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LAMBERT'S THEOREM ON THE SPHERE

Abstract. In this note, we survey a generalization of the classical Lambert's Theorem [3] to the spherical and pseudospherical Kepler problems together with some background informations.

A. Introduction.

In [12], P. Serret asked and solved the inverse question of finding a central force potential on the sphere whose orbits are spherical conic sections: Up to a multiplicative mass factor, the potential should take the form of the cotangent of the central angle between the moving particle and the attracting center. The natural mechanical system with this Serret potential is thus considered as a generalization of the classical Kepler problem to the sphere. A similar generalization of the Kepler problem to the pseudosphere has been obtained by Killing [7].

The classical and spherical/pseudospherical Kepler problems share many similarities:

- The orbits are conic sections, with one focus at the attracting center (P. Serret [12], Killing [7]);
- The energy depends only on the (geodesic) semi major axis (Killing [7], Velpry [14]);
- The assertion of Bertrand theorem (c.f. [1, Lecture 3]) holds (Liebmann [11], Higgs [5], Ikeda and Katayama[6], Kozlov and Harin [8]).

With the help of Appell's central projection, the Kepler problem on the sphere and the pseudosphere can be effectively deduced from the classical Kepler problem in Euclidean space ([1, Lecture 5]). All these phenomena can be derived from the fact that unparametrized orbits are related directly by the central projection, as part of Appell's theory.

Concerning time parametrization of Kepler problem in Euclidean space, Lambert's theorem states the following: For simplicity we consider the planar problem. For two points A_1 and A_2 in a plane, the passing time from A_1 to A_2 along an arc of the Keplerian orbit with attracting center O and semi major axis a is a multivalued function of the energy of the orbit and the three mutual distances $c = |A_1A_2|, r_1 = |A_1O|, r_2 = |OA_2|$. Indeed, when the energy is negative, the energy h determines the semi major axis length, which measures the size of the elliptic orbit, while r_1, r_2, c subsequently determine (in a multi-value way) the eccentricity of the orbit. Lambert's theorem [10] asserts that this dependence on four variables can be effectively reduced to three: the energy, or equivalently the semi major axis a, the mutual distance c and the sum $r_1 + r_2$. The two functions c and $r_1 + r_2$ are well-defined for all triples of positive real numbers (r_1, r_2, c) and are independent of the semi major axis a. We note that this fact is also true for parabolic and hyperbolic orbits, corresponding respectively to cases with zero and positive Kepler energies.

THEOREM 1. *The passing time along a Keplerian arc is a multivalued function of $c, r_1 + r_2, h$.*

In the spherical/pseudospherical problem, the passing time is analogously a multivalued function of the spherical energy and the three geodesic distances on the sphere. Again, the spherical/pseudospherical energy determines the geodesic semi major axis. An analogue of Lambert's theorem on the sphere would thus be a similar reduction of number of variables on which the passing time depends.

Question 1. In the spherical/pseudospherical Kepler problem, do there exist two functions f, g of the three geodesic distances among the start point, the end point and the attracting center, so that the passing time from the start point to the end point along a Keplerian arc of the orbit can be expressed as a multivalued function only of these two functions f, g and the geodesic semi major axis (or equivalently the energy) of the orbit?

The answer to this question was thought to be negative, due to the fact that Appell's projection changes time. It was thus quite surprising when the author received some argumentss from my collaborator A. Albouy showing in particular that the answer to this question should be positive. This leads in the end to the following theorem of [3]:

THEOREM 2. *Lambert's theorem generalizes to the spherical and pseudospherical Kepler Problems, with mutual distances replaced by the geodesic mutual distances on the sphere/pseudosphere.*

Indeed there is also a Lambert's theorem for the Hooke problem of isotropic harmonic oscillators as well as their spherical/pseudospherical analogues. This is discussed in [3] and we opt not to discuss these systems in this note.

In Section B, we shall recall the definition and some properties of the spherical Kepler problem from [1, Lecture 5]. We then analyse the passing time along Keplerian arcs and Lambert's Theorem in Section C. In Section D, we analyze the passing time in the spherical problem. In Section E, we provide our findings from [3] with some additional remarks.

By normalization, we assume throughout this note that the masses are unit.

B. Central Projection and the Spherical Kepler Problem

In this section, we shall first define the spherical Kepler Problem. The sphere has dimension 2 and is considered as embedded in a Euclidean space of dimension 3. The choice of dimensions here is just for convenience and is certainly not essential. The choice to work with a sphere of radius R instead of the unit sphere is of course inessential. A small purpose for this is to make the link between spherical problem and the classical Kepler problem somehow more transparent. The pseudospherical Kepler Problem can be defined analogously by a completely parallel construction, by considering a pseudosphere as embedded in a Minkowski Space instead of a sphere embedded in a Euclidean space, for which we shall not make detailed discussion here. Note that both the spherical and pseudospherical Kepler problems can be discussed in an intrinsic way using proper charts, say gnomonic chart as given by the central projection, or the stereographic chart projected from the antipodal point of the attracting center to have a complete and conformal chart. In the pseudospherical case these correspond to the Beltrami-Klein model and Poincaré's disc models respectively.

Now we proceed with the spherical case. Let $(\mathbf{F}, \langle , \rangle)$ be a three-dimensional Euclidean space and

$$\mathbf{E} := \{q \in \mathbf{F} :< Z, q >= R\}$$

be a plane in $\mathbf{F}$ for some $Z \in \mathbf{F}$ of unit norm, and some $R > 0$. For a particle q moving in $\mathbf{F}$, let $h = \langle Z, q \rangle / R$ be the "normalized height" of q. For those $q \in F$ such that $\langle Z, q \rangle \neq 0$, we consider the motion of its central projection $q_{\mathbf{E}} = q/h$ in $\mathbf{E}$. We have

$$\dot{q}_{\mathbf{E}} = \frac{h\dot{q} - \dot{h}q}{h^2}, \; \frac{d}{dt}(h^2 \dot{q}_{\mathbf{E}}) = h\ddot{q} - \ddot{h}q.$$

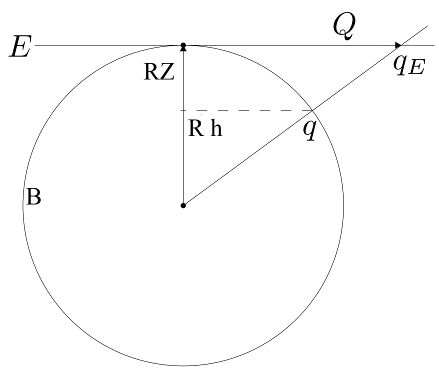

Figure 1: Central projection of a sphere (from [1]).

Let the motion of q be given by $\ddot{q} = \psi(q) + \lambda q$, where λ is a function of $q, \dot{q}$ and also possibly a function of time t. We have then

$$\frac{d}{dt}(h^2\dot{q}_{\mathbf{E}}) = h\psi(q) - \frac{< Z, \psi(q) >}{R} q. \tag{1}$$

Denote by $B := B(R)$ the sphere in $\mathbf{F}$ centered at the origin with radius R. From now on, the motion of q is restricted on B. We have

PROPOSITION 1. *(Appell [4], see also [1]) A force field on B induces a force field on $\mathbf{E} \subset \mathbf{F}$, s.t. the central projection of the moving particle $q \in B$ moves under the induced force field on $\mathbf{E}$ up to a time reparametrization $d\tau = h^{-2}dt$.*

A central field on B, with an attracting center at the contact point $O = RZ$ of $\mathbf{E}$ and B, thus naturally induces a central field on $\mathbf{E}$. When the induced central field on $\mathbf{E}$ is that of the Kepler problem, the corresponding force field on B defines the spherical Kepler problem.

The unit normal vector Z of $\mathbf{E}$ can be extended to a constant vector field in $\mathbf{F}$ normal to $\mathbf{E}$ for which we still denote it by Z. By projecting Z orthogonally to the tangent planes of B we get a central force field $Z_B = Z - \frac{h}{R}q$ on B, in which $h = \frac{\langle Z, q\rangle}{R}$. The centers of the central force field are respectively RZ and $-RZ$. Any other central force field with the same centers can be expressed as $b(q)Z_B$, where $b : B \setminus \{RZ, -RZ\} \to \mathbb{R}$ is a scalar function.

In order to calculate the central force field on $\mathbf{E}$ corresponding to bZ_B, let $Q = q_{\mathbf{E}} - RZ$, and recall that τ is a new time variable defined by $d\tau = h^{-2}dt$. From Eq.(1) with $\psi(q) = bZ_B$, we have

$$\frac{d^2Q}{d\tau^2} = \frac{d^2q_{\mathbf{E}}}{d\tau^2} = h^2\frac{d}{dt}(h^2\dot{q}_{\mathbf{E}}) = bh^2(hZ_B - \frac{\langle Z, Z_B\rangle}{R}q) = bh^2(hZ - \frac{q}{R}) = -\frac{bh^3Q}{R}$$

and

$$h = \frac{R}{\|q_{\mathbf{E}}\|} = \frac{R}{\sqrt{R^2 + \|Q\|^2}} = \frac{1}{\sqrt{1 + \frac{\|Q\|^2}{R^2}}}, \tag{2}$$

which is equivalent to

$$\|Q\| = Rh^{-1}(1-h^2)^{\frac{1}{2}}.$$

In order to have

$$\frac{d^2Q}{d\tau^2} = -\|Q\|^{-3}Q,$$

i.e. the induced central force field defines the Kepler problem on $\mathbf{E}$, we need to have

$$\frac{bh^3}{R} = \|Q\|^{-3}, \text{ that is } b = R^{-2}(1-h^2)^{-\frac{3}{2}}.$$

By Proposition 1, the Keplerian orbits on B are spherical conics (c.f. [13]) and the attracting center is one of their foci. Its energy takes the form

$$\mathcal{E} = \frac{1}{2}\|\dot{q}\|^2 + U(q),$$

in which the function $U(q)$ (that we suppose moreover to be homogeneous to eliminate the constant of integration) satisfies

$$\nabla U(q) = \psi(q) = b Z_B.$$

By a direct calculation, we find

$$U(q) = -\frac{\langle q, Z\rangle}{R\sqrt{\|q\|^2 - \langle q, Z\rangle^2}} = -\frac{h}{R\sqrt{1-h^2}}.$$

We now write the energy $\mathcal{E}$ with the semi major axis a and the eccentricity e of the projected Keplerian orbit in $\mathbf{E}$:

PROPOSITION 2.

$$\mathcal{E} = -\frac{1}{2a} + \frac{a(1-e^2)}{2R^2}. \tag{3}$$

Proof. In terms of Q, the expression of U is

$$U = -\frac{h}{R\sqrt{1-h^2}} = -\frac{1}{\|Q\|}.$$

Now we calculate $\|\dot{q}\|^2$. From $q = h q_{\mathbf{E}}$ we get $\dot{q} = \dot{h} q_{\mathbf{E}} + h\dot{q}_{\mathbf{E}}$. Note also $q_{\mathbf{E}} = RZ + Q$. We have thus $\dot{q}_{\mathbf{E}} = \dot{Q}$ and $\langle \dot{q}_{\mathbf{E}}, q_{\mathbf{E}}\rangle = \langle \dot{Q}, Q\rangle$. From (2), we also have

$$\dot{h} = -\frac{\langle \dot{Q}, Q\rangle}{R^2(\sqrt{1+\|Q\|^2/R^2})^3} = -\frac{h^3\langle \dot{Q}, Q\rangle}{R^2}.$$

Therefore

$$\begin{aligned}
\|\dot{q}\|^2 &= \langle \dot{h} q_{\mathbf{E}} + h\dot{q}_{\mathbf{E}}, \dot{h} q_{\mathbf{E}} + h\dot{q}_{\mathbf{E}}\rangle \\
&= \dot{h}^2\,\|q_{\mathbf{E}}\|^2 + h^2\,\|\dot{q}_{\mathbf{E}}\|^2 + 2h\dot{h}\,\langle q_{\mathbf{E}}, \dot{q}_{\mathbf{E}}\rangle \\
&= \dot{h}^2\,\frac{R^2}{h^2} + h^2\,\|\dot{Q}\|^2 + 2h\dot{h}\langle Q, \dot{Q}\rangle \\
&= h^2\,\|\dot{Q}\|^2 - \frac{h^4}{R^2}\,\langle Q, \dot{Q}\rangle^2 \\
&= h^{-2}\,\|\frac{dQ}{d\tau}\|^2 - \frac{1}{R^2}\,\langle Q, \frac{dQ}{d\tau}\rangle^2 \\
&= \|\frac{dQ}{d\tau}\|^2 + \frac{1}{R^2}\,(\|Q\|^2\|\frac{dQ}{d\tau}\|^2 - \langle Q, \frac{dQ}{d\tau}\rangle^2) \\
&= \|\frac{dQ}{d\tau}\|^2 + \frac{C^2}{R^2}.
\end{aligned}$$

In which $C := \|Q \times \dfrac{dQ}{d\tau}\|$ is the norm of the angular momentum of the Keplerian orbit in **E**. As by definition

$$\mathcal{E} = \frac{1}{2}\|\dot{q}\|^2 - \frac{1}{\|Q\|},$$

we have

$$\mathcal{E} = E + \frac{C^2}{2R^2},$$

where we have denoted by E the energy of the projected Kepler problem in **E**. We thus get Eq. (3) from the classical expressions $E = -\dfrac{1}{2a}$, $C^2 = a(1-e^2)$. □

On the other hand, if we denote the geodesic major axis by $R\theta_a$, where θ_a is the maximal central angle of the spherical ellipse, it is already known from Killing (c.f. [7], [14]) that θ_a is only a function of $\mathcal{E}$. Let us calculate a little more to see that this is indeed the case: By central projection, the pericenter and apocenter of the spherical ellipse in B is projected respectively to the pericenter and apocenter of the projected ellipses in **E**. The line passing through the origin of **F** and the point RZ is perpendicular to **E**, and divides the angle θ_a into two angles θ' and θ''. The major axis of the projected ellipse in **E** is divided by the focus point $O = RZ$ into two segments with length $a(1+e)$ and $a(1-e)$ respectively. As the vector RZ has length R, we have

$$\{\tan\theta', \tan\theta''\} = \{\frac{a(1-e)}{R}, \frac{a(1+e)}{R}\}.$$

Therefore

$$\tan\theta_a = \tan(\theta' + \theta'') = \frac{\frac{2a}{R}}{1 - \frac{a^2(1-e^2)}{R^2}} = \frac{1}{R(\frac{1}{2a} - \frac{a(1-e^2)}{2R^2})} = -\frac{1}{R\mathcal{E}}.$$

And thus for fixed R, the angle θ_a is only a function of $\mathcal{E}$.

C. Lambert's Theorem for the Kepler Problem

For two points A_1 and A_2 in **E**, the passing time ΔT from A_1 to A_2 along a Keplerian arc of an elliptic orbit with semi major axis a can be expressed as a function of the three mutual distances $r_1 = |OA_1|$, $r_2 = |OA_2|$, $c = |A_1A_2|$ and a. In general, there are two ellipses passing through the points A_1, A_2 with semi major axis a. We denote by u the eccentric anomaly of a point on an elliptic orbit, and denote by u_1, u_2 the eccentric anomalies of A_1, A_2 respectively. There are yet many choices for these angles and thus, it is seen that ΔT is a multivalued function of r_1, r_2, c and a.

Nevertheless, once (u_1, u_2, e, a) are chosen, we deduce directly from the Kepler equation that

Claim 1.

$$\Delta T = \int_{u_1}^{u_2} a^{\frac{3}{2}} (1 - e\cos u)\, du. \tag{4}$$

Following Lagrange [9], we define two angles ϕ, ψ by the relations

$$\begin{cases} \phi = \arccos\big(e\cos((u_1+u_2)/2)\big); \\ \psi = (u_1-u_2)/2. \end{cases} \tag{5}$$

The change of coordinates from (u_1, u_2, e) to (ϕ, ψ, e) is regular when

$$e \neq 0, e \neq 1, u_1 + u_2 \neq 0\,(mod\,2\pi).$$

In general this change of coordinates is given by a two-to-one mapping, with (u_1, u_2, e) and $(u_1' = -u_2, u_2' = -u_1, e)$ corresponding to the same value of (ϕ, ψ, e).

By substitution, we see that ΔT, as a function of (ϕ, ψ, e, a), does not depend on e:

$$\Delta T = a^{\frac{3}{2}}(u_2 - u_1 - e(\sin u_2 - \sin u_1)) = a^{\frac{3}{2}}(-2\psi + 2\sin\psi\cos\phi).$$

On the other hand, the following relations have been deduced by Lagrange in [9, pp. 564-566]

$$\begin{cases} r_1 + r_2 = 2a(1-\cos\psi\cos\phi) \\ c = 2a\,|\sin\psi\sin\phi|, \end{cases} \tag{6}$$

which allows to express ψ and ϕ as multi-valued functions only of r_1+r_2, a and c. Following Lagrange, we may thus conclude that

THEOREM 3. *(Lambert [10]) ΔT is a multi-valued function of c, r_1+r_2 and a.*

The proof presented here is just one of many proofs of Lambert's theorem that were found in the history. In [2], a detailed timeline of these proofs can be found, together with yet two new proofs of this theorem.

D. Period of periodic orbits in the spherical Kepler problem

We shall now consider the corresponding spherical Kepler problem on B defined above. The point O is the North pole of B. Let B_1, B_2 be two points on the north hemisphere. Let s_1, s_2, d be respectively the geodesic distances OB_1, OB_2 and B_1B_2 on B, and let $\theta_1 = s_1/R$, $\theta_2 = s_2/R$, $\theta = d/R$. Let $\Delta' T$ be the passing time for a particle to move from B_1 to B_2 along a spherical Keplerian orbit in a spherical ellipse with energy $\mathcal{E}$.

We shall eventually analyze $\Delta' T$ in $\mathbf{E}$ while keeping the time reparametrization in mind. We suppose that one of the ellipses determined by O, B_1, B_2 and $\mathcal{E}$ lies entirely in the north hemisphere, so that it projects to an ellipse in $\mathbf{E}$.

Let $r = \|Q\|$ be the distance of the projected particle $q_{\mathbf{E}}$ from O. In terms of the elliptic elements of the projected ellipse in $\mathbf{E}$, we have

$$r = a(1 - e\cos u)$$

and

$$h = \frac{1}{\sqrt{1+\frac{r^2}{R^2}}} = \frac{R}{\sqrt{R^2 + a^2(1-e\cos u)^2}}.$$

Also, by differentiating the Kepler equation in **E**, we obtain

$$d\tau = a^{\frac{3}{2}}(1 - e\cos u)du.$$

Thus with $dt = h^2 d\tau = \dfrac{d\tau}{1+\frac{r^2}{R^2}}$, we obtain

$$\Delta' T = \int_{t_1}^{t_2} dt = \int_{\tau_1}^{\tau_2} \frac{1}{1+\frac{r^2}{R^2}} d\tau = \int_{u_1}^{u_2} a^{\frac{3}{2}} \frac{1 - e\cos u}{1 + \frac{a^2}{R^2}(1 - e\cos u)^2} du. \tag{7}$$

By normalization we set $R = 1$ from now on.

The period of the spherical elliptic motion has been calculated in [7], [8], and it has been found to depend only on $\mathcal{E}$. Indeed, we may calculate the integral

$$\mathbf{T} = \int_0^{2\pi} a^{\frac{3}{2}} \frac{1 - e\cos u}{1 + a^2(1 - e\cos u)^2} du$$

by decomposing the integrand as

$$a^{\frac{3}{2}} \frac{1 - e\cos u}{1 + a^2(1 - e\cos u)^2} = \frac{\sqrt{a}}{2}\left(\frac{1}{i + a(1 - e\cos u)} + \frac{1}{-i + a(1 - e\cos u)}\right)$$

which allows us to integrate them out separately. We thus get

$$\mathbf{T} = \pi\sqrt{a}\left(\frac{1}{\sqrt{a^2(1-e^2) - 1 + 2ai}} + \frac{1}{\sqrt{a^2(1-e^2) - 1 - 2ai}}\right)$$

in which both complex square roots are meant to have positive real parts.

With $R = 1$, Eq. (3) now reads

$$\mathcal{E} = -\frac{1}{2a} + \frac{a(1-e^2)}{2} \tag{8}$$

thus

$$a^2(1-e^2) - 1 = 2a\mathcal{E}. \tag{9}$$

From this, we have

$$\mathbf{T} = \frac{\pi\sqrt{2}}{2}\left(\frac{1}{\sqrt{\mathcal{E}+i}} + \frac{1}{\sqrt{\mathcal{E}-i}}\right).$$

By squaring this expression and taking square root, we obtain in the end

$$\mathbf{T} = \frac{\pi\sqrt{\mathcal{E} + \sqrt{\mathcal{E}^2+1}}}{\sqrt{\mathcal{E}^2+1}}.$$

This gives a generalization of Kepler's third law to the spherical Kepler problem. A similar formula can be also obtained for the pseudospherical Kepler problem as well.

E. Spherical and Pseudospherical Lambert's Theorems

It is shown in [3] that Lambert's theorem can be generalized to the the spherical and pseudo-spherical Kepler problems, as has been stated in Theorem 2. This surprising fact yet enhances the list of similarities between Kepler problem in a Euclidean space and Kepler problem on a sphere or on a pseudo-sphere.

Referring the proof to the paper jointly written with A. Albouy [3], we shall be content to make a few comments in this section which illustrates the proof.

The generalization of Lambert theorem to the spherical and pseudospherical problems as in [3] is based on the idea of Lambert vectors, which are pair of tangent vectors (δ_A, δ_B), based on the two end points A and B of the Keplerian arc respectively, such that along the infinitesimal variation of the Keplerian arc as defined by the infinitesimal variation (δ_A, δ_B) with the same energy, the Maupertuis action functional and the passing time does not change. Thanks to the theory of Hamilton as explained in [2], this amounts to require that the relation

$$\langle \delta_A, v_A \rangle = \langle \delta_B, v_B \rangle \tag{10}$$

holds for any pair of velocities at end points (v_A, v_B) along some Keplerian arc. Isometries clearly gives rise to Lambert vectors for which we consider trivial. The phenomenon as indicated in the statement of Lambert's theorem asserts the existence of non-trivial Lambert vectors.

Now the key observation is that

PROPOSITION 3. *Non-trivial Lambert vectors in the plane give rise to non-trivial Lambert vectors on the sphere/pseudosphere by Appell's projection.*

Indeed Appell's projection when applied to a tangent vector is not only the push-forward, but the push-forward composed with a multiplicative factor depending only on the positions. Now since (10) is linear, and the inner product on tangent vectors on a sphere/pseudosphere is again a bilinear form, we may further properly adjust each transformed tangent vector by factors depending only on the positions, to get a non-trivial Lambert vector on the sphere/pseudosphere from a non-trivial Lambert vector in the plane, since each isometric symmetry of the planar central force problem lifts to the sphere and pseudosphere, and vice versa.

There is therefore a certain form of Lambert's theorem also for the spherical and pseudospherical Kepler problems. This is surprising since Appell's projection does not preserve time parametrization, nor energy hypersurfaces. The effective form of Lambert's theorem can therefore be worked out with this Lambert vectors and we found exactly the same statement as in the Euclidean case. The approach shows in particular that the property stated by Lambert's theorem for Kepler problem is a projective dynamical property.

We end by posing yet some questions:

As we have said, there are many proofs of distinct natures for Lambert's theorem for the Kepler problem on the plane. This calls for the following general question:

Question 2. Is it possible to adapt more among these proofs to the spherical/pseudo-spherical Kepler problems as well?

Another question of interest concerns the role of symmetry and the phenomenon stated by Lambert's theorem outside harmonic oscillators and Kepler problems. This can already be posed in the planar case:

Question 3. Are their non-rotational invariant natural mechanical system in the plane which has a non-trivial Lambert vector?

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