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A NOTE ON SCHOEN’S CALABI–YAU THREEFOLDS

Abstract. We study the topology of a real Lagrangian in Schoen’s Calabi–Yau threefold X and compute its mod 2 cohomology using two methods; first via a concrete Mayer–Vietoris calculation, then by an exact sequence relating the mod 2 cohomology of the real Lagrangian to the cohomology of X . We conclude that these two methods agree. This in particular corrects a previous computation made by Castaño–Bernard–Matessi.

1. Introduction

Schoen’s Calabi–Yau threefold X [13] is the fibred product of two rational elliptic surfaces over \mathbb{P}^1 , and is of particular interest from the point of view of the Strominger–Yau–Zaslow conjecture in mirror symmetry [8,9]. Kovalev has described a Lagrangian 3-torus fibration on X [11], which has further been studied by Gross in [8, §4] from the perspective of toric degenerations of Calabi–Yau toric complete intersections. In particular, Gross describes a singular torus fibration on X using constructions given in [7].

Later on, Castaño–Bernard–Matessi–Solomon studied anti-symplectic involutions on the total space of such torus fibrations in [4]. The fixed point set of such involutions are real Lagrangians. As these Lagrangians provide an algebro-geometric path to open Gromov–Witten invariants and the Fukaya category [6], understanding their topology attracts significant interest from both symplectic and algebraic geometers. The mod 2 cohomology of real Lagrangians is particularly widely studied in the literature due to its deep connections to real enumerative algebraic geometry [3,10,12].

For the case of the real Lagrangian in Schoen’s Calabi–Yau X , Castaño–Bernard–Matessi outline the computation of the cohomology of a real Lagrangian in [5, §4.2] via an argument based on the Mayer–Vietoris theorem. Here we first detail this Mayer–Vietoris computation by further investigating the topology of the torus fibration on Schoen’s Calabi–Yau threefold, noting that our result differs slightly from the one obtained in [5] (in which $H^1(\Sigma, \mathbb{Z}_2) \cong \mathbb{Z}_2^{34}$). We explain this discrepancy below, and give a detailed account of the Mayer–Vietoris calculation outlined in [5]. Our main result is the following:

THEOREM 1. *The mod 2 cohomology groups of the real Lagrangian $\Sigma \subset X$ are given by*

$$H^0(\Sigma, \mathbb{Z}_2) = H^3(\Sigma, \mathbb{Z}_2) \cong \mathbb{Z}_2, \quad \text{and} \quad H^1(\Sigma, \mathbb{Z}_2) = H^2(\Sigma, \mathbb{Z}_2) \cong \mathbb{Z}_2^{36}.$$

We provide an alternative proof to this result using techniques developed in [2], where we give a general framework to compute the mod 2 cohomology groups of real Lagrangians in a Calabi–Yau X , and its mirror \check{X} , which admit (singular) torus

fibrations

$$f: X \rightarrow B, \quad \text{and} \quad \check{f}: \check{X} \rightarrow B$$

over a \mathbb{Z}_2 homology sphere B . We let ι denote the anti-symplectic involution on X sending $x \mapsto -x$ in each fibre. Our main result in [2] involves the comparison of the following maps:

1. The map $\text{Sq}: D \mapsto D^2$, where $D \in H^1(B, R^1 \check{f}_* \mathbb{Z}_2)$.
2. The map $\beta: H^1(B, R^2 f_* \mathbb{Z}_2) \rightarrow H^2(B, R^1 f_* \mathbb{Z}_2)$; the connecting homomorphism introduced by Castaño-Bernard-Matessi in [5] associated to the short exact sequence

$$0 \rightarrow R^1 f_* \mathbb{Z}_2 \oplus \mathbb{Z}_2^2 \rightarrow \pi_* \mathbb{Z}_2 \rightarrow R^2 f_* \mathbb{Z}_2 \rightarrow 0,$$

where π denotes the restriction of f to the fixed locus $L_{\mathbb{R}}$ of ι .

We prove in [2, Theorem 1.1] that the linear maps β and Sq coincide. Applying this result to the case of Schoen's Calabi–Yau, we verify Theorem 1. This result also provides the first instance of a real Lagrangian in Calabi–Yau with Picard number greater than one which provides positive evidence for Conjecture 1.3 in [1], stating that the cohomology of real Lagrangians in Calabi–Yau toric hypersurfaces are 2-primary.

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2. A Real Lagrangian in Schoen's Calabi–Yau

Take an elliptically fibred K3 surface over \mathbb{P}^1 , with 24 singular fibres of Kodaira type I_1 . Let γ be a simple closed curve bounding a disk D^2 containing the image of 12 singular fibres and not intersecting any other critical value on the base. Let M^4 be the preimage of D^2 , and set $\bar{X} := M^4 \times T^2$, where T^2 denotes the standard 2-torus. It is shown in [8, §4] that Schoen's Calabi–Yau X is diffeomorphic to a fibered coproduct of two copies of \bar{X} along their boundary.

There is a fibration on \bar{X} over the solid torus $D^2 \times S^1$, obtained by taking the product of the fibration on M^4 with the standard S^1 fibration on T^2 . This gives a torus fibration on X over the three sphere S^3 , expressed as the union of the solid tori

$$B_i = D^2 \times S^1,$$

for $1 \leq i \leq 2$, the bases of the fibrations on the two copies of \bar{X} forming X .

Consider an involution on M^4 obtained by restricting a fiber-preserving anti-symplectic, and anti-holomorphic involution on the K3 surface and take the involution on T^2 which preserves the fibres of the fibration on S^1 . The product of these two

involutions induces an anti-symplectic involution on X , and the fixed point locus Σ , which is naturally a real Lagrangian, is an 8-to-1 cover

$$\sigma: \Sigma \rightarrow B \cong S^3,$$

branched over 24 circles, as described in [5].

2.1. Proof of Theorem 1 via Mayer-Vietoris

We set $A_i := \sigma^{-1}(B_i)$ for each $i \in \{1, 2\}$; we will apply the Mayer–Vietoris theorem to the decomposition $\Sigma = A_1 \cup A_2$.

First note that the fixed point locus on the fiber-preserving anti-symplectic involution on M^4 is obtained, as in [5, §4.2], as the disjoint union of a 2-disc E and a genus 4 surface with three disks removed, denoted by S . The boundary components of S are contained in the preimage of the boundary of D^2 . Recalling that $B_i \cong D^2 \times S^1$ for each $i \in \{1, 2\}$ we obtain that each A_i is the disjoint union of two copies of $S \times S^1$, and two copies of $E \times S^1$.

We let $E_i^1 \times S^1, S_i^1 \times S^1, E_i^2 \times S^1, S_i^2 \times S^1$ denote the four components of A_i for each $i \in \{1, 2\}$. Following [2, 5] we may identify fibres of σ with the half integral points in $\mathbb{R}^2/\mathbb{Z}^2$ and $\mathbb{R}^3/\mathbb{Z}^3$. Note that, similarly as in [5, Corollary 1], since the monodromy always leaves one point on each fiber invariant, the fixed point locus Σ has two connected components: one is homeomorphic to the base S^3 and the other is a 7-to-1 branched cover over S^3 . For each $i \in \{1, 2\}$, the boundary of A_i consists of four two-dimensional tori (given by $\partial(E_i^j \times S^1)$ and $\partial(S_i^j \times S^1)$ for $j \in \{1, 2\}$ respectively). Following the constructions described in [5, §4.1] and [8, §4], the components of $A_1 \cap A_2$ can be indexed by T_i , for $0 \leq i \leq 7$, where each T_i is the 2-torus defined as follows:

1. T_0 bounds a copy of $E_i^1 \times S^1$ in A_i for each $i \in \{1, 2\}$.
2. T_4 bounds an the solid torus $E_1^2 \times S^1$ in A_1 .
3. T_1, T_2 , and T_3 are boundary components of $S_1^1 \times S^1 \subset A_1$.
4. T_5, T_6 , and T_7 are boundary components of $S_1^2 \times S^1 \subset A_1$.
5. T_3 bounds the solid torus $E_2^2 \times S^1$ in A_2 .
6. T_1, T_4 , and T_5 are boundary components of $S_2^1 \times S^1 \subset A_2$.
7. T_2, T_6 , and T_7 are boundary components of $S_2^2 \times S^1 \subset A_2$.

Now consider the following part of the Mayer–Vietoris sequence for the decomposition $X = A_1 \cup A_2$:

$$H_1(\Sigma, \mathbb{Z}_2) \rightarrow H_0(A_1 \cap A_2, \mathbb{Z}_2) \rightarrow H_0(A_1, \mathbb{Z}_2) \oplus H_0(A_2, \mathbb{Z}_2) \rightarrow H_0(\Sigma, \mathbb{Z}_2) \rightarrow 0.$$

Using the descriptions of A_1, A_2 , and X given above, this sequence has the form

$$(1) \quad H_1(\Sigma, \mathbb{Z}_2) \rightarrow \mathbb{Z}_2^8 \rightarrow \mathbb{Z}_2^4 \oplus \mathbb{Z}_2^4 \rightarrow \mathbb{Z}_2^2 \rightarrow 0.$$

The preceding part of the Mayer–Vietoris sequence has the form

$$H_1(A_1 \cap A_2, \mathbb{Z}_2) \rightarrow H_1(A_1, \mathbb{Z}_2) \oplus H_1(A_2, \mathbb{Z}_2) \rightarrow H_1(\Sigma, \mathbb{Z}_2).$$

Computing these spaces, this sequence becomes

$$(2) \quad \mathbb{Z}_2^{16} \xrightarrow{\varphi} \mathbb{Z}_2^{24} \oplus \mathbb{Z}_2^{24} \longrightarrow H_1(\Sigma, \mathbb{Z}_2).$$

We let $\varphi = \varphi_1 \oplus \varphi_2$ denote the components of the map φ in (2). Note that it follows directly from (1) and (2) that $\dim H_1(\Sigma, \mathbb{Z}_2) = 34 + k$, where $k = \dim \ker(\varphi)$. We now describe a system of linear equations defining the kernel of φ .

Since $A_1 \cap A_2$ is the disjoint union of eight two-dimensional tori, the vector space $H_1(A_1 \cap A_2, \mathbb{Z}_2)$ is generated by eight pairs $x_i, y_i \in H_1(T_i, \mathbb{Z}_2)$ for $i \in \{0, \dots, 7\}$. Hence any element of $H_1(A_1 \cap A_2, \mathbb{Z}_2)$ can be expressed as a sum of the form $\sum_{i=0}^7 (a_i x_i + b_i y_i)$, where a_i and b_i are elements of \mathbb{Z}_2 .

We choose the bases $\{x_i, y_i\}$ of each component of $H_1(A_1 \cap A_2, \mathbb{Z}_2)$ such that $\varphi_1(y_i)$ generates the first homology group of the S^1 factor of the corresponding connected component of A_1 (recalling that this component is homeomorphic to either $E_1^j \times S^1$ or $S_1^j \times S^1$ for some $j \in \{1, 2\}$) and $\varphi_2(x_i)$ generates the first homology group of the S^1 factor of the corresponding connected component of A_2 . Moreover, the homology classes $\varphi_1(x_i)$ and $\varphi_2(y_i)$ are either meridian curves on a solid torus, or the classes of boundary circles of one of the three discs removed from the punctured genus four surface S .

LEMMA 1. *Consider a class $\sum_{i=0}^7 (a_i x_i + b_i y_i)$, for a_i and b_i in \mathbb{Z}_2 . This class is contained in the kernel of φ if and only if the following equations hold:*

1. $a_0 = b_0 = 0$.
2. $b_4 = 0$.
3. $b_1 + b_2 + b_3 = 0$ and $a_1 = a_2 = a_3$.
4. $b_5 + b_6 + b_7 = 0$ and $a_5 = a_6 = a_7$.
5. $a_3 = 0$.
6. $a_1 + a_4 + a_5 = 0$ and $b_1 = b_4 = b_5$.
7. $a_2 + a_6 + a_7 = 0$ and $b_2 = b_6 = b_7$.

Proof. These equations each follow directly from the above description of the 3-manifolds bounded by each T_i for $i \in \{0, \dots, 7\}$. For example, considering the equations in item (1), both x_0 and y_0 are the classes of longitudes of a solid torus and hence both a_0 and b_0 must vanish.

Considering the equations in item (3), we recall that T_1, T_2 , and T_3 are the boundary components of a copy of $S \times S^1 \subset A_1$. The classes y_1, y_2 , and y_3 are all

mapped (by φ_1) to the multiples of the S^1 factor of $S^1_1 \times S^1$, and hence the component of $\varphi_1(\sum_{i=0}^7(a_i x_i + b_i y_i))$ corresponding to this factor is equal to the class of $\varphi_1(b_1 y_1 + b_2 y_2 + b_3 y_3) = b_1 + b_2 + b_3$, which must vanish. The classes x_1, x_2 , and x_3 correspond to boundary components of the surface S^1_1 . The only linear combination $\sum_{i=1}^3 a_i x_i$ of these classes which are boundaries of 2-chains in S^1_1 are multiples of $x_1 + x_2 + x_3$ (which is the class of the boundary of S^1_1 itself). That is, $\sum_{i=1}^3 a_i x_i = \lambda(x_1 + x_2 + x_3)$ for some λ ; hence $a_1 = a_2 = a_3$. Similar computations verify the equations in items (4), (6), and (7). \square

Taken as equations over the integers, the equations described in Lemma 1 have no non-zero solutions, and hence $H^1(\Sigma, \mathbb{Z}) \cong \mathbb{Z}^{34}$ (in agreement with the calculation of [5, §4.2]). However, over \mathbb{Z}_2 , we find a 2-dimensional space of solutions generated by:

1. $a_i = 0$ for all $i \in \{0, \dots, 7\}$, $b_0 = b_1 = b_4 = b_5 = 0$, and $b_2 = b_3 = b_6 = b_7 = 1$,
2. $b_i = 0$ for all $i \in \{0, \dots, 7\}$, $a_0 = a_1 = a_2 = a_3 = 0$, and $a_4 = a_5 = a_6 = a_7 = 1$.

That is, we find that $k = 2$, $H_1(\Sigma, \mathbb{Z}_2) \cong \mathbb{Z}_2^{36}$, and hence $H^1(\Sigma, \mathbb{Z}_2) \cong \mathbb{Z}_2^{36}$.

2.2. Proof of Theorem 1 via the squaring map

Consider the following exact sequence of sheaf cohomology groups introduced in [5]:

$$(3) \quad 0 \rightarrow H^1(B, R^1 f_* \mathbb{Z}_2) \rightarrow H^1(B, \sigma_* \mathbb{Z}_2) \rightarrow H^1(B, R^2 f_* \mathbb{Z}_2) \xrightarrow{\beta} H^2(B, R^1 f_* \mathbb{Z}_2).$$

Noting that Schoen's Calabi–Yau X can be described as a complete intersection in a toric variety [13], it is simply-connected by the Lefschetz hyperplane theorem; in this case, $H^1(B, R^1 \check{f}_* \mathbb{Z}_2) \cong H_2(X, \mathbb{Z}_2)$. Moreover, the mirror to Schoen's Calabi–Yau X is itself homeomorphic to X (see [8, §4]). Hence the ranks of $H^1(B, R^1 \check{f}_* \mathbb{Z}_2)$ and $H^1(B, R^2 f_* \mathbb{Z}_2)$ agree and [2, Theorem 1.1] implies that the rank of the connecting homomorphism

$$(4) \quad \beta: H^1(B, R^2 f_* \mathbb{Z}_2) \rightarrow H^2(B, R^1 f_* \mathbb{Z}_2)$$

is equal to the rank of the squaring map $\text{Sq}: D \mapsto D^2$ from $H^2(X, \mathbb{Z}_2)$ to $H^4(X, \mathbb{Z}_2)$ which takes an element $D \in H^2(X, \mathbb{Z}_2)$ to $D \smile D \in H^4(X, \mathbb{Z}_2)$. Note that this map is linear over \mathbb{Z}_2 .

As discussed in the introduction, Schoen's Calabi–Yau is the fibre product of a pair of rational elliptic surfaces E_1 and E_2

$$(5) \quad \begin{array}{ccc} X & \xrightarrow{\pi_2} & E_2 \\ \downarrow \pi_1 & & \downarrow \\ E_1 & \longrightarrow & \mathbb{P}^1. \end{array}$$

In particular, $H^2(X, \mathbb{Z}_2) \cong \mathbb{Z}_2^{19}$ is spanned by vectors in the 10 dimensional subspaces $\pi_1^*(H^2(E_1, \mathbb{Z}_2)) \cong \mathbb{Z}_2^{10}$ and $\pi_2^*(H^2(E_2, \mathbb{Z}_2)) \cong \mathbb{Z}_2^{10}$. These spaces intersect in the class of a fibre of either composition $X \rightarrow \mathbb{P}^1$ in (5).

Since $(\pi_i^*(\alpha))^2 = \pi_i^*(\alpha^2)$, for any class $\alpha \in H^2(E_i, \mathbb{Z}_2)$ and $i \in \{1, 2\}$, the image of Sq is spanned by $\pi_1^*(H^4(E_1, \mathbb{Z}_2)) \cong \mathbb{Z}_2$ and $\pi_2^*(H^4(E_2, \mathbb{Z}_2)) \cong \mathbb{Z}_2$. That is, the rank of Sq is equal to 2 and, by [2, Theorem 1.1], the rank of β is also equal to 2. In particular, we can conclude from (3) that $H^1(B, \sigma_* \mathbb{Z}_2)$ has dimension $19 + (19 - 2) = 36$. Hence, from the Leray spectral sequence for σ , we have that $H^1(B, \sigma_* \mathbb{Z}_2) \cong H^1(\Sigma, \mathbb{Z}_2) \cong \mathbb{Z}_2^{36}$, verifying Theorem 1.

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