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THE LAPLACE RESONANCE: BETWEEN THEORY AND SPACE MISSIONS

Abstract. The Galilean satellites of Jupiter - Io, Europa, and Ganymede - are observed to move in a dynamical configuration known as the "Laplace resonance". It means that the pairs of satellites Io-Europa and Europa-Ganymede are characterized by a 2:1 ratio between their mean longitudes through a relation involving also the arguments of perijoves. Another dynamical configuration is known as the "de Sitter resonance" in which a certain combination of mean longitudes and arguments of perijoves librates, while it rotates in the Laplace resonance. In view of the space mission JUICE, which will be launched in 2022 towards Jupiter and three of its largest moons, we investigate the dynamics of both resonances using a suitable normal form that allows to describe the resonant Laplace and de Sitter configurations. We review a series of papers in collaboration with F. Paita and G. Pucacco ([8], [30], [9]), where we study the evolution of the Laplace librating argument and, among other results, we provide an estimate on its amplitude and frequency.

A. Introduction

The Laplace resonance is one of the most intriguing subjects of Celestial Mechanics and a continuous source of inspiration for theoretical studies. It represents a commensurability between the mean motions of the Galilean satellites of Jupiter: Io, Europa, Ganymede, Callisto. It was discovered by P.-S. Laplace at the end of the XVIII century, when he proved a theorem ([19]) that shows that on average (i.e., after averaging over fast angles) the following relation holds:

$$n_1 - 3n_2 + 2n_3 = 0 \, , \tag{1}$$

where n_1, n_2, n_3 denote, respectively, the mean motions of Io, Europa, Ganymede. The relation (1) is an important result with several consequences. In fact, if we denote by λ_j, $j = 1,2,3$, the mean longitudes and by $\tilde{\omega}_j$ the arguments of perijoves of Io, Europa, Ganymede, then one has

$$\begin{aligned} \lambda_1 - 2\lambda_2 + \tilde{\omega}_1 &= 0 \\ \lambda_1 - 2\lambda_2 + \tilde{\omega}_2 &= \pi \\ \lambda_2 - 2\lambda_3 + \tilde{\omega}_2 &= 0 \, . \end{aligned}$$

As a consequence of the previous relations, the Laplace angle $\Phi_L \equiv \lambda_1 - 3\lambda_2 + 2\lambda_3$ librates around π.

A further consequence is that from the relation

$$\lambda_1 - 3\lambda_2 + 2\lambda_3 = \pi \, , \tag{2}$$

one finds that the three satellites can never be in triple conjunction. A graphical representation of this statement is given in Figure 1.

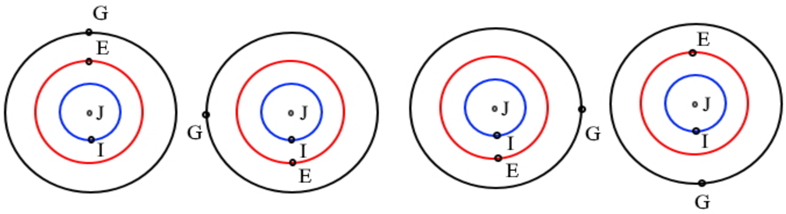

Figure 1: The four panels - from left to right - show the geometry of the Laplace resonance with the satellites denoted by the letters *I*, *E*, *G*, and Jupiter at the center.

In fact, if Europa and Ganymede are in conjunction, then $\lambda_2 = \lambda_3$ (mod. 2π) and hence from (2) it follows that $\lambda_1 - \lambda_2 = \pi$ (mod. 2π), namely Io lies on the same line, but opposite to Europa and Ganymede as in the first panel of Figure 1.

On the other hand, if Io and Europa are in conjunction, then $\lambda_1 = \lambda_2$ (mod. 2π), so that from (2) it is $\lambda_2 - \lambda_3 = \frac{\pi}{2}$ (mod. π), which means that Ganymede is in quadrature with the other two satellites (on the left or on the right), as shown in the second and third panels of Figure 1.

Finally, if Io and Ganymede are in conjunction, then $\lambda_1 = \lambda_3$ (mod. 2π), which implies $\lambda_1 - \lambda_2 = \frac{\pi}{3}$ (mod. $\frac{2}{3}\pi$) due to (2); as a consequence, Europa is in opposition with respect to Io and Ganymede, as shown in the last panel of Figure 1.

The impressive regularity of the mutual configurations of the three inner satellites of Jupiter puzzled astronomers and mathematicians since several years; we will give in Section B a historical review of the study of the Laplace resonance. Although theoretical studies and space missions have disclosed many features of such resonance, several questions are still open and deserve a careful investigation. Among the others: can we evaluate with enough precision the amplitude of libration of the resonance? Do we understand the evolutionary mechanism that led the satellites to their present state? Which are the main physical effects that are responsible for the stability of the resonance? Is the Laplace resonance a common state in which we can possibly find extra-solar planets and satellites?

In the next sections we will try to give partial answers to such questions, although we will see that the study of the Laplace resonance presents mathematical difficulties that need further elaborated studies. However, we will give some results that contribute to the understanding of the resonance, with a special attention to the space mission JUICE (JUpiter ICy moon Explorer), that the European Space Agency (ESA) scheduled for launch in September 2022 with an expected duration of 11 years.

The content of this work, which summarizes the results found in the papers [8], [30] and [9], is the following. In Section B we give a historical overview of the main results about the Laplace resonance; in Section C we give a Cartesian and Hamiltonian formulation for its treatment; a resonance related to the Laplace one, whose discovery is due to W. de Sitter, is briefly presented in Section D; a discussion of the applicability of KAM theory to Laplace and de Sitter resonances is shown in Section E; the Laplace

	Io	*Europa*	*Ganymede*	*Callisto*
$mass\ (kg)$	$8.932 \cdot 10^{22}$	$4.800 \cdot 10^{22}$	$1.482 \cdot 10^{23}$	$1.076 \cdot 10^{23}$
$radius\ (km)$	1821.5	1560.8	2631.2	2410.3
$semimajor\ axis\ (km)$	$4.218 \cdot 10^{5}$	$6.711 \cdot 10^{5}$	$1.070 \cdot 10^{6}$	$1.883 \cdot 10^{6}$
$eccentricity$	0.004	0.009	0.001	0.007
$inclination\ (degrees)$	0.04	0.47	0.18	0.19

Table 4.1: Physical and orbital elements of the Galilean satellites (https://nssdc.gsfc.nasa.gov/planetary/factsheet/joviansatfact.html)

resonance in a dissipative context - with the dissipation due to tidal interaction - is discussed in Section F.

B. A historical remark: from Galileo to JUICE

In 1610 Galileo Galilei announces in his work titled "Sidereus Nuncius" the discovery of four satellites around Jupiter. They were named Io, Europa, Ganymede and Callisto; their importance resides on the fact that they were the first objects observed to orbit around another planet and forming a Solar system in miniature. Later on, Johannes Kepler made his own observations and confirmed the discovery in 1611.

The main physical and orbital parameters of the satellites are reported in Table 4.1, which shows that the four satellites are on almost circular and planar orbits.

It is remarkable that in 1676 the Danish astronomer O. Rømer gave an estimate of the speed of light by timing the eclipses of the satellite Io. His estimate of the light speed was of about 200 000 km/sec, while the true value amounts to about 299 792 km/sec. The eclipses of the Galilean satellites were extensively used in the XVI and XVII centuries in a different context: the determination of the longitude. Indeed, the method for computing the longitude is (again) due to Galileo; the occurrence and times of the eclipses can be computed theoretically and compared with observations at a local place, so to obtain the local time and hence to compute the longitude.

A big progress in the theoretical investigation of the dynamics of the satellites was obtained through a monumental work by J.-L. Lagrange on the theory of Jupiter's satellites, which was awarded a prize in 1766 by the Paris Academy of Sciences.

At the end of the XVIII century, P.S. Laplace discovered the resonance which bears his name, showing that the first three satellites are never in conjunction, namely on collinear positions and at the same side. The *1:2:4 Laplace resonance* is implied by the fact that the periods of Io, Europa, Ganymede, say T_1, T_2, T_3, satisfy the relations

$$T_2 = 2T_1 , \qquad T_3 = 2T_2 .$$

Also F. Tisserand contributed to the study of the Galilean satellites through an outstanding work published in 1855, titled “Traité de Mécanique Céleste” ([34]), where in the fourth volume he investigates the theory of the satellites of Jupiter and Saturn.

Another step forward was made by R.A. Sampson ([32]) who published in 1921 an analytical theory of the Galilean satellites to predict their ephemerides. Sampson theory was later revisited by J.H. Lieske ([21], [22]), also by using an algebraic manipulator.

Meanwhile, a remarkable discovery was made by W. de Sitter in 1925, who proved the existence of a family of resonant, linearly stable periodic orbits, different from the Laplace resonance. The dynamical configuration and its link with the Laplace resonance will be presented in detail in Section D. It is worth mentioning the PhD thesis at Yale University in 1966 by B. Marsden ([24]), who used von Zeipel method to average the short-period terms of the Hamiltonian describing the Laplace resonance to compute long-period effects.

A remarkable contribution was given by S. Ferraz-Mello ([11]) who gave a comprehensive study of the classical results, with special reference to [34], and provided a complete first-order solution by solving integro-differential equations for the oscillations in longitude and latitude.

Estimates of the amplitude of libration of the resonance were computed by J. Henrard in [15], using an appropriate set of variables and a Hamiltonian normal form, and by Lieske in [21] (see also [22]), using the ephemerides of the Galilean satellites; this latter computation gives a result of 0.066^o with a period of 2 071 days. The libration amplitude was confirmed by Musotto et al in [29], using long-term numerical simulations. We also mention [18], where accurate ephemerides over centuries have been computed, based on a numerical integration of quite elaborated equations of motion.

The studies we have described up to now mainly rely on a setting where only conservative forces have been considered. However, dissipative forces may play a role in the evolutionary history; in this context we mention the works by Yoder ([36]), Yoder and Peale ([37]), Malhotra ([23]), which present an analytical theory for the tidal origin of the Laplace resonance (see also Section F). More recently, the stability of the de Sitter resonance has been investigated by Broer, Hanssmann, Zhao ([4], [3]), using Kolmogorov-Arnold-Moser (KAM) theory. This study is non trivial and requires a deep analytical investigation; we will mention some results in Section E.

We conclude this section by mentioning that the Galilean satellites Europa, Ganymede, Callisto will be the target of the ESA space mission JUICE with nominal launch date in September 2022. The goal of the mission will be to investigate the conditions that led to the emergence of habitable worlds around Jupiter. The Laplace resonance may play a relevant role, since it can contribute to redistribute the energy between the satellites and Jupiter, thus provoking a tidal dissipation on Io that triggers a volcanic activity on the satellite ([36], see also [2], [23], [37]).

C. Cartesian and Hamiltonian formulation

In this Section we introduce the equations of motion using Cartesian formalism (Section C.1) and a Hamiltonian approach (Section C.2). The model will include the attraction of Jupiter with the effects due to its non-spherical shape, the mutual gravitational interactions of the satellites, the influences of Callisto and the Sun.

C.1. Cartesian equations of motion

We consider the three satellites of Jupiter, Io, Europa, Ganymede, labeled as S_1, S_2, S_3, with masses m_1, m_2, m_3; the satellites move around Jupiter whose mass is denoted by m_0. We assume that the satellites are point-masses, while Jupiter is considered to have a non-spherical shape and, hence, it generates an oblateness potential on the i-th satellite that we approximate within its spherical harmonic coefficient J_2, i.e.

$$U_{i0} = \frac{R^2}{r_i^3} J_2\, P_2(\sin\phi_i)\ ,$$

where $r_i = |\underline{r}_i|$, R denotes Jupiter's mean radius, P_2 is the second order Legendre polynomial and (r_i, ϕ_i, λ_i), $i = 1,2,3$, represent the equatorial Jovicentric spherical coordinates of the satellites; in particular, ϕ_i and λ_i denote the latitude and longitude of each satellite, whose position is given by the vector $\underline{r}_i$, $i = 1,2,3$. With this notation and denoting by $\mathcal{G}$ the gravitational constant, the Cartesian equations of motion read as

$$\begin{aligned} \ddot{\underline{r}}_i &= -\frac{\mathcal{G}(m_0+m_i)}{r_i^3}\underline{r}_i + \sum_{j=1,\, j\neq i}^{4} \mathcal{G} m_j \left(\frac{\underline{r}_j - \underline{r}_i}{r_{ij}^3} - \frac{\underline{r}_j}{r_j^3} \right) \\ &+ \mathcal{G}(m_0+m_i)\nabla_i U_{i0} + \sum_{j=1,\, j\neq i}^{4} \mathcal{G} m_j \nabla_j U_{j0}\ , \end{aligned} \tag{3}$$

where $r_{ij} = |\underline{r}_j - \underline{r}_i|$. Such equations are supplemented by the energy preservation E =constant along the motion (see [30]), where

$$\begin{aligned} E &= \sum_{i=1}^{4} \frac{m_i |\dot{\underline{r}}_i|^2}{2} - \frac{1}{2M} \left(\sum_{i=1}^{4} m_i \dot{\underline{r}}_i \right)^2 \\ &+ \sum_{i=1}^{4} \mathcal{G} m_i m_0 \left(\frac{1}{r_i} + U_{i0} \right) + \sum_{i=1}^{3} \sum_{j=i+1}^{4} \frac{\mathcal{G} m_j m_i}{r_{ij}}\ , \end{aligned} \tag{4}$$

where M denotes the total mass of the system.

A remarkable result obtained in [30] is the comparison between the numerical integration of equations (3) under the energy preservation (4) with the results obtained from the ephemerides taken at the epoch J2000 from the NASA Spice toolkit. Both the ephemerides and the integration of the Cartesian equations of motion give a libration in longitude of about 0.8^o. The power spectrum of the Laplace angle Φ_L versus time shows several frequencies, beside that at about 2 000 days. To remove such frequencies, a low-pass filter (e.g., at 1 000 days) can be implemented, yielding a maximum amplitude for the libration of the Laplace angle of less than 0.02^o (compare with [30], [29], [21]). A numerical integration of the Cartesian equations of motion without taking into account the J_2 effect shows a libration of the Laplace angle of about 65^o (compare with [30]), thus highlighting the importance of the oblateness of Jupiter in shaping the Laplace resonance.

C.2. Hamiltonian formulation

The description of the equations of motion using a Hamiltonian approach can be conveniently used to explore the Laplace resonance as well as the de Sitter resonance, and their generalizations. The first task is to introduce the so-called *Jacobi coordinates* ([23]), which are defined as follows. Let $\underline{\tilde{r}}_i$, $i = 1,2,3$, be the position vectors of the satellites in an inertial reference frame with fixed origin O and let $\underline{\tilde{r}}_{ij} = \underline{\tilde{r}}_j - \underline{\tilde{r}}_i$ be the mutual distances. Let $\kappa_i = m_i/M_i$ with $M_1 = m_0 + m_1$, $M_2 = m_0 + m_1 + m_2$, $M_3 = m_0 + m_1 + m_2 + m_3$; then, the Jacobi coordinates $\underline{\rho}_i$, $i = 1,2,3$, are defined by

$$\begin{aligned} \underline{\rho}_1 &= \underline{\tilde{r}}_1 , \\ \underline{\rho}_2 &= \underline{\tilde{r}}_2 - \kappa_1 \underline{\rho}_1 , \\ \underline{\rho}_3 &= \underline{\tilde{r}}_3 - \kappa_1 \underline{\rho}_1 - \kappa_2 \underline{\rho}_2 . \end{aligned}$$

Introducing the quantities

$$\mu_1 = \frac{m_0 m_1}{M_1} , \qquad \mu_2 = \frac{M_1 m_2}{M_2} , \qquad \mu_3 = \frac{M_2 m_3}{M_3} ,$$

the Keplerian part of the Hamiltonian, providing the interaction of each satellite with Jupiter, is given by

$$\mathcal{H}_{Kep} = -\frac{\mathcal{G} M_1 \mu_1}{2a_1} - \frac{\mathcal{G} M_2 \mu_2}{2a_2} - \frac{\mathcal{G} M_3 \mu_3}{2a_3} ,$$

where a_i denotes the semimajor axis of each satellite. The Hamiltonian function providing the mutual satellite's interactions is given by

$$\begin{aligned} \mathcal{H}_{int} &= -\frac{\mathcal{G} m_1 m_2}{|\underline{\rho}_2 - (1-\kappa_1)\underline{\rho}_1|} - \frac{\mathcal{G} m_2 m_3}{|\underline{\rho}_3 - (1-\kappa_2)\underline{\rho}_2|} - \frac{\mathcal{G} m_1 m_3}{|\underline{\rho}_3 - (1-\kappa_1)\underline{\rho}_1 + \kappa_2 \underline{\rho}_2|} \\ &- \mathcal{G} M_1 m_2 \Big(\frac{m_0/M_1}{|\underline{\rho}_2 + \kappa_1 \underline{\rho}_1|} - \frac{1}{\rho_2} \Big) - \mathcal{G} M_2 m_3 \Big(\frac{m_0/M_2}{|\underline{\rho}_3 + \kappa_1 \underline{\rho}_1 + \kappa_2 \underline{\rho}_2|} - \frac{1}{\rho_3} \Big) , \end{aligned}$$

where $\rho_i = |\underline{\rho}_i|$. To study the resonance, it is important to expand $\mathcal{H}_{int}$ in terms of the orbital elements a_i, e_i (e_i denotes the eccentricity) and in terms of the angles λ_i, $\tilde{\omega}_i$, $i = 1,2,3$.

Retaining only the secular and resonant terms, one obtains the following expansions of the mutual interactions $\mathcal{H}^{1,2}$, $\mathcal{H}^{2,3}$, $\mathcal{H}^{1,3}$, truncated at second order in the eccentricities:

$$\begin{aligned}
\mathcal{H}^{1,2} &= -\frac{\mathcal{G}m_1m_2}{a_2}\big[\frac{1}{2}b^{(0)}_{1/2}(\alpha_{1,2})+f_1^{1,2}e_1\cos(2\lambda_2-\lambda_1-\tilde\omega_1)+f_2^{1,2}e_2\cos(2\lambda_2-\lambda_1-\tilde\omega_2)\\
&+ f_3^{1,2}(e_1^2+e_2^2)+f_4^{1,2}e_1e_2\cos(\tilde\omega_1-\tilde\omega_2)+f_5^{1,2}e_1e_2\cos(4\lambda_2-2\lambda_1-\tilde\omega_1-\tilde\omega_2)\\
&+ f_6^{1,2}e_1^2\cos(4\lambda_2-2\lambda_1-2\tilde\omega_1)+f_7^{1,2}\cos(4\lambda_2-2\lambda_1-2\tilde\omega_2)\big]\ ,\\
\mathcal{H}^{2,3} &= -\frac{\mathcal{G}m_2m_3}{a_3}\big[\frac{1}{2}b^{(0)}_{1/2}(\alpha_{2,3})+f_1^{2,3}e_2\cos(2\lambda_3-\lambda_2-\tilde\omega_2)+f_2^{2,3}e_3\cos(2\lambda_3-\lambda_2-\tilde\omega_3)\\
&+ f_3^{2,3}(e_2^2+e_3^2)+f_4^{2,3}e_2e_3\cos(\tilde\omega_2-\tilde\omega_3)+f_5^{2,3}e_2e_3\cos(4\lambda_3-2\lambda_2-\tilde\omega_2-\tilde\omega_3)\\
&+ f_6^{2,3}e_2^2\cos(4\lambda_3-2\lambda_2-2\tilde\omega_2)+f_7^{2,3}\cos(4\lambda_3-2\lambda_2-2\tilde\omega_3)\big]\ ,\\
\mathcal{H}^{1,3} &= -\frac{\mathcal{G}m_1m_3}{a_3}\big[\frac{1}{2}b^{(0)}_{1/2}(\alpha_{1,3})+f_3^{1,3}(e_1^2+e_3^2)+f_4^{1,3}e_1e_3\cos(\tilde\omega_1-\tilde\omega_3)\big]\ ,
\end{aligned}$$

where the functions $f_k^{i,j}$ are linear combinations of the Laplace coefficients $b_s^{(n)}$ and their derivatives (see [28]).

The Hamiltonian contributions due to the oblateness of Jupiter and to the gravitational effect of the Sun and Callisto can be reduced to the study of the secular Hamiltonian, averaged over the fast angles. Precisely, the corresponding Hamiltonians, that we denote by $\mathcal{H}_{obl}$, $\mathcal{H}_{Sun}$, $\mathcal{H}_{Cal}$, are given by the following expressions:

$$\mathcal{H}_{obl}=-\sum_{i=1}^{3}\frac{\mathcal{G}M_i\mu_i}{2a_i}\,J_2\Big(\frac{R}{a_i}\Big)^2\,\Big(1+\frac{3}{2}e_i^2\Big)\ ,$$

while $\mathcal{H}_{Sun}$ and $\mathcal{H}_{Cal}$ are given by

$$\mathcal{H}_\tau=-\sum_{i=1}^{3}\frac{\mathcal{G}m_im_\tau}{a_\tau}\,\Big[\frac{1}{2}b^{(0)}_{\frac{1}{2}}\Big(\frac{a_i}{a_\tau}\Big)-1+\frac{1}{8}\frac{a_i}{a_\tau}b^{(1)}_{\frac{3}{2}}\Big(\frac{a_i}{a_\tau}\Big)\,(e_i^2+e_\tau^2)\Big]\ ,$$

where τ denotes, respectively, the Sun and Callisto.

We sum up the different terms to obtain the Hamiltonian in the planar case:

$$\mathcal{H}=\mathcal{H}_{Kep}+\mathcal{H}_{int}+\mathcal{H}_{obl}+\mathcal{H}_{Sun}+\mathcal{H}_{Cal}\ , \tag{5}$$

in which, as before, we retain only the secular and resonant terms.

The derivation of the Hamiltonian function can be generalized to other resonances of different type in which two pairs of satellites, say S_1-S_2 and S_2-S_3, are in a $p:q$ and $m:n$ resonance for $p,q,m,n\in\mathbb{Z}_+$, whereas the Laplace resonance corresponds to 2:1 and 2:1. In [9] the resonances 3:1 and 3:1, 2:1 and 3:2, 2:1 and 3:1 have been considered; the computation of chaos indicators (most notably the Fast Lyapunov exponents) shows a marked regularity of the 2:1 and 2:1 resonance, despite a chaotic behavior of the other resonances on a relatively short time scale.

We also point out that in [9] a detailed study of the dependence of the Laplace resonance on the main parameters of the satellites (masses, eccentricities, etc.) has been performed.

D. From Laplace to the de Sitter resonance

As we mentioned in Section B, a different periodic configuration, in which three satellites orbiting a central planet can be found, was discovered by de Sitter ([10]), who showed the existence of a family of linearly stable periodic orbits in a 4:2:1 mean motion resonance. This solution was deeply studied in [4], [3] using KAM theory (see Section E). The difference between Laplace and de Sitter resonances has been conveniently explained in [8], using the following approach (we refer to [8] for full details).

From the 6 degrees of freedom (d.o.f.) Hamiltonian (5), implementing a transformation to introduce resonant angles, one obtains a 4 d.o.f. Hamiltonian, since two angles are cyclic. Then, by implementing a resonant normal form, one obtains a 1 d.o.f. Hamiltonian, whose structure clarifies the difference between Laplace and de Sitter resonances. The results will show that the Laplace periodic orbit is characterized by a locking of $\tilde{\omega}_1$, $\tilde{\omega}_2$, while the angle $2\lambda_3-\lambda_2-\tilde{\omega}_3$ rotates; on the contrary, in the de Sitter resonance also the angle $2\lambda_3-\lambda_2-\tilde{\omega}_3$ librates.

According to [8], we retain only the contributions due to the Keplerian part, the mutual interactions and the oblateness, obtaining a Hamiltonian function that we express in modified Delaunay variables defined as

$$L_i=\mu_i\sqrt{\mathcal{G}M_i a_i}\ , \qquad G_i\equiv L_i(1-\sqrt{1-e_i^2})$$

and conjugated angles λ_i, $\tilde{\omega}_i$, $i=1,2,3$. Then, we obtain a Hamiltonian of the form

$$\mathcal{H}_6=\mathcal{H}_6(L_1,L_2,L_3,G_1,G_2,G_3,\lambda_1,\lambda_2,\lambda_3,\tilde{\omega}_1,\tilde{\omega}_2,\tilde{\omega}_3)\ ,$$

that we further transform using the following resonant coordinates according to [15], [23]:

$$\begin{aligned}
q_1&=2\lambda_2-\lambda_1-\tilde{\omega}_1, & P_1&=G_1,\\
q_2&=2\lambda_2-\lambda_1-\tilde{\omega}_2, & P_2&=G_2,\\
q_3&=2\lambda_3-\lambda_2-\tilde{\omega}_3, & P_3&=G_3,\\
q_4&=3\lambda_2-2\lambda_3-\lambda_1, & P_4&=\tfrac{1}{3}\left(L_2-2(G_1+G_2)+G_3\right),\\
q_5&=\lambda_1-\lambda_3, & P_5&=\tfrac{1}{3}\left(3L_1+L_2+G_1+G_2+G_3\right),\\
q_6&=\lambda_3, & P_6&=L_1+L_2+L_3-G_1-G_2-G_3\ .
\end{aligned}$$

For the resulting Hamiltonian, it turns out that at first order in the eccentricities, the variables q_5, q_6 are cyclic, so that the associated actions P_5, P_6 are constants, yielding a 4 d.o.f. Hamiltonian of the form:

$$\mathcal{H}_4=\mathcal{H}_4(P_1,P_2,P_3,P_4,q_1,q_2,q_3,q_4)\ ,$$

depending parametrically on P_5, P_6. By solving the equations

$$\frac{\partial\mathcal{H}}{\partial P_k}(P_1,P_2,P_3,P_4,q_1,q_2,q_3,q_4)=0,\quad \frac{\partial\mathcal{H}}{\partial q_k}(P_1,P_2,P_3,P_4,q_1,q_2,q_3,q_4)=0,\qquad k=1,...,4\ ,$$

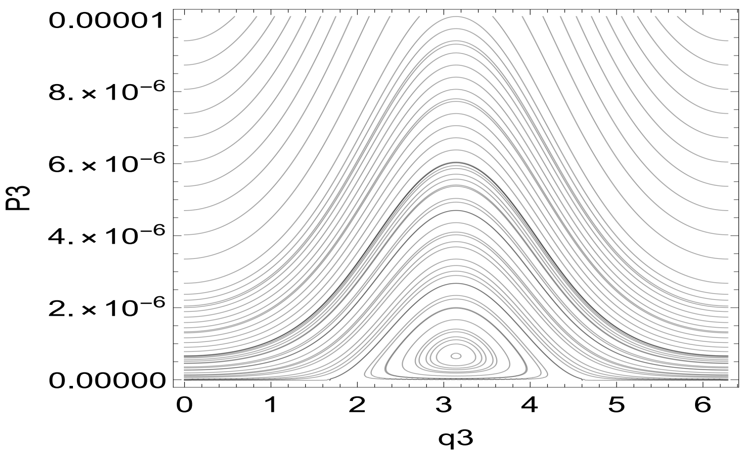

Figure 2: The de Sitter solution corresponds to the equilibrium position in the plane (q_3, P_3), while the Laplace solution corresponds to a rotational curve outside the librational region.

one obtains 16 equilibrium positions. We refer to the de Sitter solution as the only *stable* equilibrium position, which corresponds to

$$(q_1, q_2, q_3, q_4) = (0, \pi, \pi, \pi) \ ,$$

while all other combinations are unstable. We notice that in the de Sitter equilibrium, the angle q_3 librates around π.

Next, we expand the Hamiltonian truncated at first order in the eccentricity around the de Sitter equilibrium and we retain only terms up to second order in the momenta. Applying a Lie-series transformation to eliminate the variables q_1, q_2, q_4, we obtain the 1-dimensional Hamiltonian (compare with [5], [31] for a similar reduction procedure in a different context):

$$\begin{aligned} \mathcal{H}_1(P_3, q_3) &= -0.00368693 P_3 - 1.66667 P_3^2 \\ &- (6.04641 \times 10^{-6} \sqrt{P_3} - 1.14398 \cdot 10^{-5} P_3^{3/2}) \cos q_3 \ . \end{aligned}$$

The level curves of the Hamiltonian $\mathcal{H}_1$ are shown in Figure 2, which gives a straightforward explanation of the difference between the Laplace and the de Sitter resonance: the de Sitter solution corresponds to the equilibrium position with $q_3 = \pi$, while the Laplace solution corresponds to one of the rotational curves outside the librational region surrounding the de Sitter resonance, compare with Figure 2.

E. Applications of KAM theory

A powerful tool to study the existence of invariant surfaces in dynamical systems is given by KAM theory ([17], [1], [27]), which gives results on quasi-periodic motions in non-integrable systems and, in particular, on the persistence of invariant surfaces in nearly-integrable Hamiltonian systems.

The theory gives the existence of rotational (or primary) and librational (or secondary) tori (compare, e.g., with [6], [7]; see also [35] for applications to exoplanets). In nearly integrable systems, having fixed a frequency for the unperturbed system, KAM theory ensures the persistence of an invariant surface for the perturbed system on which the motion is quasi-periodic with the same frequency of the unperturbed system. The KAM theorem can be applied provided the following conditions are fulfilled:

(i) the frequency must satisfy a Diophantine condition (which is necessary to deal with the so-called *small divisor* problem);

(ii) a non-degeneracy condition (on coordinates and parameters) must be satisfied (which is necessary to solve a suitable cohomological equation that gives a sequence of approximate solutions, which converge to the true quasi-periodic solution).

As for the second assumption, there are a number of non-degeneracy conditions, due to different authors. In particular, we speak of:

- Kolmogorov's non-degeneracy, when the determinant of the Hessian matrix of the unperturbed Hamiltonian is non-zero, thus ensuring that the mapping from the actions to the frequencies is a local diffeomorphism;

- Arnold's isoenergetic non-degeneracy condition, which involves both first and second derivatives of the unperturbed Hamiltonian and which ensures that the frequency ratio is fixed and that KAM tori exist on a given energy level;

- Rüssmann non-degeneracy condition, which ensures that the span to a given order of the frequency map coincides with $\mathbb{R}^n$ (where n is the number of degrees of freedom).

It is worth mentioning that in [4], [3], a multi-scale parametrized version of KAM theory has been implemented to show the existence of librating quasi-periodic KAM orbits around the linearly stable family of periodic orbits. The results can be presented as follows (we refer to [4], [3] for full details).

THEOREM 1. *([4], [3]) In the 1+3-body problem [Jupiter-Io-Europa-Ganymede], for almost all masses among which one sufficiently dominates the others, there exists a set of positive measure of quasi-periodic orbits librating around the family of linearly stable de Sitter periodic orbits.*

Another result of [4], [3] considers the 1+4 body problem, in which Callisto is assumed to move on an almost circular orbit. This implies the appearance of an extra period which, together with the de Sitter periodic orbit, gives rise to normally elliptic, isotropic invariant 2-tori and maximal tori.

THEOREM 2. *([4], [3]) In the 1+4-body problem [Jupiter-Io-Europa-Ganymede-Callisto], there is a large measure Cantor set of sufficiently small eccentricities and masses, such that:*

- the system admits normally elliptic invariant 2-tori superposing the family of the de Sitter periodic orbits with Callisto on an almost circular orbit with a frequency incommensurable with the frequencies of the inner three satellites;

- the system admits Lagrangian invariant tori (obtained by excitation of elliptic normal modes) with Io, Europa, Ganymede close to a 1:2:4 resonance, and Callisto on an almost circular orbit with a frequency incommensurable with the frequencies of the inner three satellites.

In the above results, the applicability condition of KAM theorem is that the map from the parameters to the normal frequencies is a local diffeomorphism. In [4], the authors prove that non-degeneracy is obtained by solving two quadratic equations in the parameters given by the rescaled masses m_1, m_2 and the eccentricity e_2.

We conclude by saying that Kolmogorov and Arnold non-degeneracy conditions are not satisfied when using the actual astronomical values of the parameters and when considering the Hamiltonian function (5).

The (Kolmogorov or Arnold) non-degeneracy condition is satisfied in a different regime of values, for example taking $m_1 = m_2 = 0.01$ (in units of the mass of Jupiter) and $e = 0.1$.

We believe that an appropriate study of the application of KAM theorem to the Laplace resonance needs dedicated mathematical results for Hamiltonian systems with high-order degeneracy, for which we refer to [14], [25], [26].

F. Tidal effects

In the previous Sections we have investigated the Laplace resonance within a conservative framework. However, tidal dissipation within the satellites might be relevant for the evolutionary history of the resonance. As marked in [36], tidal heating in the inner satellite Io is the most likely source of energy for the volcanic activity, which has been observed already by the Voyager 1 space mission.

Tidal dissipation affects the evolution of the orbital elements, namely semimajor axis and eccentricity. Assuming inelastic planetary and satellite tides, the rates of variation of the orbital elements are given by the expansions (see [16], [12], [13], [20]):

$$\begin{aligned} \dot{a} &= 3\frac{k_J}{Q_J}\frac{m_S}{M}(\frac{R}{a})^5 na\,[1-(7D-\frac{51}{4})e^2] \\ \dot{e} &= -\frac{3}{2}\frac{k_J}{Q_J}\frac{m_S}{M}(\frac{R}{a})^5 n\,(7D-\frac{19}{4})e \end{aligned} \tag{6}$$

with

$$D = \frac{Q_J}{Q_S}\frac{k_S}{k_J}(\frac{R_S}{R})^5(\frac{M}{m_S})^2\ ,$$

where m_S is the mass of the satellite, n its mean motion, k_S and k_J are the second order Love numbers of the satellite and Jupiter, Q_S and Q_J are the tidal dissipation functions of the satellite and Jupiter. Equations (6) complement the equations which can be obtained from the Hamiltonian (5). It was shown in [37] that the tidal dissipation in Io provokes a damping of the libration amplitude. A formation scenario, due to tidal effects, of the Laplace resonance has been studied in [23]. Finally, tidal evolution of Laplace-like resonances has been investigated in [33], which includes seminal results on the thermal history of Ganymede. In a future work, we plan to go into a deeper detail of the dissipative effects on the Laplace resonance.

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