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ON THE CACTUS RANK OF EMBEDDED VARIETIES WITH GENERICALLY NICE OSCULATING SPACES

Abstract. Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. For any $q \in \mathbb{P}^r$ let $cr_{X_{\mathrm{reg}}}(q)$ denote the minimal degree of a zero-dimensional subscheme of X_{reg} spanning q. We give an upper bound (in terms of $n := \dim X$ and r) for the maximum of all $cr_{X_{\mathrm{reg}}}(q)$, $q \in \mathbb{P}^r$, and/or for the integer $cr_{X_{\mathrm{reg}}}(q)$ when q is general in $\mathbb{P}^r$, when the generic osculating spaces of X have the expected dimensions. For the proofs we use the results on these problems proved using apolarity by Bernardi, Ranestad, Schreyer and others when X is the Veronese embedding of a projective space.

1. Introduction

Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety defined over an algebraically closed field $\mathbb{K}$ with characteristic zero. Set $n := \dim X$. For any $q \in \mathbb{P}^r$ the *regular cactus rank* (resp. *regular local cactus rank*) $cr_{X_{reg}}(q)$ (resp. $cr_{X_{\mathrm{reg}},\mathrm{loc}}(q)$) of q is the minimal degree of a zero dimensional (resp. zero-dimensional and connected) scheme $Z \subset X_{\mathrm{reg}}$ such that $q \in \langle Z \rangle$, where X_{reg} denote the set of all smooth points of X and $\langle\ \rangle$ denote the linear span. Let $cr_{X_{\mathrm{reg}}}$ (resp. $cr_{X_{\mathrm{reg}},\mathrm{loc}}$) denote the maximum of all $cr_{X_{reg}}(q)$, $q \in \mathbb{P}^r$ (the *maximum cactus rank of* X_{reg} and the *maximum local cactus rank of* X_{reg}). There is a non-empty open subset U of $\mathbb{P}^r$ such that $cr_{X_{\mathrm{reg}}}(q) = cr_{X_{\mathrm{reg}}}(q')$ (resp. $cr_{X_{\mathrm{reg}},\mathrm{loc}}(q) = cr_{X_{\mathrm{reg}},\mathrm{loc}}(q')$) for all $q, q' \in U$. Set $cr_{X_{\mathrm{reg}}}(\mathrm{gen}) := cr_{X_{reg}}(q)$ (resp. $cr_{X_{\mathrm{reg}},\mathrm{loc}}(\mathrm{gen}) := cr_{X_{reg},\mathrm{loc}}(q)$) for any $q \in U$ (the *generic cactus rank of* X_{reg}). The aim of this paper is to give upper bounds for $cr_{X_{\mathrm{reg}}}$ and $cr_{X_{\mathrm{reg}}}(\mathrm{gen})$ in terms of geometric informations about the embedding of X in $\mathbb{P}^r$. We impose that $Z \subset X_{\mathrm{reg}}$, because if Z has nasty singularities, they would give a trivial and worthless upper bound for the integer cr_X and $cr_X(\mathrm{gen})$. For instance, $cr_X = 2$ if X has a singular point p with r-dimensional Zariski tangent space. The study of the integers $cr_{X_{\mathrm{reg}}}(q)$ (called scheme-length in [17]) and $cr_{X_{\mathrm{reg}}}$ is an important topic first studied when X is a Veronese variety and then in general (in particular after a key result, [9, Lemma 2.3], says that any $Z \subset X_{\mathrm{reg}}$ with $\deg(Z) = cr_{\mathrm{reg}}(q)$ is Gorenstein) ([1, 3–5, 9, 10, 13, 14, 17, 20]). If $n \leq 3$ all Gorenstein $Z \subset X_{\mathrm{reg}}$ are smoothable ([9, Proposition 6.1], [11]). Thus no wonder that to get non-trivial statements we need $n \geq 4$. To state our results we introduce the following function $N(n,d)$ of the integers $n \geq 3$ and $d \geq 1$.

NOTATION 1. For all $n \geq 3$ and $d \geq 3$ set $N(n,d) := 2\binom{n+\lfloor d/2 \rfloor}{n}$ if d is odd and $N(n,d) := \binom{n+\lfloor d/2 \rfloor}{n} + \binom{n+\lfloor d/2 \rfloor+1}{n}$ if d is even. Set $N(n,2) := 2$ and $N(n,1) := 1$.

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We use the case of the Veronese varieties (in particular [5, Theorem 3]) to prove our results. Let p be a smooth point of an integral variety M of positive dimension, n. For any positive integer m let (mp,M) denote the $(m-1)$-th infinitesimal neighborhood of p in M, i.e. the closed subscheme of M with $(\mathscr{I}_{p,M})^m$ as its ideal sheaf. The scheme (mp,M) is zero-dimensional, $(mp,M)_{\mathrm{red}} = \{p\}$ and $\deg((mp,M)) = \binom{m+n-1}{n}$. Now consider again an integral and non-degenerate variety $X \subset \mathbb{P}^r$ with $\dim X = n$. Let $\mu(X)$ be the maximal positive integer m such that (mp,X) is linearly independent (i.e. $\dim\langle(mp,M)\rangle = \binom{m+n-1}{n} - 1$) for some $p \in X_{\mathrm{reg}}$. The scheme $(\mu(X)o,X)$ is linearly independent for a general $o \in X_{\mathrm{reg}}$ (Remark 3). Obviously we have $\binom{n+\mu(X)-1}{n} \le r+1$. There are some varieties X such that $\binom{n+\mu(X)-1}{n} = r+1$ (Example 1), but obviously this condition gives a strong restriction on r. For these particular pairs (X,r) we prove the following result.

COROLLARY 1. *If $\mu := \mu(X_{\mathrm{reg}}) \ge 4$ and $\binom{n+\mu-1}{n} = r+1$, then $cr_{X_{\mathrm{reg}},\mathrm{loc}}(X) \le N(n,\mu-1)$.*

For arbitrary n and r there are many examples with the property that $\mu(X)$ is the maximal integer with $\binom{n+\mu(X)-1}{n} \le r+1$ (Example 2).

When we allow not connected zero-dimensional schemes we easily get the following result.

THEOREM 2. *Assume $\mu := \mu(X) \ge 4$ and set $\alpha := r+1-\binom{n+\mu-1}{n}$. Then $cr_{X_{\mathrm{reg}}}(X) \le N(n,\mu-1)+\alpha$.*

We also use several osculating spaces, according to the following definition.

DEFINITION 1. *Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. Set $n := \dim X$. Fix positive integers i and m_i, $1 \le i \le s$. We say that X satisfies $\pounds(m_1,\dots,m_s)$ if $m_i \le \mu(X)$ for all i and $\dim\langle\cup_{i=1}^s (m_i p_i, X)\rangle = \min\{r, \sum_{i=1}^s \binom{n+m_i-1}{n} - 1\}$ for a general $(p_1,\dots,p_s) \in X_{\mathrm{reg}}^s$.*

By the semicontinuity theorem for cohomology the condition in Definition 1 is satisfied if there is at least one $(p_1,\dots,p_s) \in X_{\mathrm{reg}}^s$ such that $\dim\langle\cup_{i=1}^s (m_i p_i, X)\rangle = \min\{r, \sum_{i=1}^s \binom{n+m_i-1}{n} - 1\}$. When several consecutive m_i are equal, say $m_i = m$ for $s-x+1 \le i \le x$, we write $\pounds(m_1,\dots,m_{s-x+1},m^x)$.

REMARK 1. Assume that X satisfies $\pounds(m_1,\dots,m_s)$. Since X is non-degenerate, it satisfies $\pounds(m_1,\dots,m_s,1^x)$ for all $x > 0$.

THEOREM 3. *Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety satisfying $\pounds(m_1,\dots,m_s)$ for some integer $m_i > 0$ such that $m_i \le \mu(X)$ for all i. Assume $\sum_{i=1}^s \binom{n+m_i-1}{n} \ge r+1$. Then $cr_{X_{\mathrm{reg}}} \le \sum_{i=1}^s N(n,m_i-1)$.*

Corollary 2 and Theorem 2 also easily follows from Theorem 3.

For the generic cactus rank we prove the following result.

THEOREM 4. *Fix positive integers n, m and r such that* $\binom{n+m-1}{n} \le r < \binom{n+m}{n}$. *Let* $X \subset \mathbb{P}^r$ *be an integral and non-degenerate variety such that* $\mu(X) = m$ *and* $\langle((m+1)p, X)\rangle = \mathbb{P}^r$ *for a general* $p \in X_{\mathrm{reg}}$. *Then* $cr_{X_{\mathrm{reg}},\mathrm{loc}}(\mathrm{gen}) \le N(n, m-1)$.

Theorem 3 is an improvement by $\binom{n+m}{n} - r - 1$ of Theorem 2, but only for the generic cactus rank.

2. General remarks

When X is smooth we often write X instead of X_{reg}.

REMARK 2. For all positive integers n, d set $r_{n,d} := \binom{n+d}{n} - 1$. Let $\nu_{n,d} : \mathbb{P}^n \to \mathbb{P}^{r_{n,d}}$ denote the order d Veronese embedding of $\mathbb{P}^n$. Set $X_{n,d} := \nu_{n,d}(\mathbb{P}^n)$. Fix $p \in X_{n,d}$. We have $\dim\langle(mp, X_{n,d})\rangle = \min\{\binom{m+n-1}{n} - 1, r_{n,d}\}$, i.e. the zero-dimensional scheme $(mp, X_{n,d})$ is linearly independent for all $m \le d+1$ and it spans $\mathbb{P}^{r_{n,d}}$ for all $m \ge d+1$. Thus $\mu(X_{n,d}) = d+1$.

REMARK 3. Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. For any $p \in X_{\mathrm{reg}}$ let $\mu_p(X)$ be the maximal integer $t > 0$ such that the scheme (tp, X) is linearly independent. By the upper semicontinuity theorem for cohomology ([16, III.12.8]) the function $X_{\mathrm{reg}} \to \mathbb{N}$ defined by the formula $p \mapsto \mu_p(X)$ is lower semicontinuous. Thus $\mu_p(X) \le \mu(X)$ for all $p \in X_{\mathrm{reg}}$.

EXAMPLE 1. Let Y be an integral n-dimensional projective variety. Fix an integer $m \ge 3$ and set $r := \binom{n+m-1}{n} - 1$. Let $\mathscr{L}$ be a very ample line bundle on Y. Fix an integer $t \ge m-1$. Since $\mathscr{L}$ is very ample, we have $h^0(\mathscr{I}_{mp,Y} \otimes \mathscr{L}^{\otimes t}) = h^0(\mathscr{L}^{\otimes t}) - \binom{m+n-1}{n}$ for each $p \in Y_{\mathrm{reg}}$. Let $W \subset \mathbb{P}^N$, $N := h^0(\mathscr{L}^{\otimes t}) - 1$, be the image of Y by the complete linear system $|\mathscr{L}^{\otimes t}|$. We saw that each scheme (mo, W), $o \in W_{\mathrm{reg}}$, is linearly independent. For large t we may assume that $N \ge r$. If $N = r$, take $X := W$. Now assume $N > r$. Fix $o \in W_{\mathrm{reg}}$. After fixing o we take a general $(N-r-1)$-dimensional linear subspace $A \subset \mathbb{P}^N$. Call $\ell : \mathbb{P}^N \setminus A \to \mathbb{P}^r$ the linear projection from A. Since A is general and $\dim\langle(mo, W)\rangle = r$, we have $A \cap \langle(mo, W)\rangle = \emptyset$. Since $n = \dim W < r$, we also have $W \cap A = \emptyset$ and so $X := \ell(W) \subset \mathbb{P}^r$ is an integral and non-degenerate variety. Thus $\ell_{|\langle(mo,W)\rangle} : \langle(mo, W)\rangle \to \mathbb{P}^r$ is an isomorphism. Thus the same is true for any general $o' \in W$. We get $\mu(X) \ge m$. Since $r = \binom{n+m-1}{n} - 1$, we have $\mu(X) \le m$. Thus $\mu(X) = m$.

EXAMPLE 2. Let Y be an integral n-dimensional projective variety. Fix an integer $m \ge 3$ and an integer r such that $\binom{n+m-1}{n} - 1 \le r \le \binom{n+m}{n} - 2$. Let $\mathscr{L}$ be a very ample line bundle on Y and an integer $t \ge m-1$. Since $\mathscr{L}$ is very ample, we have $h^0(\mathscr{I}_{mp,Y} \otimes \mathscr{L}^{\otimes t}) = h^0(\mathscr{L}^{\otimes t}) - \binom{m+n-1}{n}$ for each $p \in Y_{\mathrm{reg}}$. Let $W \subset \mathbb{P}^N$,

$N := h^0(\mathscr{L}^{\otimes t} - 1)$, be the image of Y by the complete linear system $|\mathscr{L}^{\otimes t}|$. We saw that each scheme (mo, W), $o \in W_{\mathrm{reg}}$, is linearly independent. For large t we may assume that $N \geq r$. If $N = r$, take $X := W$. Now assume $N > r$. Fix $o \in W_{\mathrm{reg}}$. After fixing o we take a general $(N-r-1)$-dimensional linear subspace $A \subset \mathbb{P}^N$. Call $\ell : \mathbb{P}^N \setminus A \to \mathbb{P}^r$ the linear projection from A. Since A is general and $\dim\langle (mo, W)\rangle = r$, we have $A \cap \langle (mo, W)\rangle = \emptyset$. Since $n = \dim W < r$, we also have $W \cap A = \emptyset$ and so $X := \ell(W) \subset \mathbb{P}^r$ is an integral and non-degenerate variety. Thus $\ell_{|\langle (mo,W)\rangle} : \langle (mo, W)\rangle \to \mathbb{P}^r$ is an isomorphism. Thus the same is true for any general $o' \in W$. We get $\mu(X) \geq m$. Since $r \leq \binom{n+m}{n} - 2$, we have $\mu(X) \leq m$. Thus $\mu(X) = m$. Now assume $t \geq m$. In this case we have $h^0(\mathscr{I}_{(m+1)p,Y} \otimes \mathscr{L}^{\otimes t}) = h^0(\mathscr{L}^{\otimes t}) - \binom{m+n}{n}$ for each $p \in Y_{\mathrm{reg}}$ and hence each scheme $((m+1)o, W)$, $o \in W_{\mathrm{reg}}$, is linearly independent. For a fixed $o \in W_{\mathrm{reg}}$ we may take a general A with the additional condition that $\dim\langle ((m+1)o, W)\rangle \cap A = \binom{m+n}{n} - r - 1$. We get $\langle ((m+1)p, X)\rangle = \mathbb{P}^r$ for a general $p \in X_{\mathrm{reg}}$.

REMARK 4. Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. Set $n := \dim X$. We explain why using only zero-dimensional subschemes of X_{reg} there is no upper bound $u_n(r)$ for $cr_{X_{\mathrm{reg}}}(\mathrm{gen})$ which is exponentially better (or even quadratically better) for $r \gg 0$ than the expected generic rank $\lceil (r+1)/(n+1)\rceil$, i.e. the lower bound for $r_{X_{\mathrm{gen}}}(\mathrm{gen})$, which is the actual value of $r_{X_{\mathrm{reg}}}(\mathrm{gen})$ if X is not defective. For any integer $t > 0$ let $a(n,t)$ denote the dimension of the set of all colength t ideals of the power series ring $\mathbb{K}[[x_1, \dots, x_n]]$, i.e. the set of all degree connected zero-dimensional subschemes of X_{reg} supported by a fixed $p \in X_{\mathrm{reg}}$. By [8, Theorem at page 436] there are positive real numbers a_n and b_n such that

$$a_n t^{2-2/n} \leq a(n,t) \leq b_n t^{2-2/n} \tag{1}$$

for $t \gg 0$ (see [6, 15] for more on punctual Hilbert schemes). Hence we cannot have more than a quadratic improvement if we require connected zero-dimensional schemes with as reduction a prescribed point of X_{reg}. If we continue to prescribe that the zero-dimensional scheme is connected, but we allow as its reduction all $p \in X_{\mathrm{reg}}$ (possibly with different p's for different $q \in \mathbb{P}^r$) we only need to add a $+n$ to the right term in (1). If we allow non-connected zero-dimensional schemes we do not get much better, since $a(n,t_1) + a(n,t_2) \leq a(n,t_1+t_2) + 1$.

We do not know if the upper bound in (1) may be improved if we restrict to Gorenstein zero-dimensional schemes. It is known that this is a very strong restriction, which shows for instance that we do not get better upper bounds for the generic (cactus) rank when $n = 3$ ([9, Proposition 6.1], [11]).

3. Proofs of the main results

LEMMA 1. *Fix $q \in \mathbb{P}^{r_{n,d}}$, $d \geq 3$ and take a connected zero-dimensional scheme $Z \subset X_{n,d}$ with $q \in \langle Z\rangle$. Set $\{p\} := Z_{\mathrm{red}}$. Then there is $Z' \subseteq ((d+1)p, X_{n,d}) \cap Z$ such that $q \in \langle Z'\rangle$. For any $o \in X_{n,d}$ and any $q \in \mathbb{P}^{r_{n,d}}$ there is a zero-dimensional scheme $A \subset X_{n,d}$ such that $A_{\mathrm{red}} = \{o\}$, $A \subseteq ((d+1)o, X_{n,d})$, $q \in \langle A\rangle$ and $\deg(A) \leq N(n,d)$.*

Proof. Write $(d+1)p$ instead of $((d+1)p, X_{n,d})$. We need to find Z' such that $\mathscr{I}_{Z'} \supset \mathscr{I}_Z + \mathscr{I}_{(d+1)p}$ and $q \in \langle Z' \rangle$. Let F be the homogeneous degree d polynomial (uniquely determined up to a non-zero multiplicative constant) representing q. Let $I_{(d+1)p}$ and I_Z be the homogeneous ideals of $(d+1)p$ and Z in the vector space of homogenous degree d polynomials on $\mathbb{P}^n$. By apolarity we have $I_Z \subseteq F^\perp$. Since $\langle (d+1)p \rangle = \mathbb{P}^{r_{n,d}}$, we have $I_{(d+1)p} \subseteq F^\perp$. Thus $I_Z + I_{(d+1)p} \subseteq F^\perp$. Hence $I_{Z \cap (d+1)p} \subset F^\perp$. Since $Z_{\mathrm{red}} = \{p\}$, we have $Z' := Z \cap (d+1)p \neq \emptyset$. Z' is a non-empty zero-dimensional scheme such that $I_{Z'} \subset F^\perp$. By apolarity we have $q \in \langle Z' \rangle$. The second assertion of the lemma follows from the first one and [5, Theorem 1.3]. □

PROPOSITION 1. *Take $p \in X_{\mathrm{reg}}$ and an integer $m \geq 3$ such that (mp, X) is linearly independent, i.e., assume* $\dim\langle (mp, X) \rangle = r_{n,m-1}$. *Then for each $q \in \langle (mp, X) \rangle$ and any $o \in X_{n,m-1}$ there is $q' \in \mathbb{P}^{r_{n,m-1}}$ such that $cr_{X_{n,m-1},\mathrm{loc}}(q') = cr_{X,p}(q)$,*

Proof. By assumption X is smooth at p and $\dim X = \dim X_{n,m-1}$. Hence there is a formal isomorphism $u : \hat{\mathscr{O}}_{X_{n,m-1},o} \to \hat{\mathscr{O}}_{X,p}$ induced by any linear isomorphism between the tangent spaces of $X_{n,m-1}$ at o and the tangent space of X at p. Since u is linear and $\dim\langle (mp, X) \rangle = r_{n,m-1}$, u induces a linear isomorphism φ between $\mathbb{P}^{r_{n,m-1}}$ and $\langle (mp, X) \rangle$ mapping $(mo, X_{n,m-1})$ onto (mp, X). Take $q' := \varphi(q)$. □

Proof of Corollary 1: This is a consequence of Proposition 1 and Remark 2 (i.e. [5, Theorem 3]), because $\langle (mp, X) \rangle = \mathbb{P}^r$ under the assumptions of Corollary 1. □

Proof of Theorem 2: Fix $q \in \mathbb{P}^r$. Fix a general $p \in X_{\mathrm{reg}}$ and set $M := \langle (\mu(X)p, X) \rangle$. Since p is general, the definition of $\mu(X)$ gives $\dim M = \binom{n+\mu(X)-1}{n} - 1$. Since X is non-degenerate, for a general $S \subset X$ such that $\#(S) = \alpha$ we have $\langle S \cup M \rangle = \mathbb{P}^r$. If $q \in \langle S \rangle$ (resp. $q \in M$) we have $cr_{X_{\mathrm{reg}}}(q) \leq \alpha$ (resp. $cr_{X_{\mathrm{reg}}}(q) \leq N(n, \mu(X) - 1)$). If $q \notin (\langle S \rangle \cup M)$ there are $q' \in \langle S \rangle$ and $q'' \in M$ such that $q \in \langle \{q', q''\} \rangle$. Thus $cr_{X_{\mathrm{reg}}}(q) \leq N(n, \mu(X) - 1) + \alpha$. □

Theorem 2 also follows from Remark 1 and the statement of Theorem 3.

PROPOSITION 2. *Assume $\mu(X) \geq 4$ and $r = \binom{n+\mu(X)-1}{n}$. Then $cr_{X_{\mathrm{reg}},\mathrm{loc}}(q) \leq N(d, \mu(X) - 1)$ for a general $q \in \mathbb{P}^r$.*

Proof. Since $r = \binom{n+\mu(X)-1}{n}$, for a general $p \in X_{\mathrm{reg}}$ the linear space $M_p := \langle (mp, X) \rangle$ is a hyperplane. By Proposition 1 we have $cr_{X_{\mathrm{reg}},\mathrm{loc}}(q) \leq N(n, \mu(X) - 1)$ for all $q \in M_p$. Since X is non-degenerate, a general $q \in \mathbb{P}^r$ is contained in some hyperplane M_p. □

Proof of Theorem 3: Fix a general $(p_1, \dots, p_s) \in X^s_{\mathrm{reg}}$. By $\pounds(m_1, \dots, m_s)$ the linear spaces $M_i := \langle (m_i p_i, X) \rangle$ span $\mathbb{P}^r$. Hence for each $q \in \mathbb{P}^r$ there is $S_q \in \{1, \dots, s\}$, $S_q \neq \emptyset$ and $q_i \in M_i$, $i \in S_q$, such that $q \in \langle \cup_{i \in S_q} q_i \rangle$. Apply Proposition 1 to each q_i with $m_i \geq 3$. If $m_i = 1$ we have $q_i = o_i$ and $N(n, 0) = 1$. If $m_i = 2$ the linear space $\langle (2o_i, X) \rangle$ is the tangent space of X at o_i and any point of it has cactus rank ≤ 2. □

As a particular case of Theorem 3 we get the following results.

COROLLARY 2. *Set* $\alpha := r+1-\binom{n+\mu(X)-1}{n}$, $x := \lceil \alpha/(n+1) \rceil$. *Assume* $\pounds(X,\mu(X),2^x))$. *Then* $cr_{X_{\mathrm{reg}}}(X) \le N(n,\mu(X)-1)+\lceil \alpha/(n+1) \rceil$

COROLLARY 3. *Let* $X \subset \mathbb{P}^r$ *be an integral and non-degenerate variety. Set* $n := \dim X$. *Fix integers* $s \ge 2$ *and* $m > 0$ *such that* $m \le \mu(X)$ *and* $(s-1)\binom{n+m-1}{n} \le r < s\binom{n+m-1}{n}$. *Assume that X satisfies* $\pounds(m^s)$. *Then* $cr_{X_{\mathrm{reg}}} \le sN(n,m-1)$.

Thus, at least if $n \ge 4$ when $\mu(X)$ is almost maximal knowing the truth of $\pounds(m_1,\dots,m_s)$ for very large m_1 is quite useful. The case of the Segre-Harbourne-Hirschowitz conjecture when $X = \mathbb{P}^2$ ([12]) and the similar Laface-Ugaglia conjecture for $\mathbb{P}^3$ ([18]) show that there is no hope to have $\pounds(m_1,\dots,m_s)$ for arbitrary X and $m_1,\dots,m_s$ with $m_i \le \mu(X)$ for all i. However, by Proposition 3 just knowing $\pounds(m_1,m_2)$ with $m_2 \le m_1 \le \mu(X)$ we get $cr_{X_{\mathrm{reg}}} \le N(n,m_1)+N(n,m_2)$ if $r \le \binom{n+m_1-1}{n}+\binom{n+m_2-1}{n}-1$.

REMARK 5. Let X and Y be quasi-projective varieties. Fix $p \in X_{\mathrm{reg}}$ and $o \in Y_{\mathrm{reg}}$ and assume $\dim X = \dim Y = n$. Let μ be the maximal ideal of the local ring $\mathscr{O}_{X,p}$ and ν the maximal ideal of the local ring $\mathscr{O}_{Y,o}$. By assumption X is smooth at p and $\dim X = \dim Y = n$. Hence there is a formal isomorphism $u : \hat{\mathscr{O}}_{X,p} \to \hat{\mathscr{O}}_{Y,o}$ induced by any linear isomorphism between the dual of tangent spaces of μ/μ^2 of X at p and the dual of the tangent space ν/ν^2 of Y at p. Call $x_1,\dots,x_n$ a regular system of parameters of $\mathscr{O}_{X,p}$. Since X and Y are smooth at p and o and they have dimension n, $\hat{\mathscr{O}}_{X,p}$ is isomorphic to the power series ring $\mathbb{K}[[x_1,\dots,x_n]]$ in n variables $x_1,\dots,x_n$ over $\mathbb{K}$ and $\hat{\mathscr{O}}_{Y,o}$ is isomorphic to the power series ring in the n variables $y_1,\dots,y_n$, where $y_i := u(x_i)$.

Proof of Theorem 3: Set $w := \binom{n+m}{n}-1$. Let $Y := X_{n,m} \subset \mathbb{P}^w$ be the Veronese n-dimensional variety associated to the complete linear system $|\mathscr{O}_{\mathbb{P}^n}(m)|$. We fix a general $(w-r-1)$-dimensional linear subspace $V \subset \mathbb{P}^w$ and let $\ell : \mathbb{P}^u \setminus V \to \mathbb{P}^r$ denote the linear projection from V. Since V is general and $r \ge 2n+1$, ℓ_Y is an embedding. We fix $o \in Y$ and a general $p \in X_{\mathrm{reg}}$. As in Remark 5 let ν be the maximal ideal of $\mathscr{O}_{Y,o}$ and μ the maximal ideal of $\mathscr{O}_{X,p}$.

Claim 1: Up to an automorphism of $\mathbb{P}^r$ we may assume that $\ell(o) = p$ and that ℓ induces an isomorphism $u : \hat{\mathscr{O}}_{X,p} \to \hat{\mathscr{O}}_{Y,o}$.

Proof of Claim 1: Since $m = \mu(X)$ and $\langle((m+1)p,X)\rangle = \mathbb{P}^r$, there is a regular system of parameters $x_1,\dots,x_n$ of the regular local ring $\mathscr{O}_{X,p}$ such that we may take as homogeneous coordinates of $\mathbb{P}^r$ all monomials in $x_1,\dots,x_n$ of degree $\le m-1$ and $r+1-\binom{n+m-1}{n}$ degree m homogeneous polynomials $u_h(x_1,\dots,x_n)$, $1 \le h \le r+1-\binom{n+m-1}{n}$ in the variables $x_1,\dots,x_n$. We take a regular systems of parameters $y_1,\dots,y_n$

of $\mathscr{O}_{Y,o}$ formed by the dehomogeneization of n linear forms on $\mathbb{P}^w$. Since Y is the order m Veronese embedding, we may take as homogeneous coordinates of $\mathbb{P}^w$ the homogeneizations of all polynomials of degree $\le m$ in the variables $y_1,\dots,y_n$. We may take as V a linear space not intersecting the zero loci of all polynomials of degree $\le m-1$ in $y_1,\dots,y_n$ and projecting the linear space spanned by the degree m homogeneous polynomial in $y_1,\dots,y_n$ to the linear span of the polynomials $u_i(y_1,\dots,y_n)$, $1\le i\le r+1-\binom{n+m-1}{n}$.

For any integer $t>0$ the isomorphism u induces an isomorphism between the schemes (tp,X) and (to,Y). The vector space ν^m/ν^{m+1} (resp. μ^k/μ^{m+1}) has dimension $\binom{n+m-1}{n-1}$ and it has as a basis the degree m monomials in the variables $x_1,\dots,x_n$ (resp. $y_1,\dots,y_n$). Fix an integer x such that $\binom{n+m-1}{n}<x<\binom{n+m}{n}$. There is a bijection between the set $\mathscr{B}_X$ of all degree x subschemes of $((m+1)p,X)$ containing (mp,X) and the set $\mathscr{A}_X$ of all ideals J of $\mathscr{O}_{((m+1)p,X)}$ containing the ideal $\mathscr{I}_{(mp,X)}$ of $\mathscr{O}_{(mp,X)}$ and with $\dim_{\mathbb{K}} J/\mathscr{I}_{(mp,X)}=x-\binom{n+m-1}{n}$. Since the product in $\mathscr{O}_{((m+1)p,X)}$ of any two elements of $\mathscr{I}_{(mp,X)}$ is zero, there is a bijection between $\mathscr{A}$ and the Grassmannian of all $(x-\binom{n+m-1}{n})$-dimensional linear subspaces of ν^m/ν^{m+1}. The same description holds for the set of all degree x subschemes of $((m+1)p,X)$ containing (mp,X). Since $V\cap\langle(mo,Y)\rangle=\emptyset$, we may represent V by a $(w-r)$-dimensional linear subspace V' of ν^m/ν^{m+1}.

Fix a general $q'\in\mathbb{P}^w$. Since q' is general, $q'\notin V$ and hence $q:=\ell(q')$ is defined. Since q' is general, q is a general point of $\mathbb{P}^r$. By Proposition 1 there is a connected zero-dimensional scheme $Z\subset((m+1)o,Y)$ such that $\deg(Z)\le N(n,m)$ and $q'\in\langle Z\rangle$. If $\ell(Z)\subset X$, then $cr_{X_{\mathrm{reg}},\mathrm{loc}}(q)\le N(n,m)$. If $Z\subset(mo,Y)$, then we may repeat the proof of Corollary 1. Now assume $Z\nsubseteq(mo,Y)$.

Let $W\subset((m+1)o,Y)$ be the union of (mo,Y) and Z, i.e. the closed subscheme of $((m+1)o,Y)$ with $I_Z\cap I_{mo}$ as its ideal sheaf. Since $Z\nsubseteq(mo,Y)$, we have $(mo,Y)\subsetneq W\subseteq(mo,Y)$. Thus W is associated to a linear subspace W' of ν^m/ν^{m+1}. Since Z is Gorenstein ([9, Lemma 2.3]), the annihilator $(0:\mu_Z)$ of the maximal ideal of $\mathscr{O}_Z$ is one-dimensional. The annihilator $(0:\mu_W)$ of the maximal ideal of W has dimension $\dim W'$. Hence $\dim W'=1$. To conclude the proof it is sufficient to prove the following claim.

Claim 2: For a general q' we have $W'\cap V'=\{0\}$.

Proof of Claim 2: Let $W_1\subset((m+1)o,Y)$ be the zero-dimensional scheme containing (mo,Y) and with V' as its associated subspace. Since $\dim W'=1$, we have $W'\cap V'\ne\{0\}$ if and only if $W'\subseteq V'$, i.e. if and only if $W\subseteq W_1$, i.e. if and only if $Z\subseteq W_1$. If we take $q'\notin\langle W_1\rangle$, then the associated scheme Z is not contained in W_1, because $q'\in\langle Z\rangle$. □

For each integer $m>0$ let $\mathrm{Osc}(m,X)$ denote the m-osculating variety of X, i.e. the closure in $\mathbb{P}^r$ of the union of all $\langle((m+1)p,X)\rangle$, $p\in X_{\mathrm{reg}}$. The integer $\min\{r,n+\binom{n+m-1}{m}-1\}$ is the expected dimension of the m-osculating variety of the

n-dimensional variety $X \subset \mathbb{P}^r$. We do not think it is very useful (for this topic) to study the more difficult problem of the dimension of joins of several osculating varieties $\mathrm{Osc}(X, m_i)$ of X (as done in [2] in a very particular case) is effort-efficient for giving upper bounds for the cactus X-rank (it seems better to prove $\pounds(m_1, \dots, m_s)$) for some more $m_1, \dots, m_s$. We offer the following example in which it gives more than $\pounds(m, 2)$.

EXAMPLE 3. Take r such that $\binom{n+m-1}{n} \le r \le \binom{n+m-1}{n} + n - 1$ and assume $m = \mu(X)$ and $\mathrm{Osc}(X, m) = \mathbb{P}^r$. We have $cr_{X_{\mathrm{reg}}}(\mathrm{gen}) \le \binom{n+m-1}{n}$, because the assumption $\mathrm{Osc}(X, m) = \mathbb{P}^r$ implies that for a general $q \in \mathbb{P}^r$ there is some $p \in X_{\mathrm{reg}}$ with $q \in \langle (mp, X) \rangle$.

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