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MULTI-VALUED FIXED POINTS AND THE INVERSE PROBLEM FOR CENTRAL CONFIGURATIONS

Abstract. Central configurations play an important role in the dynamics of the n-body problem, and have been studied as relative equilibria, critical points, or projective fixed points of maps on configuration spaces. We describe some results on central configuration as fixed points of quotient maps, and then on the inverse problem in dimension 1, i.e. finding (positive or real) masses which make a given collinear configuration central. We study the inverse problem as a fixed point problem for multi-valued maps.
Keywords: n-body problem; multi-valued map; central configuration; inverse problem.

A. Introduction

Let $n \geq 2$ and $d \geq 1$ be an integer (the number of bodies and the dimension); let $m_1, \ldots, m_n$ be positive parameters, the masses. A configuration of n points in $E = \mathbb{R}^d$ is a n-tuple $\boldsymbol{q} = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n)$ with $\boldsymbol{q}_j \in E$ for all j and $\boldsymbol{q}_i \neq \boldsymbol{q}_j$ whenever $i \neq j$. The *collision set* is defined as $\Delta = \bigcup_{i<j} \{\boldsymbol{q} \in E^n : \boldsymbol{q}_i = \boldsymbol{q}_j\}$, and hence the space of all configurations, the *configuration space*, is equal to

$$\mathbb{F}_n(E) = E^n \smallsetminus \Delta.$$

Configuration spaces have been the subject of different kinds of study until recent decades (see Fadell–Husseini [6] for a topological point of view and some deep and interesting analytical results). The topological/global approach in studying configuration spaces has direct consequences to the understanding of dynamical systems of n point particles interacting under mutual forces (the n-body problem: see for example [22]).

One of the first problems is the problem of finding and classifying *central configurations*: given a potential U defined, for a given homogeneity parameter $\alpha > 0$, as

$$U = \sum_{i<j} \frac{m_i m_j}{\|\boldsymbol{q}_i - \boldsymbol{q}_j\|^\alpha},$$

(for $\alpha = 1$, it is the Newtonian gravitational interaction) a central configurations is $\boldsymbol{q} \in \mathbb{F}_n(E)$ such that there exists a (negative) constant λ such that

$$\lambda m_i \boldsymbol{q}_i = -\alpha \sum_{j \neq i} \frac{m_i m_j}{\|\boldsymbol{q}_i - \boldsymbol{q}_j\|^{\alpha+2}} (\boldsymbol{q}_i - \boldsymbol{q}_j) = \frac{\partial U}{\partial \boldsymbol{q}_i} \tag{1}$$

for $i = 1, \ldots, n$.

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For more details and an introduction to the problem of central configurations, see e.g. [17], [4] [22]), [23], [15], [12], [2], [16], [1].

B. Central configurations as fixed points: self-maps and multi-valued self maps

Let the mass-metric $\langle -,- \rangle_M$ be defined on the tangent vectors of $\mathbb{F}_n(E)$ as

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle_M = \sum_{i=1}^{n} m_i \boldsymbol{v}_i \cdot \boldsymbol{w}_i,$$

where $\boldsymbol{v}_i \cdot \boldsymbol{w}_i$ is the standard euclidean scalar product in $\mathbb{R}^d$. Let $\|-\|_M$ be the norm associated to the scalar product $\langle -,- \rangle_M$, and ∇_M denote the relative gradient, with components

$$(\nabla_M)_j = \frac{1}{m_j} \frac{\partial}{\partial \boldsymbol{q}_j}, \quad j = 1, \ldots, n.$$

Equation (1) can be re-written as

$$-\frac{\lambda}{\alpha} \boldsymbol{q}_i = \sum_{j \neq i} \frac{m_j}{\|\boldsymbol{q}_i - \boldsymbol{q}_j\|^{\alpha+2}} (\boldsymbol{q}_i - \boldsymbol{q}_j) = -\frac{1}{\alpha m_i} \frac{\partial U}{\partial \boldsymbol{q}_i}, \quad \iff \lambda \boldsymbol{q} = \nabla_M U, \tag{2}$$

and since the potential U is homogeneous in $\boldsymbol{q}$ of degree $-\alpha$, $\langle \nabla_M U(\boldsymbol{q}), \boldsymbol{q} \rangle_M = -\alpha U(\boldsymbol{q}) < 0$, from which $\lambda \|\boldsymbol{q}\|_M^2 = -\alpha U(\boldsymbol{q}) \implies \lambda < 0$.

Equation (2) can therefore also be written as

$$\boldsymbol{q} = \frac{-\nabla_M U}{\|\nabla_M U\|_M} = F(\boldsymbol{q}), \tag{3}$$

for the self-map $F \colon \mathbb{F}_n(E) \to E^n$ defined above.

The image of S is the unit sphere S in E^n with respect to the mass-metric norm $\|-\|_M$ (called also the *inertia ellipsoid*) $S = \{\boldsymbol{q} \in E^n : \|\boldsymbol{q}\|_M^2 = 1\}$. Hence, central configurations are fixed points of the map F restricted to the collisionless ellipsoid $S \cap \mathbb{F}_n(E)$: $F \colon S \cap \mathbb{F}_n(E) \to S$.

The group $\mathrm{Iso}(E)$ of euclidean isometries in E acts diagonally on E^n, and the subspace Δ is invariant: hence $\mathrm{Iso}(E)$ acts on the configuration space $\mathbb{F}_n(E)$. Moreover, the potential U is invariant with respect to $\mathrm{Iso}(E)$, hence $\nabla_M U$ is invariant with respect to translations and equivariant with respect to $O(d)$.

This implies that the map F is invariant with respect to (diagonal) translations: $F(\boldsymbol{q}) = F(\boldsymbol{q} + \boldsymbol{v})$, whenever $\boldsymbol{v}$ is a vector of type $\boldsymbol{v}_1 = \boldsymbol{v}_2 = \ldots = \boldsymbol{v}_n$. Moreover, the image $F(\boldsymbol{q})$ is orthogonal to any such a diagonal $\boldsymbol{v}$ (with respect to $\langle -,- \rangle_M$), i.e. it belongs to the subspace

$$X_0 = \{\boldsymbol{v} \in E^n : \sum_{i=1}^{n} m_i \boldsymbol{v}_i = \mathbf{0}\}.$$

Thus any central configuration must belong to X_0, and we can consider the restricted map $F_0 \colon S_0 \cap \mathbb{F}_n(E) \to S_0$, with $S_0 = S \cap X_0$.

Since F is $SO(d)$-equivariant (being $\nabla_M U$ so), for each $g \in SO(d)$ the equality $F(g\boldsymbol{q}) = gF(\boldsymbol{q})$ holds. Therefore, if $\pi\colon X_0 \to X_0/SO(d)$ denotes the projection onto the quotient, the map F induces a map f on the quotient $S_0/SO(d)$ (maybe not defined on collisions)

$$\begin{array}{ccc} S_0 & \xrightarrow{\;F\;} & \overline{S}_0 \subset X_0 \\ \Big\downarrow \pi & & \Big\downarrow \pi \\ S_0/SO(d) & \xrightarrow{\;f\;} & \overline{S}_0/SO(d). \end{array} \tag{4}$$

The results in following theorem were proved in [11], [7], [9]. See also [8]), where the Jacobi change of coordinates with mutual differences is analyzed from a topological point of view, so that simplicial cohomology is used to to simplify the mass-metric projections, as shown later in (5).

THEOREM 1. *The map f is well-defined, compactly fixed, and*

$$\pi \operatorname{Fix}(F) = \operatorname{Fix} f.$$

If $[\boldsymbol{q}] = \pi(\boldsymbol{q})$ is an isolated fixed-point of f, with maximal isotropy stratum of $S_0/SO(d)$, then its fixed point index is $(-1)^\mu$, where μ is the Morse index of U at $\boldsymbol{q}$.

REMARK 1. Consider the following variables, for all $i, j = 1, \dots, n$

$$\boldsymbol{q}_{ij} = \boldsymbol{q}_i - \boldsymbol{q}_j; \quad \boldsymbol{Q}_{ij} = \begin{cases} \frac{\boldsymbol{q}_{ij}}{\|\boldsymbol{q}_{ij}\|^{\alpha+2}} & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases} \tag{5}$$

Then

$$F(\boldsymbol{q})_i = \frac{\sum_{j=1}^n m_j \boldsymbol{Q}_{ij}}{\|\sum_{j=1}^n m_j \boldsymbol{Q}_{ij}\|_M}.$$

For each $\boldsymbol{q}$, its image $F(\boldsymbol{q})$ is a linear combination with positive coefficients

$$\frac{m_j}{\|\sum_{j=1}^n m_j \boldsymbol{Q}_{ij}\|_M}, j = 1, \dots, n$$

of the column vectors of the skew-symmetric matrix with entries $\boldsymbol{Q}_{ij}$.

REMARK 2. It is possible to extend the above results by replacing $SO(d)$ with $O(d)$, i.e. by composition with a double branched cover, as we will show in the following examples.

EXAMPLE 1. If $d = 1$, then $SO(1)$ is trivial, X_0 is a hyperplane of dimension $n-1$ in E^n and S_0 is an ellipsoid of dimension $n-2$: $S_0 \cap \mathbb{F}_n(E)$ has $n!$ connected components, one for each strict ordering of the real coordinates $q_1, \dots, q_n$. The self-map f coincides with F and it has one fixed point in each one of the $n!$ components.

Now consider the full group $O(1) = \pm 1$ of one-dimensional linear isometries of $E = \mathbb{R}$, and the associated group action on E and S_0: it acts as the antipodal map $a\colon S_0 \to S_0$, the corresponding group action, and the quotient $S_0/\pm$. As shown above, f is $\pm$-equivariant, and therefore it induces a map

$$\bar{f}\colon S_0/\pm \subset \mathbb{P}^{n-2}(\mathbb{R}) \to S_0/\pm\,.$$

This map now can be extended to a continuous map $S_0/\pm \to S_0/\pm$, for $n = 3$. For any $n > 3$, there is not such a continuous extension; in this case $S_0 \cap \mathbb{F}_n(E)$ is an open subspace of the sphere of dimension $n-2$ which is projected onto an open subspace of the projective space $\mathbb{P}^{n-2}(\mathbb{R})$ (with $\frac{n!}{2}$ components). Binary collisions can be projectively regularized, but triple collisions or more not. In each of the components there is an isolated central configuration.

EXAMPLE 2. If $d = 2$, then the action of $SO(2)$ on X_0 corresponds to the action of the unit circle in $\mathbb{C}$ on $\mathbb{C}^{n-1}$, and hence $S_0/SO(2) \cong \mathbb{P}^{n-2}(\mathbb{C})$. The collision set Δ projects onto the union of $n!/2$ projective hyperplanes. For $n = 3$, $\mathbb{P}^{n-2}(\mathbb{C})$ is the Riemann sphere $\mathbb{P}^1(\mathbb{C})$, also termed the *shape sphere*. Collisions are codimension 1 hyperplanes, i.e. points. The group $O(2)/SO(2) = G$ acts on $\mathbb{P}^{n-2}(\mathbb{C})$ by complex conjugation on the homogeneous variables, and its fixed subspace is homeomorphic to $\mathbb{P}^1(\mathbb{R})$, of the former example. Furthermore, the corresponding self-map $F\colon \mathbb{P}^{n-2}(\mathbb{C}) \to \mathbb{P}^{n-2}(\mathbb{C})$ is also equivariant with respect to complex conjugation, and hence its restriction to the subspace fixed by G is equal to the self-map of the former example, i.e. the collinear case. The quotient $\mathbb{P}^1(\mathbb{C})/G$ for $n = 3$ is a disc, with $\mathbb{P}^1(\mathbb{R})$ in its boundary.

An interesting feature of the planar case is that in this dimension binary collisions can be projectively regularized already with the $SO(2)$-action (i.e. it is not necessary to go to the $O(2)$-action as with the collinear case). Triple collisions are blown-up to subspaces on S_0 of positive dimension, hence they cannot be regularized.

REMARK 3. It is possible to compute fixed point indices of the central configurations, and to consider ther sum as (local) Lefschetz number, or also the Morse indices of the corresponding critical points of the reduced potential on the inertia ellipsoid. It is possible to use both to compute topological estimates on the number of central configurations, assuming that they are non-degenerate (and Morse non-degenerate on the quotient). The computations follow from results on the homology of configuration spaces or projective configuration spaces. See [20], [21], [19], [13], and [14].

Now, it can be proven in many ways that for each choice of masses there is a non-empty (compact) subset of central configurations in S_0. The *inverse* problem of central configurations is as follows. Following Albouy–Moeckel [3], given a configuration of n bodies, one needs to find positive (or real) masses making it central. In the following we consider only collinear configurations ($d = 1$). For details and further references we refer to [3], [18], [24], [5], [10]. In the next sections we are going to show how it can be written as a fixed-point problem for multi-valued map, and illustrate some consequences.

C. Inverse collinear central configurations and hyperplane-valued maps

As in 1, let Q denote the skew-symmetric matrix with entries

$$Q_{ij} = \begin{cases} \dfrac{q_i - q_j}{\|q_i - q_j\|^{\alpha+2}} & \text{if } i \neq j \\ 0 & \text{otherwise;} \end{cases}$$

then equation 2 can be written as

$$-\frac{\lambda}{\alpha}\boldsymbol{q} = Q\boldsymbol{m},$$

where $\boldsymbol{m}$ is the vector with components $m_1, \dots, m_n$. Hence $\boldsymbol{q} \in X_0 \subset \mathbb{F}_n(\mathbb{R})$ is a central configuration for a choice of (real or positive) masses $\boldsymbol{m}$ if and only if

$$\begin{bmatrix} 0 & Q_{12} & Q_{13} & \dots & Q_{1n} \\ -Q_{12} & 0 & Q_{23} & \dots & Q_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -Q_{1n} & -Q_{2n} & \dots & -Q_{n-1,n} & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{bmatrix} = \lambda' \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}$$

for some $\lambda' \in \mathbb{R}$. If the masses are all positive, then $\lambda' > 0$. The problem now is that the subspace X_0 depends upon the choice of masses, and so it is better to consider a different formulation. As it is shown in [10], the problem is equivalent to the following: re-define $X_0 = \{\boldsymbol{q} \in \mathbb{F}_n(\mathbb{R}) : \sum_{i=1}^n q_i = 0\}$, and consider the orthogonal projection of $\Pi\colon \mathbb{F}_n(\mathbb{R}) \to X_0$ (with respect to the standard euclidean scalar product). With *this* definition of X_0, which does not depend on the choice of masses, it is no longer true that all central configurations belong to X_0. But $\boldsymbol{q}$ is a central configuration (with center of mass not necessarily in 0) for a choice of masses $\boldsymbol{m}$ if and only if there exists $c, \lambda' \in \mathbb{R}$ such that

$$Q\boldsymbol{m} + c\boldsymbol{e} = \lambda'\boldsymbol{q}, \tag{6}$$

where $\boldsymbol{e} \in E^n$ is the vector with constant entries 1. Hence, given $\boldsymbol{q} \in \mathbb{F}_n(\mathbb{R})$, the inverse central configuration problem has solutions (i.e. there exist positive or real masses $\boldsymbol{m}$ such that the configuration is central with respect with this choice of masses in its center of mass) if and only if there exists $\boldsymbol{m}, \lambda'$ such that

$$\Pi Q\boldsymbol{m} = \lambda'\Pi\boldsymbol{q} \tag{7}$$

since $\boldsymbol{e}$ is a generator of the kernel of the orthogonal projection Π. Again, if masses are positive, then so is $\lambda' > 0$.

Given the $n \times n$ skew-symmetric matrix Q

$$Q = \begin{bmatrix} 0 & Q_{12} & Q_{13} & \dots & Q_{1n} \\ -Q_{12} & 0 & Q_{23} & \dots & Q_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -Q_{1n} & -Q_{2n} & \dots & -Q_{n-1,n} & 0 \end{bmatrix}$$

let Q^{b} denote the $(n+1)\times(n+1)$ skew-symmetric bordered matrix

$$Q^{\mathrm{b}} = \begin{bmatrix} 0 & Q_{12} & Q_{13} & \dots & Q_{1n} & 1 \\ -Q_{12} & 0 & Q_{23} & \dots & Q_{2n} & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -Q_{1n} & -Q_{2n} & \dots & -Q_{n-1,n} & 0 & 1 \\ -1 & -1 & \dots & -1 & -1 & 0 \end{bmatrix}.$$

Now, the inverse problem can be set as searching for *real* masses, or for *positive* masses (the mechanical case). The case of real masses can be seen as a fixed point problem for the multi-valued map $\Psi\colon \mathbb{F}_n(\mathbb{R}) \multimap \mathbb{R}^n$ which sends $\boldsymbol{q} \in \mathbb{F}_n(\mathbb{R})$ to the subspace of $\mathbb{R}^n$ generated by the n columns of Q and the vector $\boldsymbol{e}$. Equivalently, it is the multi-valued map $X_0 \multimap X_0$ sending $\boldsymbol{q} \in X_0$ the subspace of X_0 generated by the n columns of ΠQ.

As the matrix Q is skew-symmetric, its determinant is zero when n is odd, and it is equal to the square of its Pfaffian when n is even, that is $\det Q = (\mathrm{Pf}\, Q)^2$. The same holds for Q^{b}, which is skew-symmetric. In [3] Albouy and Moeckel (formulating the problem in terms of exterior products of forms) conjectured that (at least for $\alpha = 1$) the determinant of Q (for n even) and of Q^{b} (for n odd) are always non-zero. For the n for which the conjecture is true, (6) admits real solutions and a kind of uniqueness hold. In [3], Albouy and Moeckel proved it to be true for $n \leq 4$ and $\alpha > 0$ (rigorous), or $\alpha = 1$ and $n \leq 6$ (with a computer-assisted proof). Xie in [24] proved it to be true for $n \leq 6$ and $\alpha = 1$. In [10] it is proved to be true for $n \leq 6$ and $\alpha > 0$, or for $n \leq 8$ and $\alpha = 1$ (with a computer-assisted proof in maple). The maple code could not perform any useful computation for $n = 10$, while a python-sage (sage-math) script could give a computer assisted proof of the conjecture also for $n = 10$, computing all the 488783941 non-zero integer coefficients of a polynomial in 8 variables of degree 64, in about 10 days on a normal desktop computer. The sage-math code is as follows. Comments and the description of the algorithm can be found in [10], where the maple version is listed.

```
import sys
import time
from timeit import default_timer as clock

start_time = time.time()
t1 = clock()

def main_init(n):
    global qq
    polyRing = PolynomialRing(ZZ, 'x', n)
    x = polyRing.gens()
    qq = [[0 for j in range(n + 1)] for i in range(n + 1)]
    for i in range(1, n):
        for j in range(i + 1, n):
            qq[i][j] = (sum([x[k] for k in range(i, j)]))
        qq[i][n] = 1

def partitions(seq):
    # seq is a list of 2k elements
    N = len(seq)
    if N == 0:
        yield []
```

```
    elif N == 2:
        yield [(seq[0], seq[1])]
    else:
        for i in range(N - 1):
            for rest in partitions([seq[x] for x in range(N - 1) if x != i]):
                yield [(seq[i], seq[N - 1]), ] + rest

def P(p, N):
    i, j = p
    if not i < j:
        raise Exception("i<j Error!")
    result = []
    for x in range(N):
        if x < i:
            result.append((x, i))
        if x > i and x != j:
            result.append((i, x))
        if x < j and x != i:
            result.append((x, j))
        if x > j:
            result.append((j, x))
    return result

def square_monomial(part, all_pairs):
    result = 1
    for i, j in [pa for pa in all_pairs if pa not in part]:
        result *= ((qq[i + 1][j + 1]))
    return ((result)**2)

def square_term(pair, all_pairs):
    global qq
    result = 1
    for i, j in [pa for pa in all_pairs
                 if (pair[0] in pa or pair[1] in pa) and (pair != pa)]:
        result *= (qq[i + 1][j + 1])
    return result**2

def all_seq_pairs(list):
    """generator to list all possible ordered index pairs..."""
    N = len(list)
    for j in range(N):
        for k in range(j + 1, N):
            yield (list[j], list[k])

def reduced_list(j, list):
    return [x for x in list if x not in (j, list[-1])]

def recursive_pfaffian(seq):
    N = len(seq)
    sys.stdout.write(".")
    sys.stdout.flush()
    if len(seq) == 2:
        return 1
    this_all_seq_pairs = [x for x in all_seq_pairs(seq)]
    result = 0
    for j in range(N - 1):
        result += (-1)**(j) *\
            square_term((seq[j], seq[-1]), this_all_seq_pairs) *\
            recursive_pfaffian(reduced_list(seq[j], seq))
    return result

def readable_time(t):
    number_of_days = t * 1.0 / (60 * 60 * 24)
    return "({} days) {}".format(int(number_of_days),
                                 time.strftime("%H:%M:%S", time.gmtime(t)))

def main(N):
```

```
    main_init(N)
    all_pas = [(i, j) for i in range(N - 1) for j in range(i + 1, N)]
    pfaffian_polynomial = recursive_pfaffian(range(N))
    coeffs = [v for v in pfaffian_polynomial.coefficients() if v != 0]
    print
    print(min(coeffs), max(coeffs), len(coeffs))

main(10)
elapsed_time = time.time() - start_time
print("\n => ELAPSED TIME: %s\n" % readable_time(elapsed_time))
print("    Totale time: %6.2fs" % (clock() - t1))
```

D. Inverse collinear central configurations and simplex-valued maps

Now, we consider the inverse problem for *positive* masses. The space $X_0 = \{\boldsymbol{q} \in \mathbb{R}^n : \sum_{j=1}^n q_j = 0\}$, orthogonal to the vector $\boldsymbol{e}$ in E^n, is a $(n-1)$-dimensional real vector space. Recall that $\Pi\colon E^n \to X_0$ is the orthogonal projection, and the inverse problem has solutions if and only if (7) has solutions positive solutions $\boldsymbol{m}, \lambda'$; without loss of generality $\boldsymbol{q} \in X_0$, and (7) has positive solutions for $\boldsymbol{q} \in X_0$ if and only if $\boldsymbol{q}$ is a linear combination of the n columns of the matrix ΠQ with positive coefficients (i.e. it belongs to a cone). Now, without loss of generality we can assume that $\boldsymbol{q}$ belongs to the open cone

$$X_0^+ = \{\boldsymbol{q} \in X_0 : q_1 > q_2 > \ldots > q_n\}.$$

Otherwise, a suitable permutation of coordinates sends any $\boldsymbol{q} \in \mathbb{F}_n(E) \cap X_0$ to X_0^+.

The cone X_0^+ is homeomorphic to $X_1^+ \times (0, +\infty)$, where

$$X_1^+ = \{\boldsymbol{q} \in X_0^+ : q_1 - q_n = 1\},$$

and X_1^+ is the interior of a $(n-2)$-dimensional simplex. Its $n-1$ faces are given by the equations

$$q_1 = q_2, \quad \ldots \quad q_{n-1} = q_n;$$

the closure of X_1^+ is described by the inequalities $q_1 \geq q_2 \geq \ldots \geq q_n$ with $\sum_{i=1}^n q_i = 0$ and $q_1 - q_n = 1$.

The following lemma (cf. [10] for details and [8] for more on mutual differences), gives a system of natural coordinates for X_0.

LEMMA 1. *Let $x_i = q_i - q_{i+1}$, for $i = 1, \ldots, n-1$. Then $\boldsymbol{x} = (x_i) \in \mathbb{R}^{n-1}$ are a linear system of coordinates on X_0, and $\boldsymbol{q} \in X_0^+$ if and only if $x_i > 0$ for $i = 1, \ldots, n-1$. Moreover, the simplex X_1^+ is the standard $n-2$ simplex in $\mathbb{R}^{n-1}$, and x_i are (positive) barycentric coordinates:*

$$X_1^+ = \{\boldsymbol{x} \in X_0 \cong \mathbb{R}^{n-1} : \forall i = 1, \ldots, n-1, \quad x_i > 0 \ \& \ \sum_{j=1}^{n-1} x_j = 1\}.$$

Let $X_1 = \{\boldsymbol{x} \in X_0 : \sum_{j=1}^{n-1} x_j = 1\}$. In such $\boldsymbol{x}$-coordinates, for a configuration $\boldsymbol{x} \in X_0$ there is a solution to the inverse problem if and only if there exists a positive vector of masses $\boldsymbol{m} \in \mathbb{R}^n$ such that the equality

$$(8) \qquad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} Q_{11} - Q_{21} & Q_{12} - Q_{22} & \dots & Q_{1n} - Q_{2n} \\ Q_{21} - Q_{31} & Q_{22} - Q_{32} & \dots & Q_{2n} - Q_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{n-1,1} - Q_{n,1} & Q_{n-1,2} - Q_{n,2} & \dots & Q_{n-1,n} - Q_{n,n} \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{bmatrix}$$

holds. Let Y be the matrix with entries $Q_{i,j} - Q_{i+1,j}$.

Consider now the case $\boldsymbol{x} \in X_1 i^+$, hence $q_1 > q_2 > \ldots > q_n$ and therefore the sums of the entries each column of Y are positive

$$(9) \qquad \sum_{i=1}^{n-1} Q_{ij} - Q_{i+1,j} = Q_{1,j} - Q_{n,j} = Q_{1,j} + Q_{j,n} > 0.$$

In other words, if $\boldsymbol{Y}_j$ denotes the j-th column of the matrix Y, by equation (9), $\boldsymbol{Y}_k$ belongs to the half-space of X_0 defined by the inequality $\sum_{i=1}^{n-1} x_i > 0 (\iff q_1 - q_n > 0)$. It is clear that X_1^+ belongs to the same half-space, and for each $j = 1, \ldots, n$ the vector

$$\frac{\boldsymbol{Y}_j}{Q_{1j} + Q_{jn}}$$

has the sum of its components equal to 1, and therefore belongs to X_1.

THEOREM 2. *Let $\psi \colon X_1^+ \multimap X_1$ be the multi-valued map defined as follows: for each $\boldsymbol{x} \in X_1^+$ the image $\psi(\boldsymbol{x}) \subset X_1$ is the convex hull (which is the union of simplices n simplices) of the n points*

$$\frac{\boldsymbol{Y}_j}{Q_{1j} + Q_{jn}} \in X_1$$

for $j = 1, \ldots, n$. Then the map ψ is continuous and there is a solution to the inverse central configuration problem for the configuration $\boldsymbol{x} \in X_1^+$ if and only if $\boldsymbol{x} \in \psi(\boldsymbol{x})$.

REMARK 4. The dimension of the simplices in $\psi(\boldsymbol{x})$ can be $< n-2$, *a priori*; a consequence of the pfaffian Albouy-Moeckel conjecture is that the columns of Q are always in general position, and hence which the dimension of the n simplices of $\psi(\boldsymbol{x})$ are $n-2$ (cf. [10]).

Another interesting results that follows from some estimates on the entries of Y is the following:

THEOREM 3 (Theorem (3.15) of [10]). *Let $\boldsymbol{q} \in X_1^+ \mathbb{F}_n(\mathbb{R})$ be a collinear configuration, such that $x_j > \frac{1}{2}$ for $2 \leq j \leq n-2$. Then the inverse problem does not have solutions for positive masses.*

EXAMPLE 3 ($n = 3$). The matrix Y is the 2×3 matrix (in positive coordinates x_1, x_2)

$$\begin{bmatrix} x_1^{-\alpha-1} & x_1^{-\alpha-1} & 1 - x_2^{-\alpha-1} \\ 1 - x_1^{-\alpha-1} & x_2^{-\alpha-1} & x_2^{-\alpha-1} \end{bmatrix}.$$

To visualize the graph of a multi-valued map, consider that $n-2=1$ and hence $X_1^+ \subset X_1 \cong \mathbb{R}$ is one-dimensional, with homeomorphism the projection $(x_1,x_2) \mapsto x_1$. The three points on X_1 correspond (projecting) to

$$\left(\frac{\boldsymbol{Y}_j}{Q_{1j}+Q_{jn}}\right)_{j=1,2,3} \mapsto \left(x_1^{-\alpha-1}, \frac{x_1^{-\alpha-1}}{x_1^{-\alpha-1}+x_2^{-\alpha-1}}, 1-x_2^{-\alpha-1}\right);$$

now, since $x_1+x_2=1$ and $0<x_1<1$, the inequalities

$$1-x_2^{-\alpha-1} < 0 < \frac{x_1^{-\alpha-1}}{x_1^{-\alpha-1}+x_2^{-\alpha-1}} < 1 < x_1^{-\alpha-1},$$

hold, then for each $\boldsymbol{x} \in X_1^+$, if CH denotes the convex-hull operator,

$$\begin{aligned}\psi(\boldsymbol{x}) &= \mathrm{CH}(\{\boldsymbol{Y}_1,\boldsymbol{Y}_2,\boldsymbol{Y}_3\}) = \mathrm{CH}\left(\left\{x_1^{-\alpha-1}, \frac{x_1^{-\alpha-1}}{x_1^{-\alpha-1}+x_2^{-\alpha-1}}, 1-x_2^{-\alpha-1}\right\}\right)\\ &= \mathrm{CH}\left(\left\{x_1^{-\alpha-1}, 1-x_2^{-\alpha-1}\right\}\right)\\ &\implies X_1^+ \subset \psi(\boldsymbol{x}),\end{aligned}$$

and this means that the inverse problem has solutions for all $\boldsymbol{x}$.

EXAMPLE 4 ($n=4$). In this case the matrix (8) is

$$\begin{bmatrix} Q_{11}-Q_{21} & Q_{12}-Q_{22} & Q_{13}-Q_{23} & Q_{14}-Q_{24}\\ Q_{21}-Q_{31} & Q_{22}-Q_{32} & Q_{23}-Q_{33} & Q_{24}-Q_{34}\\ Q_{31}-Q_{41} & Q_{32}-Q_{42} & Q_{33}-Q_{43} & Q_{34}-Q_{44}\end{bmatrix}$$

Its columns are

$$\begin{aligned}\boldsymbol{Y}_1 &= (x_1^{-\alpha-1}, -x_1^{-\alpha-1}+(x_1+x_2)^{-\alpha-1}, 1-(x_1+x_2)^{-\alpha-1})\\ \boldsymbol{Y}_2 &= (x_1^{-\alpha-1}, x_2^{-\alpha-1}, -x_2^{-\alpha-1}+(x_2+x_3)^{-\alpha-1})\\ \boldsymbol{Y}_3 &= ((x_1+x_2)^{-\alpha-1}-x_2^{-\alpha-1}, x_2^{-\alpha-1}, x_3^{-\alpha-1})\\ \boldsymbol{Y}_4 &= (1-(x_2+x_3)^{-\alpha-1}, (x_2+x_3)^{-\alpha-1}-x_3^{-\alpha-1}, x_3^{-\alpha-1}).\end{aligned}$$

The convex hull $\mathrm{CH}(\boldsymbol{Y}_1,\boldsymbol{Y}_2,\boldsymbol{Y}_3,\boldsymbol{Y}_4)$ is the union of four 2-simplices, and it is possible to compute the regions in X_1 covered by each of the simplices. The 2-simplex X_1^+ and the plane X_1 can be radially projected on the hemisphere in the unit sphere, as in figure

The simplex is represented together with the lines with equations $x_i+x_j=0$ in X_1, and the regions with inverse solutions are shaded (one for each of the four simplices). It is clear that for $x_2>1/2$ there are no solutions, as predicted by Theorem 3.

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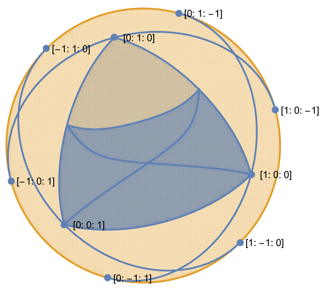

Figure 1: The simplex X_1^+ in X_1, for $n = 4$, with the regions covered by the four simplices and barycentric coordinates

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