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HARMONIC METALLIC STRUCTURES

Abstract. The concept of harmonic metallic structure on a metallic pseudo-Riemannian manifold is introduced. In the case of compact manifolds we prove that harmonicity of a metallic structure J , with $J^2 = pJ + qI$ and $p^2 + 4q \neq 0$, is equivalent to $dJ = 0$. Conditions for a harmonic metallic structure to be preserved by harmonic maps are also given. Moreover, we consider harmonic metallic structures on the generalized tangent bundle, provide a Weitzenböck formula for the dual metallic structure and express the Hodge-Laplace operator on $TM \oplus T^*M$.

1. Introduction

Inspired by the paper of W. Jianming [12], we introduce the notion of *harmonic metallic structure* and underline the connection between harmonic metallic structures and harmonic maps. It is well known that harmonic maps play an important role in many areas of mathematics. They often appear in nonlinear theories because of the nonlinear nature of the corresponding partial differential equations. In theoretical physics, harmonic maps are also known as sigma models. Remark also that harmonic maps between manifolds endowed with different geometrical structures have been studied in many contexts: S. Ianuș and A. M. Pastore treated the case of contact metric manifolds [10], C.-L. Bejan and M. Benyounes the almost para-Hermitian manifolds [1], B. Sahin the locally conformal Kähler manifolds [14], S. Ianuș, R. Mazzocco and G. E. Vilcu the quaternionic Kähler manifolds [9], J. P. Jaiswal the Sasakian manifolds [11], D. Fetcu the complex Sasakian manifolds [7], J. Li the Finsler manifolds [13] etc. A. Fotiadis studied the noncompact case, describing the problem of finding a harmonic map between noncompact manifolds [8].

In the present paper, first, we consider the case of compact Riemannian manifolds and prove that harmonicity of a metallic structure J with $J^2 = pJ + qI$ and $p^2 + 4q \neq 0$, is equivalent to $dJ = 0$, then we relate harmonicity to integrability. Conditions for a harmonic metallic structure to be preserved by harmonic maps are also given. Moreover, we consider harmonic metallic structures on the generalized tangent bundle, provide a Weitzenböck formula for the dual metallic structure and express the Hodge-Laplace operator on $TM \oplus T^*M$.

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2. Harmonic metallic structures

2.1. Preliminaries

Let (M, g) be an n -dimensional pseudo-Riemannian manifold. We recall that a metallic pseudo-Riemannian structure J on M is a g -symmetric $(1, 1)$ -tensor field on M such that $J^2 = pJ + qI$, for some p and q real numbers, and (M, J, g) is called a *metallic pseudo-Riemannian manifold* ([4], [5], [6]).

Let ∇ be the Levi-Civita connection associated to g . Consider the exterior differential and codifferential operators defined for any tangent bundle-valued p -form $T \in \Gamma(\Lambda^p T^*M \otimes TM)$ by

$$(dT)(X_1, \dots, X_{p+1}) := - \sum_{i=1}^{p+1} (-1)^i (\nabla_{X_i} T)(X_1, \dots, \widehat{X}_i, \dots, X_{p+1})$$

and

$$(\delta T)(X_1, \dots, X_{p-1}) := - \sum_{i=1}^n (\nabla_{E_i} T)(E_i, X_1, \dots, X_{p-1}),$$

for $\{E_i\}_{1 \leq i \leq n}$ a g -orthonormal frame field, and the Hodge-Laplace operator on $\Gamma(\Lambda^p T^*M \otimes TM)$ by

$$\Delta := d \circ \delta + \delta \circ d.$$

W. Jianming studied in [12] some properties of harmonic complex structures and we discussed in [3] the almost tangent case and in [2] the para-cosymplectic case.

We pose the following:

DEFINITION 1. *A metallic structure J is called harmonic if $\Delta J = 0$.*

If M is compact and g is positive definite, from the definition it follows that J is harmonic if and only if $dJ = 0$ and $\delta J = 0$ which is equivalent to $(\nabla_X J)Y = (\nabla_Y J)X$, for any $X, Y \in C^\infty(TM)$ and $\text{trace}(\nabla J) = 0$.

2.2. Properties of harmonic metallic structures

Let (M, J, g) be an n -dimensional metallic pseudo-Riemannian manifold, let ∇ be the Levi-Civita connection associated to g and let $\{E_i\}_{1 \leq i \leq n}$ be a g -orthonormal frame field.

We have the followings:

LEMMA 1.

$$\sum_{i=1}^n g((dJ)(X, E_i), E_i) = \sum_{i=1}^n g((\nabla_X J)E_i, E_i) + g(X, \delta J),$$

for any $X \in C^\infty(TM)$.

Proof. We have:

$$\begin{aligned}
& \sum_{i=1}^n g((dJ)(X, E_i), E_i) = \sum_{i=1}^n g((\nabla_X J)E_i - \nabla_{E_i} JX + J(\nabla_{E_i} X), E_i) = \\
& = \sum_{i=1}^n [g((\nabla_X J)E_i, E_i) - E_i(g(JX, E_i)) + g(JX, \nabla_{E_i} E_i) + g(\nabla_{E_i} X, JE_i)] = \\
& = \sum_{i=1}^n [g((\nabla_X J)E_i, E_i) - E_i(g(X, JE_i)) + g(X, J(\nabla_{E_i} E_i)) + g(\nabla_{E_i} X, JE_i)] = \\
& = \sum_{i=1}^n [g((\nabla_X J)E_i, E_i) - g(X, \nabla_{E_i} JE_i) + g(X, J(\nabla_{E_i} E_i))] = \\
& = \sum_{i=1}^n [g((\nabla_X J)E_i, E_i) - g(X, (\nabla_{E_i} J)E_i)] = \sum_{i=1}^n g((\nabla_X J)E_i, E_i) + g(X, \delta J).
\end{aligned}$$

□

COROLLARY 1.

$$dJ = 0 \implies g(X, \delta J) = - \sum_{i=1}^n g((\nabla_X J)E_i, E_i), \text{ for any } X \in C^\infty(TM).$$

LEMMA 2.

$$\sum_{i=1}^n g((dJ)(X, E_i), JE_i) = \frac{1}{2} p \sum_{i=1}^n g((\nabla_X J)E_i, E_i) + pg(X, \delta J) - g(JX, \delta J),$$

for any $X \in C^\infty(TM)$.

Proof.

$$\begin{aligned}
& \sum_{i=1}^n g((dJ)(X, E_i), JE_i) = \sum_{i=1}^n [g((\nabla_X J)E_i, JE_i) - g((\nabla_{E_i} J)X, JE_i)] = \\
& = \sum_{i=1}^n [g(\nabla_X JE_i, JE_i) - g(J(\nabla_X E_i), JE_i) - g(\nabla_{E_i} JX, JE_i) + g(J(\nabla_{E_i} X), JE_i)] = \\
& = \sum_{i=1}^n [\frac{1}{2} X(g(JE_i, JE_i)) - pg(\nabla_X E_i, JE_i) - qg(\nabla_X E_i, E_i) - E_i(g(JX, JE_i)) + \\
& \quad + g(JX, \nabla_{E_i} JE_i) + pg(\nabla_{E_i} X, JE_i) + qg(\nabla_{E_i} X, E_i)] = \\
& = \sum_{i=1}^n [\frac{1}{2} pX(g(E_i, JE_i)) - pX(g(E_i, JE_i)) + pg(\nabla_X JE_i, E_i) + \\
& \quad + pg(X, J(\nabla_{E_i} E_i)) - pg(X, \nabla_{E_i} JE_i)] - g(JX, \delta J) =
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^n \left[-\frac{1}{2} pg(\nabla_X E_i, J E_i) + \frac{1}{2} pg(E_i, \nabla_X J E_i) \right] + pg(X, \delta J) - g(JX, \delta J) = \\
&= \sum_{i=1}^n \frac{1}{2} pg((\nabla_X J) E_i, E_i) + pg(X, \delta J) - g(JX, \delta J).
\end{aligned}$$

□

As an application, we get the following:

PROPOSITION 1. *Let (M, J, g) be a metallic pseudo-Riemannian manifold such that $J^2 = pJ + qI$ with $p^2 + 4q \neq 0$. Then $dJ = 0$ implies $\delta J = 0$.*

Proof. If $dJ = 0$, then:

$$\sum_{i=1}^n g((\nabla_X J) E_i, E_i) = -g(X, \delta J)$$

and

$$\sum_{i=1}^n \frac{1}{2} pg((\nabla_X J) E_i, E_i) = -pg(X, \delta J) + g(JX, \delta J).$$

Hence

$$g(JX - \frac{1}{2} pX, \delta J) = 0,$$

for any $X \in C^\infty(TM)$. If $p^2 + 4q \neq 0$, then $\frac{1}{2}p$ is not an eigenvalue of J and so $J - \frac{1}{2}pI$ is invertible. In particular, $g(Y, \delta J) = 0$, for any $Y \in C^\infty(TM)$ and this implies $\delta J = 0$. □

Furthermore:

COROLLARY 2. *Let (M, J, g) be a compact metallic Riemannian manifold such that $J^2 = pJ + qI$ with $p^2 + 4q \neq 0$. Then J is harmonic if and only if $dJ = 0$.*

Proof. Let J be harmonic. Since M is compact and g is positive definite, $\Delta J = 0$ implies $dJ = 0$. Conversely, if $dJ = 0$, then also $\delta J = 0$ and furthermore $\Delta J = 0$. Then we get the statement. □

The vanishing of dJ is also related to integrability, namely we have the following:

LEMMA 3.

$$(dJ)(JX, Y) + (dJ)(X, JY) - p(dJ)(X, Y) = N_J(X, Y),$$

for any $X, Y \in C^\infty(TM)$, where N_J is the Nijenhuis tensor of J .

Proof. Let $X, Y \in C^\infty(TM)$. Then

$$(dJ)(X, Y) = (\nabla_X J)Y - (\nabla_Y J)X = [X, JY] + \nabla_{JY}X - [Y, JX] - \nabla_{JX}Y - J([X, Y]);$$

$$\begin{aligned}
(dJ)(JX, Y) &= [JX, JY] + \nabla_{JY}JX - [Y, J^2X] - \nabla_{J^2X}Y - J([JX, Y]) = \\
&= [JX, JY] + \nabla_{JY}JX - p[Y, JX] - q[Y, X] - p\nabla_{JX}Y - q\nabla_XY - J([JX, Y]); \\
(dJ)(X, JY) &= [X, J^2Y] + \nabla_{J^2Y}X - [JY, JX] - \nabla_{JX}JY - J([X, JY]) = \\
&= p[X, JY] + q[X, Y] + p\nabla_{JY}X + q\nabla_YX - [JY, JX] - \nabla_{JX}JY - J([X, JY]).
\end{aligned}$$

Hence:

$$(dJ)(JX, Y) + (dJ)(X, JY) - p(dJ)(X, Y) = [JX, JY] - J([JX, Y]) - J([X, JY]) + J^2([X, Y])$$

and the proof is complete. \square

As a consequence, we get the following:

PROPOSITION 2. *Let (M, J, g) be a compact metallic Riemannian manifold. If J is harmonic, then it is integrable.*

Proof. Since M is compact and g is positive definite, $\Delta J = 0$ implies $dJ = 0$, then we get the statement. \square

We recall the following:

DEFINITION 2. *Let (M, J, g) be a metallic Riemannian manifold and let ∇ be the Levi-Civita connection of g . (M, J, g) is called locally metallic if*

$$\nabla J = 0.$$

Moreover:

DEFINITION 3. *Let (M, J, g) be a metallic Riemannian manifold and let ∇ be the Levi-Civita connection of g . (M, J, g) is called nearly Kähler, or nearly locally metallic, if the following condition holds:*

$$(\nabla_X J)Y + (\nabla_Y J)X = 0,$$

for any $X, Y \in C^\infty(TM)$.

We have the following:

PROPOSITION 3. *Let (M, J, g) be a compact metallic Riemannian manifold. Then (M, J, g) is locally metallic if and only if J is harmonic and (M, J, g) is nearly Kähler.*

Proof. If (M, J, g) is locally metallic then J is harmonic and (M, J, g) is nearly Kähler.

Conversely, if (M, J, g) is nearly Kähler, then $(\nabla_X J)Y = -(\nabla_Y J)X$ and from $dJ = 0$ we get $\nabla J = 0$. \square

From $\nabla J = 0$ we have $\Delta J = 0$ and we get the followings:

PROPOSITION 4. *Let (M, J, g) be a metallic pseudo-Riemannian manifold such that $J^2 = pJ + qI$ with $p^2 + 4q > 0$ and let $J_p = \frac{1}{\sqrt{p^2+4q}}(2J - pI)$ be the almost product structure on M associated to J . Then J is harmonic if and only if J_p is harmonic.*

Proof. $\Delta J_p = \frac{2}{\sqrt{p^2+4q}}\Delta J$ and the proof is complete. \square

PROPOSITION 5. *Let (M, J, g) be a metallic pseudo-Riemannian manifold such that $J^2 = pJ + qI$ with $p^2 + 4q < 0$ and let $J_c = \frac{1}{\sqrt{-p^2-4q}}(2J - pI)$ be the Norden structure on M associated to J . Then J is harmonic if and only if J_c is harmonic.*

Proof. $\Delta J_c = \frac{2}{\sqrt{-p^2-4q}}\Delta J$ and the proof is complete. \square

2.3. Bochner formula

We know that for any tangent bundle-valued differential form, $T \in \Gamma(\Lambda^1 T^*M \otimes TM)$, the following Weitzenböck formula holds [15]:

$$\Delta T = -\nabla^2 T - S,$$

where $\nabla^2 T := \sum_{i=1}^n (\nabla_{E_i} \nabla_{E_i} T - \nabla_{\nabla_{E_i} E_i} T)$ and $SX := \sum_{i=1}^n (R(E_i, X)T)E_i$, $X \in C^\infty(TM)$, for $\{E_i\}_{1 \leq i \leq n}$ a g -orthonormal frame field and $R(X, Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}$, $X, Y \in C^\infty(TM)$, the Riemann curvature tensor field. We shall also use the notation $R(X, Y, Z, W) := g(R(X, Y)Z, W)$, $X, Y, Z, W \in C^\infty(TM)$. On the metallic pseudo-Riemannian manifold (M, J, g) , taking T equal to J , for any vector field X , we have

$$SX := \sum_{i=1}^n (R(E_i, X)J)E_i = \sum_{i=1}^n [R(E_i, X)JE_i - J(R(E_i, X)E_i)].$$

We can state:

PROPOSITION 6. *Let (M, J, g) be an n -dimensional metallic pseudo-Riemannian manifold. If J is harmonic, then*

$$|\nabla J|^2 = \sum_{1 \leq i, j \leq n} R(E_i, E_j, JE_i, JE_j) + p \cdot \text{trace}(J \circ Q) - q \cdot \text{scal},$$

for $\{E_i\}_{1 \leq i \leq n}$ a g -orthonormal frame field in a neighborhood of a point $x \in M$ such that $(\nabla_{E_i} E_j)(x) = 0$, $1 \leq i, j \leq n$, Q the Ricci operator defined by $g(QX, Y) := \text{Ric}(X, Y)$ and scal the scalar curvature of (M, g) .

Proof. A similar computation like in [12] leads us to

$$\langle \nabla^2 J, J \rangle = \sum_{i=1}^n \langle \nabla_{E_i} \nabla_{E_i} J, J \rangle = -|\nabla J|^2$$

and

$$\begin{aligned}
\langle S, J \rangle &= \sum_{j=1}^n g(SE_j, JE_j) = \sum_{1 \leq i, j \leq n} g(R(E_i, E_j)JE_i, JE_j) - \sum_{1 \leq i, j \leq n} g(J(R(E_i, E_j)E_i), JE_j) = \\
&= \sum_{1 \leq i, j \leq n} R(E_i, E_j, JE_i, JE_j) - \sum_{1 \leq i, j \leq n} R(E_i, E_j, E_i, J^2E_j) = \\
&= \sum_{1 \leq i, j \leq n} R(E_i, E_j, JE_i, JE_j) + p \sum_{j=1}^n Ric(E_j, JE_j) - q \cdot scal = \\
&= \sum_{1 \leq i, j \leq n} R(E_i, E_j, JE_i, JE_j) + p \cdot trace(J \circ Q) - q \cdot scal.
\end{aligned}$$

Therefore

$$\begin{aligned}
0 &= \langle \Delta J, J \rangle = -\langle \nabla^2 J, J \rangle - \langle S, J \rangle = \\
&= |\nabla J|^2 - \sum_{1 \leq i, j \leq n} R(E_i, E_j, JE_i, JE_j) - p \cdot trace(J \circ Q) + q \cdot scal.
\end{aligned}$$

□

REMARK 1. If (M, J, g) is a locally metallic pseudo-Riemannian manifold, then $\langle \Delta J, J \rangle = 0$.

3. Harmonic maps and harmonic metallic structures

Let (M, J, g) and $(\bar{M}, \bar{J}, \bar{g})$ be two n -dimensional metallic Riemannian manifolds. Denote by ∇ and respectively, $\bar{\nabla}$ the Levi-Civita connections associated to g and respectively, \bar{g} .

Consider $\Phi : (M, J, g) \rightarrow (\bar{M}, \bar{J}, \bar{g})$ a smooth map and let

$$\tau(\Phi) := \sum_{i=1}^n [\bar{\nabla}_{\Phi_* E_i} \Phi_* E_i - \Phi_*(\nabla_{E_i} E_i)]$$

be the tension field of Φ , where $\{E_i\}_{1 \leq i \leq n}$ is a g -orthonormal frame field on TM .

PROPOSITION 7. Let $\Phi : (M, J, g) \rightarrow (\bar{M}, \bar{J}, \bar{g})$ be a metallic isometry. Then

$$\bar{J}(\tau(\Phi)) + \Phi_*(\delta J) - \delta \bar{J} = \sum_{i=1}^n [\bar{\nabla}_{\Phi_* E_i} \Phi_*(JE_i) - \Phi_*(\nabla_{E_i} JE_i)],$$

for $\{E_i\}_{1 \leq i \leq n}$ a g -orthonormal frame field on TM .

Proof. Express $\delta J = -\sum_{i=1}^n (\nabla_{E_i} J)E_i = -\sum_{i=1}^n [\nabla_{E_i} JE_i - J(\nabla_{E_i} E_i)]$ and replace it in the left-side of the relation. □

COROLLARY 3. Let $\Phi : (M, g, J) \rightarrow (\bar{M}, \bar{g}, \bar{J})$ be a metallic isometry. If there exists a g -orthonormal frame field $\{E_i\}_{1 \leq i \leq n}$ on TM such that $\Phi_*((\nabla_{E_i} J)E_i) = (\bar{\nabla}_{\Phi_* E_i} \bar{J})(\Phi_* E_i)$, then

$$\delta \bar{J} = \Phi_*(\delta J).$$

Proof. We have $\bar{\nabla}_{\Phi_* E_i} \bar{J}(\Phi_* E_i) - \Phi_*(\nabla_{E_i} J E_i) = \bar{J}(\bar{\nabla}_{\Phi_* E_i} \Phi_* E_i) - \Phi_*(J(\nabla_{E_i} E_i))$ and we get

$$\bar{J}(\tau(\Phi)) + \Phi_*(\delta J) - \delta \bar{J} = \bar{J}\left(\sum_{i=1}^n [\bar{\nabla}_{\Phi_* E_i} \Phi_* E_i - \Phi_*(\nabla_{E_i} E_i)]\right) = \bar{J}(\tau(\Phi)).$$

□

DEFINITION 4. A smooth map $\Phi : (M, J, g) \rightarrow (\bar{M}, \bar{J}, \bar{g})$ is said to be harmonic if its tension field $\tau(\Phi)$ vanishes.

PROPOSITION 8. Let $\Phi : (M, J, g) \rightarrow (\bar{M}, \bar{J}, \bar{g})$ be a metallic isometry. If Φ is a harmonic map, then

$$\Phi_*(\delta J) = \delta \bar{J} + \sum_{i=1}^n [\bar{\nabla}_{\Phi_* E_i} \Phi_*(J E_i) - \Phi_*(\nabla_{E_i} J E_i)],$$

for $\{E_i\}_{1 \leq i \leq n}$ a g -orthonormal frame field on TM .

Moreover, if there exists a g -orthonormal frame field $\{E_i\}_{1 \leq i \leq n}$ such that $\Phi_*((\nabla_{E_i} J)E_i) = (\bar{\nabla}_{\Phi_* E_i} \bar{J})(\Phi_* E_i)$, then

$$\sum_{i=1}^n [\bar{\nabla}_{\Phi_* E_i} \Phi_*(J E_i) - \Phi_*(\nabla_{E_i} J E_i)] = 0.$$

COROLLARY 4. Let $\Phi : (M, g, J) \rightarrow (\bar{M}, \bar{g}, \bar{J})$ be a metallic isometry and assume that J is a harmonic metallic structure.

1. If there exists a g -orthonormal frame field $\{E_i\}_{1 \leq i \leq n}$ on TM such that $\Phi_*((\nabla_{E_i} J)E_i) = (\bar{\nabla}_{\Phi_* E_i} \bar{J})(\Phi_* E_i)$, then $\delta \bar{J} = 0$, hence \bar{J} is a harmonic metallic structure, too.
2. If Φ is a harmonic map, then

$$\delta \bar{J} = - \sum_{i=1}^n [\bar{\nabla}_{\Phi_* E_i} \Phi_*(J E_i) - \Phi_*(\nabla_{E_i} J E_i)],$$

for $\{E_i\}_{1 \leq i \leq n}$ a g -orthonormal frame field on TM .

REMARK 2. If $\Phi : (M, g, J) \rightarrow (\bar{M}, \bar{g}, \bar{J})$ is a metallic isometry, then either $p = \bar{p}$ and $q = \bar{q}$ or J and \bar{J} are trivial metallic structures, namely equal to $\frac{\bar{q}-q}{p-\bar{p}}I$, for $p \neq \bar{p}$. Indeed, for any $X, Y \in C^\infty(TM)$, we have:

$$pg(JX, Y) + qg(X, Y) = g(JX, JY) = \bar{g}(\Phi_*(JX), \Phi_*(JY)) = \bar{g}(\bar{J}(\Phi_* X), \bar{J}(\Phi_* Y)) =$$

$$\begin{aligned} &= \overline{p}g(\overline{J}(\Phi_*X), \Phi_*Y) + \overline{q}g(\Phi_*X, \Phi_*Y) = \overline{p}g(\Phi_*(JX), \Phi_*Y) + \overline{q}g(\Phi_*X, \Phi_*Y) = \\ &= \overline{p}g(JX, Y) + \overline{q}g(X, Y), \end{aligned}$$

which implies $(p - \overline{p})J = (\overline{q} - q)I$ and similarly, $(p - \overline{p})\overline{J} = (\overline{q} - q)I$.

4. Harmonic generalized metallic structures

4.1. Metallic structures on the generalized tangent bundle

Let $TM \oplus T^*M$ be the generalized tangent bundle of M and let (\hat{J}, \hat{g}) be the generalized metallic pseudo-Riemannian structure induced by (J, g) [4]. In block matrix form, \hat{J} is written as

$$\hat{J} = \begin{pmatrix} J & 0 \\ \flat_g & -J^* + pI \end{pmatrix}$$

and the metric \hat{g} is given by

$$\begin{aligned} \hat{g}(X + \alpha, Y + \beta) &= g(X, Y) + g(\sharp_g \alpha, \sharp_g \beta) + \frac{1}{p^2 + 4q} [p(\alpha(Y) + \beta(X)) - 2(\alpha(JY) + \beta(JX))] = \\ &= g(X, Y) + g(\sharp_g \alpha, \sharp_g \beta) + \frac{\sqrt{|p^2 + 4q|}}{p^2 + 4q} [\alpha(\tilde{J}Y) + \beta(\tilde{J}X)], \end{aligned}$$

for any $X, Y \in C^\infty(TM)$ and $\alpha, \beta \in C^\infty(T^*M)$, where \flat_g and \sharp_g are the musical isomorphisms induced by g .

Let ∇ be the Levi-Civita connection of g and let $\hat{\nabla}$ be the induced generalized connection [4]:

$$\begin{aligned} \hat{\nabla} : C^\infty(TM \oplus T^*M) \times C^\infty(TM \oplus T^*M) &\rightarrow C^\infty(TM \oplus T^*M) \\ \hat{\nabla}_{X+\alpha}(Y+\beta) &:= \nabla_X Y + \nabla_X \beta, \end{aligned}$$

for any $X, Y \in C^\infty(TM)$ and $\alpha, \beta \in C^\infty(T^*M)$.

4.2. Harmonicity of generalized metallic structures

Define the exterior differential and codifferential operators for any $TM \oplus T^*M$ -valued p -form $T \in \Gamma(\Lambda^p(TM \oplus T^*M)^* \otimes (TM \oplus T^*M))$ by

$$(dT)(\sigma_1, \dots, \sigma_{p+1}) := - \sum_{i=1}^{p+1} (-1)^i (\hat{\nabla}_{\sigma_i} T)(\sigma_1, \dots, \hat{\sigma}_i, \dots, \sigma_{p+1})$$

and

$$(\delta T)(\sigma_1, \dots, \sigma_{p-1}) := - \sum_{i=1}^n (\hat{\nabla}_{\xi_i} T)(\xi_i, \sigma_1, \dots, \sigma_{p-1}),$$

for $\{\xi_i\}_{1 \leq i \leq n}$ a \hat{g} -orthonormal frame field, and the Hodge-Laplace operator on $\Gamma(\Lambda^p(TM \oplus T^*M)^* \otimes (TM \oplus T^*M))$ by

$$\Delta := d \circ \delta + \delta \circ d.$$

PROPOSITION 9.

$$(d\hat{J})(X + \alpha, Y + \beta) = (dJ)(X, Y) + (\nabla_Y J^*)\alpha - (\nabla_X J^*)\beta,$$

for any $X, Y \in C^\infty(TM)$ and $\alpha, \beta \in C^\infty(T^*M)$.

Proof. A direct computation gives:

$$\begin{aligned} (d\hat{J})(X + \alpha, Y + \beta) &= (\hat{\nabla}_{X+\alpha}\hat{J})(Y + \beta) - (\hat{\nabla}_{Y+\beta}\hat{J})(X + \alpha) = \\ &= \hat{\nabla}_{X+\alpha}\hat{J}(Y + \beta) - \hat{J}(\hat{\nabla}_{X+\alpha}(Y + \beta)) - \hat{\nabla}_{Y+\beta}\hat{J}(X + \alpha) + \hat{J}(\hat{\nabla}_{Y+\beta}(X + \alpha)) = \\ &= \nabla_X JY + \nabla_X \flat_g(Y) - \nabla_X J^*\beta + p\nabla_X \beta - J\nabla_X Y - \flat_g(\nabla_X Y) + J^*(\nabla_X \beta) - p\nabla_X \beta - \\ &- \nabla_Y JX - \nabla_Y \flat_g(X) + \nabla_Y J^*\alpha - p\nabla_Y \alpha + J\nabla_Y X + \flat_g(\nabla_Y X) - J^*(\nabla_Y \alpha) + p\nabla_Y \alpha = \\ &= (\nabla_X J)Y - (\nabla_Y J)X + (\nabla_Y J^*)\alpha - (\nabla_X J^*)\beta = \\ &= (dJ)(X, Y) + (\nabla_Y J^*)\alpha - (\nabla_X J^*)\beta. \end{aligned}$$

□

PROPOSITION 10. $d\hat{J} = 0$ if and only if (M, J, g) is locally metallic.

Proof. From Proposition 9 it follows that $d\hat{J} = 0$ if and only if $dJ = 0$ and $\nabla J^* = 0$. Since $\nabla J^* = 0$ if and only if $\nabla J = 0$ and moreover, $\nabla J = 0$ implies $dJ = 0$, we get the statement. □

In order to compute the codifferential operator we need a \hat{g} -orthonormal basis of $TM \oplus T^*M$.

First we recall the following. Let (M, J, g) be a metallic pseudo-Riemannian manifold such that $J^2 = pJ + qI$ with $p^2 + 4q \neq 0$. Define

$$\tilde{J} := \frac{-1}{\sqrt{|p^2 + 4q|}}(2J - pI).$$

A direct computation gives the following:

LEMMA 4.

$$\tilde{J}^2 = \frac{p^2 + 4q}{|p^2 + 4q|} I.$$

In particular, \tilde{J} is an almost product structure for $p^2 + 4q > 0$ and an almost complex structure for $p^2 + 4q < 0$.

We shall consider the case when g is a Riemannian metric, which, in particular, implies that $p^2 + 4q > 0$ (because if g is positive definite, then \tilde{J} has necessarily real eigenvalues, and $\tilde{J}^2 = I$).

Let n be the real dimension of M and let $\{E_i\}_{1 \leq i \leq n}$ be a g -orthonormal frame field for TM . If we consider $\{b_g(E_i)\}_{1 \leq i \leq n}$ and $\sigma_i := \tilde{J}E_i + b_g(E_i)$, we have:

$$\hat{g}(\sigma_i, \sigma_j) = g(\tilde{J}E_i, \tilde{J}E_j) + g(E_i, E_j) + g(E_i, \tilde{J}^2 E_j) + g(E_j, \tilde{J}^2 E_i) = 4g(E_i, E_j).$$

In particular, defining

$$\xi_i := \frac{1}{2}\sigma_i,$$

we have that ξ_i , for $1 \leq i \leq n$, are \hat{g} -orthonormal on $TM \oplus T^*M$.

PROPOSITION 11.

$$\delta\hat{J} = \frac{1}{4}[\delta J + b_g(\sum_{i=1}^n (\nabla_{\tilde{J}E_i} J)E_i)],$$

where $\{E_i\}_{1 \leq i \leq n}$ is a g -orthonormal frame field for TM .

Proof. A direct computation gives:

$$\begin{aligned} \delta\hat{J} &= -\sum_{i=1}^n (\hat{\nabla}_{\xi_i} \hat{J})\xi_i = -\sum_{i=1}^n [\hat{\nabla}_{\xi_i} \hat{J}\xi_i - \hat{J}(\hat{\nabla}_{\xi_i} \xi_i)] = \\ &= -\frac{1}{4} \sum_{i=1}^n [\hat{\nabla}_{\tilde{J}E_i} \hat{J}(\tilde{J}E_i + b_g(E_i)) - \hat{J}(\hat{\nabla}_{\tilde{J}E_i} (\tilde{J}E_i + b_g(E_i)))] = \\ &= -\frac{1}{4} \sum_{i=1}^n [\hat{\nabla}_{\tilde{J}E_i} (J\tilde{J}E_i + b_g(\tilde{J}E_i) - J^*(b_g(E_i)) + p b_g(E_i)) - \hat{J}(\nabla_{\tilde{J}E_i} \tilde{J}E_i + \nabla_{\tilde{J}E_i} b_g(E_i))] = \\ &= -\frac{1}{4} \sum_{i=1}^n [(\nabla_{\tilde{J}E_i} J)\tilde{J}E_i - b_g((\nabla_{\tilde{J}E_i} J)E_i)] = \\ &= \frac{1}{4} [\delta J + b_g(\sum_{i=1}^n (\nabla_{\tilde{J}E_i} J)E_i)]. \end{aligned}$$

□

LEMMA 5.

$$4(d\delta\hat{J})(X + \alpha) = (d\delta J)(X) + b_g(\sum_{i=1}^n \nabla_X ((\nabla_{\tilde{J}E_i} J)E_i)),$$

for any $X \in C^\infty(TM)$ and $\alpha \in C^\infty(T^*M)$, where $\{E_i\}_{1 \leq i \leq n}$ is a g -orthonormal frame field for TM .

Proof. We have:

$$4(d\delta\hat{J})(X + \alpha) = 4\hat{\nabla}_{X+\alpha} \delta\hat{J} = 4\hat{\nabla}_X \delta\hat{J} = \nabla_X \delta J + b_g(\sum_{i=1}^n \nabla_X ((\nabla_{\tilde{J}E_i} J)E_i)) =$$

$$= (d\delta J)(X) + \flat_g \left(\sum_{i=1}^n \nabla_X ((\nabla_{\tilde{J}E_i} J) E_i) \right).$$

□

LEMMA 6.

$$\begin{aligned} 4(\delta d\hat{J})(X + \alpha) &= (\delta dJ)(X) + (\nabla^2 J^*)(\alpha) + \\ &+ \flat_g \left(\sum_{i=1}^n [-\nabla_{\tilde{J}E_i} ((\nabla_X J) E_i) + (\nabla_X J)(\nabla_{\tilde{J}E_i} E_i) + (\nabla_{\nabla_{\tilde{J}E_i} X} J) E_i] \right), \end{aligned}$$

for any $X \in C^\infty(TM)$ and $\alpha \in C^\infty(T^*M)$, where $\nabla^2 J^* := \sum_{i=1}^n (\nabla_{E_i} \nabla_{E_i} J^* - \nabla_{\nabla_{E_i} E_i} J^*)$ and $\{E_i\}_{1 \leq i \leq n}$ is a g -orthonormal frame field for TM .

Proof. We have:

$$\begin{aligned} 4(\delta d\hat{J})(X + \alpha) &= - \sum_{i=1}^n (\hat{\nabla}_{\tilde{J}E_i} d\hat{J})(\tilde{J}E_i + \flat_g(E_i), X + \alpha) = \\ &= - \sum_{i=1}^n [\hat{\nabla}_{\tilde{J}E_i} (d\hat{J})(\tilde{J}E_i + \flat_g(E_i), X + \alpha) - (d\hat{J})(\hat{\nabla}_{\tilde{J}E_i} (\tilde{J}E_i + \flat_g(E_i)), X + \alpha) - \\ &\quad - (d\hat{J})(\tilde{J}E_i + \flat_g(E_i), \hat{\nabla}_{\tilde{J}E_i} (X + \alpha))] = \\ &= - \sum_{i=1}^n [\nabla_{\tilde{J}E_i} ((dJ)(\tilde{J}E_i, X)) + \nabla_{\tilde{J}E_i} ((\nabla_X J^*) \flat_g(E_i)) - \nabla_{\tilde{J}E_i} ((\nabla_{\tilde{J}E_i} J^*) \alpha) - \\ &\quad - (dJ)(\nabla_{\tilde{J}E_i} \tilde{J}E_i, X) + (\nabla_{\nabla_{\tilde{J}E_i} \tilde{J}E_i} J^*) \alpha - (\nabla_X J^*)(\nabla_{\tilde{J}E_i} \flat_g(E_i)) - \\ &\quad - (dJ)(\tilde{J}E_i, \nabla_{\tilde{J}E_i} X) - \nabla_{\nabla_{\tilde{J}E_i} X} \flat_g(E_i) + (\nabla_{\tilde{J}E_i} J^*) \alpha] = \\ &= (\delta dJ)(X) + \sum_{i=1}^n [(\nabla_{\tilde{J}E_i} \nabla_{\tilde{J}E_i} J^*) \alpha - (\nabla_{\nabla_{\tilde{J}E_i} \tilde{J}E_i} J^*) \alpha] + \\ &+ \flat_g \left(\sum_{i=1}^n [-\nabla_{\tilde{J}E_i} ((\nabla_X J) E_i) + (\nabla_{\nabla_{\tilde{J}E_i} X} J) E_i + (\nabla_X J)(\nabla_{\tilde{J}E_i} E_i)] \right) = \\ &= (\delta dJ)(X) + (\nabla^2 J^*)(\alpha) + \\ &+ \flat_g \left(\sum_{i=1}^n [-\nabla_{\tilde{J}E_i} ((\nabla_X J) E_i) + (\nabla_X J)(\nabla_{\tilde{J}E_i} E_i) + (\nabla_{\nabla_{\tilde{J}E_i} X} J) E_i] \right). \end{aligned}$$

□

Defining

$$(\Delta J^*)(\alpha) := \flat_g((\Delta J)(\sharp_g \alpha)),$$

we have the following Weitzenböck formula for J^* :

LEMMA 7.

$$(\Delta J^*)(\alpha) = -(\nabla^2 J^*)(\alpha) - \flat_g \left(\sum_{i=1}^n (R(E_i, \sharp_g \alpha) J) E_i \right),$$

for any $\alpha \in C^\infty(T^*M)$, where $\{E_i\}_{1 \leq i \leq n}$ is a g -orthonormal frame field for TM .

Proof. It follows immediately from the Weitzenböck formula for J [15]. \square

Then the expression of the Hodge-Laplace operator on $\Gamma(\Lambda^p(TM \oplus T^*M)^* \otimes (TM \oplus T^*M))$ computed on \hat{J} is given by:

PROPOSITION 12. *Let (M, J, g) be an n -dimensional metallic Riemannian manifold such that $J^2 = pJ + qI$ with $p^2 + 4q > 0$. Then for any $X \in C^\infty(TM)$ and $\alpha \in C^\infty(T^*M)$:*

$$4(\Delta \hat{J})(X + \alpha) = (\Delta J)(X) - (\Delta J^*)(\alpha) + \\ + \flat_g \left(\sum_{i=1}^n [(R(X, \tilde{J}E_i)J)E_i - (R(E_i, \sharp_g \alpha)J)E_i + (\nabla_{\tilde{J}E_i} J)(\nabla_X E_i)] \right),$$

where $\{E_i\}_{1 \leq i \leq n}$ is a g -orthonormal frame field for TM .

Proof. Let us consider a g -orthonormal frame field $\{F_i\}_{1 \leq i \leq n}$ in a neighborhood of a point $x \in M$ such that $(\nabla_{F_i} F_j)(x) = 0$. Remark that $E_i := \tilde{J}F_i$ defines a g -orthonormal frame field and $\tilde{J}E_i = \tilde{J}^2 F_i = F_i$. Then $(\nabla_{\tilde{J}E_i} \tilde{J}E_j)(x) = 0$.

From the previous computations we get:

$$4(\Delta \hat{J})(X + \alpha) = (d\delta J)(X) + \flat_g \left(\sum_{i=1}^n \nabla_X ((\nabla_{\tilde{J}E_i} J)E_i) + (\delta dJ)(X) + (\nabla^2 J^*)(\alpha) + \right. \\ \left. + \flat_g \left(\sum_{i=1}^n [-\nabla_{\tilde{J}E_i} ((\nabla_X J)E_i) + (\nabla_X J)(\nabla_{\tilde{J}E_i} E_i) + (\nabla_{\nabla_{\tilde{J}E_i} X} J)E_i] \right) = \right. \\ = (\Delta J)(X) - (\Delta J^*)(\alpha) + \\ \left. + \flat_g \left(\sum_{i=1}^n [(R(X, \tilde{J}E_i)J)E_i - (R(E_i, \sharp_g \alpha)J)E_i + (\nabla_{\tilde{J}E_i} J)(\nabla_X E_i)] \right). \right.$$

\square

COROLLARY 5. \hat{J} is harmonic if and only if the following conditions hold:

1. J is harmonic;
2. $\sum_{i=1}^n (R(E_i, X)J)E_i = 0$;
3. $\sum_{i=1}^n [(R(X, \tilde{J}E_i)J)E_i + (\nabla_{\tilde{J}E_i} J)(\nabla_X E_i)] = 0$,

for any $X \in C^\infty(TM)$, where $\{E_i\}_{1 \leq i \leq n}$ is a g -orthonormal frame field for TM .

5. Harmonic maps between generalized tangent bundles and harmonic generalized metallic structures

Let (M, J, g) and $(\bar{M}, \bar{g}, \bar{J})$ be two n -dimensional metallic Riemannian manifolds. Denote by ∇ and respectively, $\bar{\nabla}$ the Levi-Civita connections associated to g and respectively, \bar{g} .

Consider $\Phi : (M, J, g) \rightarrow (\bar{M}, \bar{J}, \bar{g})$ a diffeomorphism and define:

$$\hat{\Phi} : TM \oplus T^*M \rightarrow T\bar{M} \oplus T^*\bar{M}, \quad \hat{\Phi}(X + \alpha) := \Phi_*X + (\Phi^*)^{-1}\alpha,$$

for any $X \in C^\infty(TM)$ and $\alpha \in C^\infty(T^*M)$.

REMARK 3. If $\Phi : (M, J, g) \rightarrow (\bar{M}, \bar{J}, \bar{g})$ is a metallic map, then

$$\hat{J} \circ \hat{\Phi} = \hat{\Phi} \circ J,$$

where (\hat{J}, \hat{g}) and (\bar{J}, \bar{g}) are the generalized metallic Riemannian structures induced by the Riemannian structures (J, g) and (\bar{J}, \bar{g}) [4].

The tension field of $\hat{\Phi}$ is defined by

$$\tau(\hat{\Phi}) := \sum_{i=1}^n [\hat{\nabla}_{\hat{\Phi}_* \xi_i} \hat{\Phi}_* \xi_i - \hat{\Phi}_*(\hat{\nabla}_{\xi_i} \xi_i)],$$

where $\{\xi_i := \frac{1}{2}JE_i + \frac{1}{2}b_g(E_i)\}_{1 \leq i \leq n}$ for $\{E_i\}_{1 \leq i \leq n}$ a g -orthonormal frame field on TM .

PROPOSITION 13. Let $\Phi : (M, J, g) \rightarrow (\bar{M}, \bar{J}, \bar{g})$ be an isometry. Then $\hat{\Phi}$ is a harmonic map if and only if Φ is a harmonic map and

$$\sum_{i=1}^n [\bar{\nabla}_{\Phi_*(JE_i)}(\Phi_*E_i) - \Phi_*(\nabla_{JE_i}E_i)] = 0,$$

for $\{E_i\}_{1 \leq i \leq n}$ a g -orthonormal frame field on TM .

Proof. Let $\tilde{\xi}_i := \frac{1}{2}\tilde{J}E_i + \frac{1}{2}b_g(E_i)$ where $\tilde{J} := \frac{-1}{\sqrt{p^2+4q}}(2J - pI)$ and $\{E_i\}_{1 \leq i \leq n}$ a g -orthonormal frame field on TM . A direct computation gives:

$$\begin{aligned} \tau(\hat{\Phi}) &= \frac{1}{4} \sum_{i=1}^n [\bar{\nabla}_{\Phi_*(\tilde{J}E_i)}(\Phi_*(\tilde{J}E_i)) - \Phi_*(\nabla_{\tilde{J}E_i}\tilde{J}E_i)] + \\ &\quad + \bar{\nabla}_{\Phi_*(\tilde{J}E_i)}(\Phi^*)^{-1}(b_g(E_i)) - (\Phi^*)^{-1}(\nabla_{\tilde{J}E_i}b_g(E_i)) := \\ &= \frac{1}{4} \tau(\Phi) + \frac{1}{4} \sum_{i=1}^n [\bar{\nabla}_{\Phi_*(\tilde{J}E_i)}b_{\bar{g}}(\Phi_*E_i) - (\Phi^*)^{-1}b_g(\nabla_{\tilde{J}E_i}E_i)] = \\ &= \frac{1}{4} \tau(\Phi) + \frac{1}{4} \sum_{i=1}^n [b_{\bar{g}}(\bar{\nabla}_{\Phi_*(\tilde{J}E_i)}(\Phi_*E_i)) - b_{\bar{g}}(\Phi_*(\nabla_{\tilde{J}E_i}E_i))] = \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4}\tau(\Phi) + \frac{1}{4}b_{\bar{g}}\left(\sum_{i=1}^n[\bar{\nabla}_{\Phi_*(\tilde{J}E_i)}(\Phi_*E_i) - \Phi_*(\nabla_{\tilde{J}E_i}E_i)]\right) = \\
&= \frac{1}{4}\tau(\Phi) + \frac{p}{4\sqrt{p^2+4q}}b_{\bar{g}}(\tau(\Phi)) - \frac{1}{2\sqrt{p^2+4q}}b_{\bar{g}}\left(\sum_{i=1}^n[\bar{\nabla}_{\Phi_*(JE_i)}(\Phi_*E_i) - \Phi_*(\nabla_{JE_i}E_i)]\right).
\end{aligned}$$

□

PROPOSITION 14. Let $\Phi : (M, J, g) \rightarrow (\bar{M}, \bar{J}, \bar{g})$ be a metallic isometry and let (\hat{J}, \hat{g}) and (\tilde{J}, \tilde{g}) be the generalized metallic Riemannian structures induced by the non trivial metallic Riemannian structures (J, g) and (\bar{J}, \bar{g}) [4]. Then

$$\begin{aligned}
\hat{J}(\tau(\hat{\Phi})) &= -\frac{1}{4}[\Phi_*(\delta J) - \delta\bar{J}] + \frac{1}{4}\sum_{i=1}^n[\bar{\nabla}_{\Phi_*E_i}\Phi_*(JE_i) - \Phi_*(\nabla_{E_i}JE_i)] + \\
&+ \frac{p^2 + \sqrt{p^2+4q}}{4\sqrt{p^2+4q}}b_{\bar{g}}(\tau(\Phi)) + \frac{p}{4\sqrt{p^2+4q}}b_{\bar{g}}(\Phi_*(\delta J) - \delta\bar{J}) - \\
&- \frac{p}{4\sqrt{p^2+4q}}b_{\bar{g}}\left(\sum_{i=1}^n[\bar{\nabla}_{\Phi_*E_i}\Phi_*(JE_i) - \Phi_*(\nabla_{E_i}JE_i)]\right) - \\
&- \frac{p}{2\sqrt{p^2+4q}}b_{\bar{g}}\left(\sum_{i=1}^n[\bar{\nabla}_{\Phi_*(JE_i)}\Phi_*E_i - \Phi_*(\nabla_{JE_i}E_i)]\right) + \\
&+ \frac{1}{2\sqrt{p^2+4q}}b_{\bar{g}}\left(\bar{J}\left(\sum_{i=1}^n[\bar{\nabla}_{\Phi_*(JE_i)}\Phi_*E_i - \Phi_*(\nabla_{JE_i}E_i)]\right)\right),
\end{aligned}$$

for $\{E_i\}_{1 \leq i \leq n}$ a g -orthonormal frame field on TM .

Proof. It follows from Propositions 7 and 13. □

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