invo-clean.

To see this curious claim, we consider the invertible matrix $x=\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}$. Direct calculations show that $x^3\neq x$ and that $x^4=I_3$, the identity 3×3 matrix. Now, setting, $y:=1-x$, one directly computes that $y=\begin{pmatrix}0&1&1\\0&0&1\\0&0&0\end{pmatrix}$ and that $y^3=0\neq y$.

Further straightforward computations guarantee that any involution matrix v in $\mathbb{T}_3(\mathbb{Z}_2)$, i.e., $v^2=I_3$, is of the form $v=\begin{pmatrix}1&a&b\\0&1&c\\0&0&1\end{pmatrix}$, where $a,b,c\in\mathbb{Z}_2$ such that $ac=0$ (this certainly excludes the case $a=c=1$), and that any element z in the Jacobson radical of $\mathbb{T}_3(\mathbb{Z}_2)$ is of the form $z=\begin{pmatrix}0&x&y\\0&0&z\\0&0&0\end{pmatrix}$, where $x,y,z\in\mathbb{Z}_2$.

To show the truthfulness of the second part, in accordance with Proposition 3, one needs to find $z\in J(R)$ with $z^2\neq 0$ since $2=0$. Indeed, in conjunction with the presented above calculations, such elements are the matrices $\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}$ and $\begin{pmatrix}0&1&1\\0&0&1\\0&0&0\end{pmatrix}$ which are nilpotents of order 4 (notice that the Jacobson radical of any finite ring is always nil).

(3) The full matrix ring $\mathbb{M}_2(\mathbb{Z}_2)$ is invo-clean but *not* weakly tripotent.

Indeed, the first assertion for invo-cleanness follows directly from [3]. As for the second assertion concerning weak tripotentness, by considering the invertible matrix $u = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ having the property $u^3 = 1 \neq u$, one calculates that $1 - u = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = u^2 \neq (1-u)^3$ whence $(1-u)^2 = u$ and $(1-u)^3 = 1$. This shows that $\mathbb{M}_2(\mathbb{Z}_2)$ is, definitely, not weakly tripotent. This fact, however, is *not* directly deducible from the corresponding results presented in [1].

(4) The (upper) triangular matrix ring $\mathbb{T}_2(\mathbb{Z}_3)$ is *not* weakly tripotent.

To substantiate this surprising assertion, we look at the invertible matrix $A = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$ which is of order 6, that is, $u^6 = 1$. For E being the identity matrix, namely $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we consider the difference $B := E - A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$, which again inverts in the whole ring $\mathbb{T}_2(\mathbb{Z}_3)$. As $B^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the claim is sustained.

(5) The (upper) triangular matrix ring $\mathbb{T}_3(\mathbb{Z}_3)$ is *not* weakly tripotent.

This follows by analogy with points (2) and (4) examined above.

This example leads us to the following strengthening of the behavior of the triangular matrices.

THEOREM 3. *Let R be a ring and $n \in \mathbb{N}$. Then the (upper) triangular matrix ring $\mathbb{T}_n(R)$ is weakly tripotent if, and only if, either*

(1) $n = 1$ and R is weakly tripotent

or

(2) $n = 2$ and $R \cong \mathbb{Z}_2$.

Proof. "$\Rightarrow$". The first point is straightforward as $\mathbb{T}_1(R) \cong R$, so that we will be concentrated on the second one. In fact, as in the proof of [2, Corollary 4.5 (2)], it will follow that either $n = 2$ or $n = 3$, and R is a division ring. Also, since there is an epimorphism $\mathbb{T}_n(R) \to R$ for any natural number $n \geq 1$, one finds that R is weakly tripotent, too. Hence, concerning the case when $n = 2$, we claim that $R \cong \mathbb{Z}_2$. In fact, as in division rings there are no non-trivial idempotents, according to Proposition 2, it must be that $r^2 = 1$ or $r^2 = 2r$ for any $r \in R$. The second equality insures that $r = 2$ provided $r \neq 0$, because r must invert in R. That is why, for $r = 3 \neq 0$, we have that $3^2 = 1$, i.e., $8 = 2^3 = 0$. This means that $2 = 0$, because of the lack of non-trivial nilpotents in

R. Therefore, $r^2 = 1$ is equivalent to $(r-1)^2 = 0$, that is, $r = 1$. Whence either $r = 0$ or $r = 1$, as desired. If now $3 = 0$, Theorem 1 (i) enables us that $R \cong \mathbb{Z}_3$. However, Example 2 (4) tells us that $\mathbb{T}_2(\mathbb{Z}_3)$ is not weakly tripotent, as expected. So, the claim follows.

Concerning the case when $n = 3$, we assert that the ring $\mathbb{T}_3(R)$ cannot be weakly tripotent as the rings $\mathbb{T}_3(\mathbb{Z}_2)$ and $\mathbb{T}_3(\mathbb{Z}_3)$ are not so appealing to points (2) and (5) of Example 2 quoted above. Indeed, there are two sequences of epimorphisms $\mathbb{T}_3(R) \to \mathbb{T}_3(R/M) \to \mathbb{T}_3(\mathbb{Z}_2)$ and/or $\mathbb{T}_3(R) \to \mathbb{T}_3(R/M) \to \mathbb{T}_3(\mathbb{Z}_3)$ for some maximal ideal M of R, induced by the sequence of epimorphisms $R \to R/M \to \mathbb{Z}_2$ and/or $R \to R/M \to \mathbb{Z}_3$ by taking into account that R/M is a weakly tripotent field and so it does have $\mathbb{Z}_2$ and/or $\mathbb{Z}_3$ as epimorphic images.

"$\Leftarrow$". As (1) trivially ensures that $\mathbb{T}_n(R) \cong R$, we are concentrating on (2). What needs to prove is that $\mathbb{T}_2(\mathbb{Z}_2)$ is weakly tripotent. To this aim, Example 2(1) guarantees that $\mathbb{T}_2(\mathbb{Z}_2)$ is really such a ring. □

Comparing this theorem with [11, Theorem 3.6], one says that $\mathbb{T}_n(R)$ is an invo-clean ring for some $n \geq 2$ exactly when R is a Boolean ring.

Moreover, the next comments are worthwhile.

REMARK 2. As already noted above, in the paper [1] were investigated those rings R for which either (at least one of) r or $1-r$ is a solution of the equation $x^3 = x$ whenever $r \in R$. In the commutative case, the authors obtain a complete characterization like this: *R is a subring of the direct product $K_1 \times K_2 \times K_3$ such that $K_1/J(K_1) \cong \mathbb{Z}_2$, for every $z \in J(K_1)$: $z^2 = 2z$, K_2 is a Boolean ring (i.e., a subring of a direct product of copies of $\mathbb{Z}_2$), and K_3 is a subring of a direct product of copies of $\mathbb{Z}_3$.*

We shall now illustrate how some of this can be somewhat deduced from already well-known results established in [6] and [7], respectively. Indeed, at the beginning, we assert that all indecomposable weakly tripotent rings of characteristic 3 are always isomorphic to the field $\mathbb{Z}_3$ and thus they are commutative. To see that, one sees that $x = x^3$ implies that x^2 is an idempotent and so either $x^2 = 0$, which leads to $x = 0$, or $x^2 = 1$. On the other vein, as already seen above, $1-x = (1-x)^3$ implies that $x = x^3$, which is nothing new. Now, an application of [7] is a guarantor of our initial claim. Furthermore, treating the case when $8 = 0$ (and hence 2 is a nilpotent – this case also includes the cases $2 = 0$ and $4 = 0$), we assert that $U(R) = 1 + Nil(R)$. In fact, the containment $\supseteq$ being obvious, we need to show the reciprocal one. So, given $u \in U(R)$, it will follow that $u^2 = 1$ or that $(1-u)^2 = 1$ (as $u^3 = u$ or $(1-u)^3 = 1-u$). Thus, in the first possibility, $(1-u)^2 = 2(1-u) \in Nil(R)$ whence $1-u \in Nil(R)$ and $u \in 1 + Nil(R)$, as required. In the second possibility, one gets that $u^2 = 2u \in Nil(R)$, which is inadequate. Finally, the assertion is sustained. That is why, as above demonstrated, $x = x^3$ yields that x^2 is an idempotent and $(x-x^2)^2 = 2(-x+x^2)$ is a nilpotent, so that $x \in Id(R) + Nil(R)$. In the remaining case, $1-x = (1-x)^3$ yields that $(1-x)^2$ is an idempotent and using the same trick once again by replacing x with $1-x$, we arrive at $1-x \in Id(R) + Nil(R)$ giving up that $x \in (1 - Id(R)) + Nil(R) \subseteq Id(R) + Nil(R)$, as asked for. This means that R is nil-clean and, employing to [6, Theorem 4.3], the ring R must be even strongly nil-clean possessing the property that $R/J(R)$ is Boolean and $J(R)$ is nil.

For some applicable purposes of the established above results, we are now ready to state the following:

PROBLEM 1. *Find a necessary and sufficient condition when a (commutative) group ring is weakly tripotent.*

Let us notice that a criterion for a commutative group ring to be strongly invo-clean was recently obtained in [4].

In closing, with [1, Theorem 14] in mind, we also pose the following:

Conjecture. A ring R is weakly tripotent if, and only if, $R \cong R_1 \times R_2$, where either $R_2 = \{0\}$ or R_2 is a ring of characteristic 3 which is a subdirect product of a family of copies of the field $\mathbb{Z}_3$, and either $R_1 = \{0\}$ or R_1 is a ring with $3 \in U(R_1)$ which is a subdirect product of the ring $P \times B$, where B is a Boolean ring and either $P = \{0\}$ or P is an indecomposable ring with Boolean quotient $P/J(P)$ such that the equality $z^2 = 2z$ holds for all $z \in J(P)$.

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