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X. Xu*

DOUBLY-SYMMETRIC PERIODIC ORBITS IN THE SPATIAL HILL'S LUNAR PROBLEM WITH OBLATE SECONDARY PRIMARY

Abstract. In this article we consider the existence of a family of doubly-symmetric periodic orbits in the spatial circular Hill's lunar problem, in which the secondary primary at the origin is oblate. The existence is shown by applying a fixed point theorem to the equations with periodical conditions expressed in Poincaré-Delaunay elements for the double symmetries after eliminating the short periodic effects in the first-order perturbations of the approximated system.

A. Introduction

The classical spatial Hill's lunar problem is a limiting case derived from the spatial *circular restricted three-body problem* (CRTBP) [1]. Consider a modified version of the classical CRTBP, in such a problem, one small primary $\mathtt{M}_2$ is an oblate Maclaurin ellipsoid and the other primary $\mathtt{M}_1$ is a standard spheroid, so $\mathtt{M}_1$ can be considered as a mass point while $\mathtt{M}_2$ has a shape. Set the masses of $\mathtt{M}_1$ and $\mathtt{M}_2$ to be m_1 and m_2, respectively. Denote the radius of the equator of $\mathtt{M}_2$ as a_e, the polar radius b_e. The relative position of $\mathtt{M}_1$ relative to $\mathtt{M}_2$ is $\vec{r}$, and denote r be the length of the vector $\vec{r}$. In order to make the differential system about $\ddot{\vec{r}}$ integrable, suppose $\mathtt{M}_1$ moves on the equator plane of $\mathtt{M}_2$. The angular velocity ω_e of the relative circular motion of the two primaries can be calculated by

$$\omega_e^2 = \mathtt{G}(m_1+m_2)\left(1-\sum_{n=1}^{\infty}(2n+1)\frac{a_e^{2n}}{r^{2n+2}}J_{2n}P_{2n}(0)\right). \tag{1}$$

where $\mathtt{G}$ is the universal gravitational constant, $\{J_{2n}\}$ are the even zonal harmonic coefficients, and $P_n(x)$ is the n-th Legendre polynomial. By the symbol computations, J_{2n} for the Maclaurin ellipsoid can be calculated as

$$J_{2n} = (-1)^{n-1}\left(1-\frac{b_e^2}{a_e^2}\right)^n \prod_{k=1}^{n}\frac{2k-1}{2k+3} = \frac{2n-1}{2n+3}\left(1-\frac{b_e^2}{a_e^2}\right)J_{2n-2}, \quad n \geq 1. \tag{2}$$

Set the total masses as the mass unit, the radius and the period of relative circular motion of the two primaries as the distance unit, and 2π, respectively. In such units, the universal gravitational constant $\mathtt{G}=1$, and the masses of $\mathtt{M}_1$ and $\mathtt{M}_2$ are $1-\mu$ and μ respectively. Set the origin at the center of $\mathtt{M}_2$, the positive x axis is the direction

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from $\mathtt{M}_2$ to $\mathtt{M}_1$, the position $\vec{r} = \mathbf{x} = (x_1, x_2, x_3)^T$ is a column vector. In the uniform rotating frame, the conjugate momentum of $\mathbf{x} = \vec{r}$ is $\mathbf{y} = (\dot{x}_1 - x_2, \dot{x}_2 + x_1, \dot{x}_3)^T$. The Hamiltonian for the motion of the infinitesimal body moving nearby $\mathtt{M}_2$ can be written as

$$
\begin{aligned}
H_a &= \frac{1}{2}\left[y_1^2 + (y_2 - 1 + \mu)^2 + y_3^2\right] - \left[(x_1 - 1 + \mu)(y_2 - 1 + \mu) - x_2 y_1\right] - U_1 - U_2,\\
U_1 &= \frac{1-\mu}{\sqrt{1 + r^2 - 2x_1}} = (1-\mu)\sum_{n=0}^{\infty} P_n\left(\frac{x_1}{r}\right) r^n,\\
U_2 &= \frac{\mu}{r} - \frac{\mu}{r}\sum_{n=1}^{\infty}\left(\frac{a_e}{r}\right)^{2n} J_{2n} P_{2n}\left(\frac{x_3}{r}\right).
\end{aligned} \tag{3}
$$

Expand the Hamiltonian, neglect the constant term. Make symplectic scaling, $\mathbf{x} \to \mu^{1/3}\mathbf{x}$, $\mathbf{y} \to \mu^{1/3}\mathbf{y}$, at the mean while, set

$$
J_{2n} = -C_{2n}, \quad r \to \mu^{1/3} r, \quad a_e \to \mu^{1/3} a_e, \quad b_e \to \mu^{1/3} b_e.
$$

This is a conformal symplectic transformation with a multiplier $\mu^{-2/3}$, then let $\mu \to 0$, consider the limiting case, the Hamiltonian for the Hill's lunar problem with the oblateness of the second primary is obtained as

$$
\begin{aligned}
H_b &= \frac{1}{2}\|\mathbf{y}\|^2 - (x_1 y_2 - x_2 y_1) - \frac{1}{r} - P_2\left(\frac{x_1}{r}\right) r^2 - \tilde{U},\\
\tilde{U} &= \frac{1}{r}\sum_{n=1}^{\infty}\left(\frac{a_e}{r}\right)^{2n} C_{2n} P_{2n}\left(\frac{x_3}{r}\right).
\end{aligned} \tag{4}
$$

The Hill's lunar problem is studied by both analytical and numerical methods, see Michalodimitrakis (1980)[2], Howision & Meyer (2000I, 2000II)[3, 4], Maciejewski & Rybicki (2001)[5], Llibre & Roberto (2011)[6], Belbruno et.al.(2019)[7]. Special attention is paid to the effects of the oblateness on the motion of the infinitesimal body by Sharma(1990)[8], Vashkov'yak & Teslenko (2000,2001) [9, 10], Markellos et.al.(2001)[11], Perdiou (2008)[12], Bustos et.al.(2018)[13].

In the paper of X.B. Xu (2019)[14], a family of doubly symmetric periodic orbits of lunar type in the spatial CRTBP is shown to exist. Following a similar way of proof, such a family still exist in the Hill's lunar problem with the second primary oblate. The paper is organized as follows. In Sect. 2, orbital elements and canonical elements are introduced into the scaled Hamiltonian system with a small parameter. In Sect. 3, the short periodic effects of the first-order perturbation terms are eliminated by the Lie transform method. In Sect. 4, the doubly-symmetric periodic solution is introduced and the continuation is given by the use of a fixed point theorem. In the last section, several discussions are proposed.

B. The Hamiltonian in mixed elliptical elements

Set $\mathbf{x} \to \varepsilon^2\xi$, $\mathbf{y} \to \varepsilon^{-1}\eta$, $r \to \varepsilon^2 r$, $a_e \to \varepsilon^2 a_e$, $C_{2n} \to \varepsilon^{6n}\tilde{J}_{2n}$, this is a symplectic transformation with a multiplier ε^{-1}, and the small parameter ε represents the closeness of the infinitesimal body to the second primary. The Hamiltonian with ε is

$$H_c = \varepsilon^{-3}\left(\frac{\|\eta\|^2}{2} - \frac{1}{\|\xi\|}\right) - (\xi_1\eta_2 - \xi_2\eta_1) - \varepsilon^3 r^2 \mathcal{P}_2(\frac{\xi_1}{r})$$
$$- \varepsilon^{-3}\frac{1}{\|\xi\|}\sum_{n=1}^{\infty}\left(\frac{a_e}{\|\xi\|}\right)^{2n}\varepsilon^{6n}\tilde{J}_{2n}P_{2n}(\frac{\xi_3}{\|\xi\|}). \tag{5}$$

The perturbation caused by the oblateness is supposed to be as small as the third-body perturbation from infinity, so the Hamiltonian H_c can be splitted into four parts,

$$-H_c = \varepsilon^{-3}F_{01}(\xi,\eta) + F_{02}(\xi,\eta) + \varepsilon^3 F_1(\xi) + \varepsilon^9 F_R(\xi,\varepsilon). \tag{6}$$

If the time is also scaled by $t = \varepsilon^3\tau$, then the Hamiltonian is converted to $\varepsilon^3 H_c$. As ε is small, $-F_{01}$ is a Kepler problem, so the Hamiltonian $\varepsilon^3 H_c$ is a perturbed Keplerian system.

The relations between the rectangular coordinates $(\vec{r},\dot{\vec{r}})$ and the instantaneous orbital elements of a Keplerian orbit can refer to some fundamental books [15, 16] or some papers [14, 17] in celestial mechanics. The orbital elements $a, e, i, \Omega, \omega, M$, f are the semiaxis, eccentricity, inclination, the longitude of the ascending node, the argument of the pericenter, the mean anomay and the eccentric anomaly, respectively. Then the Delaunay elements,

$$\begin{array}{lll} \mathcal{L} = \sqrt{a}, & \mathcal{G} = \sqrt{a(1-e^2)}, & \mathcal{H} = \mathcal{G}\cos i, \\ h = \Omega, & g = \omega, & \ell = M, \end{array} \tag{7}$$

and the Poincaré-Delaunay elements,

$$\begin{array}{lll} Q_1 = \ell + g + h, & Q_2 = -\sqrt{2(\mathcal{L}-\mathcal{G})}\sin(g+h), & Q_3 = \ell + g, \\ \mathcal{P}_1 = \mathcal{L} - \mathcal{G} + \mathcal{H}, & \mathcal{P}_2 = \sqrt{2(\mathcal{L}-\mathcal{G})}\cos(g+h), & \mathcal{P}_3 = \mathcal{G} - \mathcal{H}. \end{array}$$

Parts of the Hamiltonian H_c can be written as

$$F_{01} = \frac{1}{2\mathcal{L}^2} = \frac{1}{2(\mathcal{P}_1+\mathcal{P}_3)^2}, \quad F_{02} = \mathcal{H} = \mathcal{P}_1 - \frac{\mathcal{P}_2^2 + Q_2^2}{2},$$
$$F_1 = r^2 P_2(\frac{\xi_1}{r}) + \tilde{J}_{2n}\frac{a_e^2}{r^3}P_2(\frac{\xi_3}{r}), \tag{8}$$

but F_1 cannot be expressed in a finite form of these canonical elements because it contains the eccentric anomaly or the true anomaly, which is a function of the eccentricity $e(\mathcal{L},\mathcal{G})$ and the mean anomaly $M = \ell$. The first-order perturbation terms in F_1 can be expanded by Tisserand expansion [18], one has

$$P_2(\frac{\xi_1}{r}) = I_1 + I_2\cos 2(f+\omega-\Omega) + I_3\cos 2(f+\omega+\Omega)$$
$$+ I_4[\cos 2(f+\omega) + \cos 2\Omega], \tag{9}$$

where

$$I_1 = \frac{3}{8}\frac{\mathcal{H}^2}{\mathcal{G}^2} - \frac{1}{8}, I_2 = \frac{3}{16}(1 - \frac{\mathcal{H}}{\mathcal{G}})^2, I_3 = \frac{3}{16}(1 + \frac{\mathcal{H}}{\mathcal{G}})^2, I_4 = \frac{3}{8}(1 - \frac{\mathcal{H}^2}{\mathcal{G}^2}), \tag{10}$$

and one gets

$$P_2(\frac{\xi_3}{r}) = -\frac{3}{4}\cos 2(f + \omega) + I_5, \quad I_5 = \frac{1}{4} - \frac{3}{4}\frac{\mathcal{H}^2}{\mathcal{G}^2}. \tag{11}$$

According to the formulas of the Hansen coefficients [18], one has

$$\left(\frac{r}{a}\right)^n \exp(\mathtt{i}mf) = \sum_{k=-\infty}^{\infty} X_k^{n,m}(e)\exp(\mathtt{i}kM),$$

so $r^2 P_2(\frac{\xi_1}{r})$ can be expanded as

$$\begin{aligned} r^2 P_2(\frac{\xi_1}{r}) = & I_1 \sum_{k=-\infty}^{\infty} X_k^{2,0}(e)\cos(kQ_1 - k(g+h)) \\ & + I_2 \sum_{k=-\infty}^{\infty} X_k^{2,2}(e)\cos((k-4)Q_1 + 4Q_3 + (2-k)(g+h)) \\ & + I_3 \sum_{k=-\infty}^{\infty} X_k^{2,2}(e)\cos(kQ_1 + (2-k)(g+h)) \\ & + I_4 \sum_{k=-\infty}^{\infty} X_k^{2,2}(e)\cos((k-2)Q_1 + 2Q_3 + (2-k)(g+h)) \\ & + I_4 \sum_{k=-\infty}^{\infty} X_k^{2,0}(e)\cos(kQ_1 - k(g+h))\cos 2(Q_1 - Q_3), \end{aligned} \tag{12}$$

and $r^{-3} P_2(\frac{\xi_3}{r})$ can be expanded as

$$\begin{aligned} r^{-3} P_2(\frac{\xi_3}{r}) = & -\frac{3}{4}\sum_{k=-\infty}^{\infty} X_k^{-3,2}(e)\cos((k-2)Q_1 + 2Q_3 + (2-k)(g+h)) \\ & + I_5 \sum_{k=-\infty}^{\infty} X_k^{-3,0}(e)\cos(kQ_1 - k(g+h)). \end{aligned} \tag{13}$$

In addition, $g + h$ can be gotten from Q_2 and $\mathcal{P}_2$. The e and $\cos i$ can be expressed as

$$e^2 = 1 - \left(1 - \frac{\mathcal{P}_2^2 + Q_2^2}{2(\mathcal{P}_1 + \mathcal{P}_3)}\right)^2, \quad \frac{\mathcal{G}}{\mathcal{H}} = 1 - \frac{\mathcal{P}_3}{\mathcal{P}_1 + \mathcal{P}_3 - \frac{\mathcal{P}_2^2 + Q_2^2}{2}}. \tag{14}$$

The Hansen coefficients above are very small if $|k|$ is large and e small, so only finite terms are needed to match the precision. As there are infinite short-period terms, the first-order perturbation system is complicate and can be simplified by the averaging method.

C. Averaging in the first-order system

Averaging method can be used to eliminate short periodic terms in the perturbed dynamical system. In celestial mechanics, Lie transforms is an explicit near-identy canonical transformation, and is taken in use here. There are two variables Q_1, Q_3 which change as fast as the time because they both contain the mean anomaly. Two averaging procedures are needed in order to average F_1 over Q_1 and Q_3 in a period of 2π successively.

One has

$$\begin{aligned}\bar{F}_1 &= \frac{1}{2\pi}\int_0^{2\pi} F_1 dQ_1 \\ &= \Big[I_1 X_0^{2,0} + I_2 X_4^{2,2}\cos(4Q_3 - 2(g+h)) + I_3 X_0^{2,2}\cos(2g+2h) \\ &\quad + I_4 X_2^{2,2}\cos 2Q_3 + \frac{1}{2} I_4 (X_{-2}^{2,0} + X_2^{2,0})\cos(2Q_3 - 2g - 2h)\Big] \\ &\quad + \tilde{J}_2 a_e^2 \left[-\frac{3}{4} X_2^{-3,2}\cos 2Q_3 + I_5 X_0^{-3,0}\right], \end{aligned} \tag{15}$$

and

$$\begin{aligned}\bar{\bar{F}}_1 &= \frac{1}{2\pi}\int_0^{2\pi} \bar{F}_1 dQ_3 \\ &= \left[(1+\frac{3}{2}e^2)I_1 + \frac{5}{2}e^2 I_3 \cos(2g+2h)\right] + \tilde{J}_2 a_e^2 (1-e^2)^{-3/2} I_5. \end{aligned} \tag{16}$$

It is seen that $\bar{\bar{F}}_1$ does not contain terms about ℓ any more, so $\mathcal{L}$ would be a constant in the first-order system, and this makes it easier to give error estimates for the continuation obeying to a fixed point theorem.

Set $\tilde{\varepsilon} = \varepsilon^3$ and let $\tilde{\varepsilon}$ be the small parameter in the Lie transforms. The doubly averaged Hamiltonian is

$$-\tilde{\varepsilon} H_d = F_{01} + \tilde{\varepsilon} F_{02} + \frac{\tilde{\varepsilon}^2}{2}\cdot 2\bar{\bar{F}}_1 + O(\tilde{\varepsilon}^4), \tag{17}$$

As there are two averaging procedures, there are two generating functions

$$\begin{aligned} W &= \frac{\tilde{\varepsilon}^2}{2}\cdot 2W_2, \quad W_2 = W_2^{(1)} + W_2^{(2)}, \\ \bar{F}_1 - F_1 + \bar{\bar{F}}_1 - \bar{F}_1 &= -\frac{\partial F_{01}}{\partial \mathcal{P}_1}\frac{\partial W_2^{(1)}}{\partial Q_1} - \frac{\partial F_{01}}{\partial \mathcal{P}_3}\frac{\partial W_2^{(2)}}{\partial Q_3}, \end{aligned} \tag{18}$$

the $W_2^{(1)}$ and $W_2^{(2)}$ can be achived by integrations,

$$\begin{aligned} W_2^{(1)} &= (\mathcal{P}_1 + \mathcal{P}_3)^3 \int (\bar{F}_1 - F_1)\, dQ_1, \\ W_2^{(2)} &= (\mathcal{P}_1 + \mathcal{P}_3)^3 \int \left(\bar{\bar{F}}_1 - \bar{F}_1\right) dQ_3, \end{aligned} \tag{19}$$

Neglecting the constants, one has

$$
\begin{aligned}
-(\mathcal{P}_1+\mathcal{P}_3)^{-3}W_2^{(1)} = {} & I_1 \sum_{k\neq 0} \frac{1}{k} X_k^{2,0} \sin(kQ_1 - k(g+h)) \\
& + I_2 \sum_{k\neq 4} \frac{1}{k-4} X_k^{2,2} \sin((k-4)Q_1 + 4Q_3 + (2-k)(g+h)) \\
& + I_3 \sum_{k\neq 0} \frac{1}{k} X_k^{2,2} \sin(kQ_1 + (2-k)(g+h)) \\
& + I_4 \sum_{k\neq 2} \frac{1}{k-2} X_k^{2,2} \sin((k-2)Q_1 + 2Q_3 + (2-k)(g+h)) \\
& + \frac{1}{2} I_4 \sum_{k\neq -2} \frac{1}{k+2} X_k^{2,0} \sin((k+2)Q_1 - k(g+h) - 2Q_3) \\
& + \frac{1}{2} I_4 \sum_{k\neq 2} \frac{1}{k-2} X_k^{2,0} \sin((k-2)Q_1 - k(g+h) + 2Q_3),
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
-(\mathcal{P}_1+\mathcal{P}_3)^{-3}W_2^{(2)} = {} & \frac{1}{4} I_2 X_4^{2,2} \sin(4Q_3 - 2(g+h)) + \frac{1}{2} I_4 X_2^{2,2} \sin 2Q_3 \\
& + \frac{1}{4} I_4 \left(X_{-2}^{2,0} + X_2^{2,0}\right) \sin(2Q_3 - 2(g+h)) - \frac{3}{2} \tilde{J}_2 a_e^2 X_2^{-3,2} \sin 2Q_3.
\end{aligned} \tag{21}
$$

Two generating functions are given above, and there are two times of Lie transforms. Both generating functions can be truncated according to a high order of the eccentricity. Theoretically, Lie transforms are inversible and can be calculated with the help of symbol calculation software and numerical calculation software [20, 21].

D. Continuation of the doubly-symmetric periodic orbits

The truncated Hamiltonian $H_{appr} = -\tilde{\varepsilon}^{-1} F_{01} - F_{02}$ is integrable, and is set as the approximated system of the full system (17). The aim of this section is to give the outline of the proof on the continuation of the doubly symmetric periodic solutions.

There exists a lemma about the proposition about the doubly symmetric periodic solution,

LEMMA 1. *Consider an one-order autonomous ordinary differential system in $\mathbb{R}^6$, and it is invariant under two anti-symplectic reflections:*

$$
\begin{aligned}
\mathscr{R}_1 &: (x_1,x_2,x_3,y_1,y_2,y_3,t) \to (x_1,-x_2,-x_3,-y_1,y_2,y_3,-t), \\
\mathscr{R}_2 &: (x_1,x_2,x_3,y_1,y_2,y_3,t) \to (x_1,-x_2,x_3,-y_1,y_2,-y_3,-t).
\end{aligned} \tag{22}
$$

That is to say, the system is symmetric about two Lagrangian planes,

$$
\begin{aligned}
\mathscr{L}_1 &= \{Z | Z = (x_1,0,0,0,y_2,y_3)^T\}, \\
\mathscr{L}_2 &= \{Z | Z = (x_1,0,x_3,0,y_2,0)^T\}.
\end{aligned} \tag{23}
$$

If one solution hits the two Lagrangian planes $\mathscr{L}_1$ and $\mathscr{L}_2$ successively with a time interval $T > 0$, then this solution is periodic with period $4T$ and doubly symmetric.

These two Lagrangian planes can be expressed in Poincaré-Delaunay elements $Z = (Q_1, Q_2, Q_3, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)^T$,

$$\begin{aligned} \mathscr{L}_a &= \{Z = (i\pi, 0, j\pi, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)^T\}, \\ \mathscr{L}_b &= \{Z = (i\pi + k\pi, 0, j\pi + m\pi + \frac{\pi}{2}, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)^T\}, \end{aligned} \tag{24}$$

Denote the initial solution of the approximated system as $Z_0^* \in \mathscr{L}_a$, the differential equations are

$$\begin{array}{lll} \frac{dQ_1}{dt} = \frac{\varepsilon^{-3}}{(\mathcal{P}_1+\mathcal{P}_3)^3} - 1, & \frac{dQ_2}{dt} = \mathcal{P}_2, & \frac{dQ_3}{dt} = \frac{\varepsilon^{-3}}{(\mathcal{P}_1+\mathcal{P}_3)^3}, \\ \frac{d\mathcal{P}_1}{dt} = 0, & \frac{d\mathcal{P}_2}{dt} = -Q_2, & \frac{d\mathcal{P}_3}{dt} = 0. \end{array} \tag{25}$$

If $Z(t; Z_0^*)$ is a doubly-symmetric periodic solution of the integrable approximated system, one has

$$Z_0^* = (i\pi, 0, j\pi, \mathcal{P}_1^*, 0, \mathcal{P}_3^*)^T, \tag{26}$$

and one fourth of the period is $T_0^* = (m + \frac{1}{2} - k)\pi$.

Consider that the initial solution of the full system (17) belongs to $\mathscr{L}_a$, and is near Z_0^*, that is

$$\begin{aligned} Z_0 &= Z_0^* + (0, 0, 0, \delta_{P1}, \delta_{P2}, \delta_{P3})^T \\ &= (i\pi, 0, j\pi, \mathcal{P}_1^* + \delta_{P1}, 0 + \delta_{P2}, \mathcal{P}_3^* + \delta_{P3})^T, \end{aligned} \tag{27}$$

where δ_{P1}, δ_{P2}, δ_{P3} are small.

The full system corresponds to the following kind of differential system,

$$\frac{dZ}{dt} = \mathcal{F}(Z, \tilde{\varepsilon}) = \tilde{\varepsilon}^{-1}\mathcal{F}_{01}(Z) + \mathcal{F}_{02}(Z) + \tilde{\varepsilon}\mathcal{F}_1(Z) + \tilde{\varepsilon}^{2+\kappa}\mathcal{F}_R(Z, \tilde{\varepsilon}), \tag{28}$$

where $Z \in \mathbb{R}^n$, $\kappa \geq 0$. The differential system of the approximated system

$$\frac{dZ}{dt} = \mathcal{F}_0(Z, \tilde{\varepsilon}) = \tilde{\varepsilon}^{-1}\mathcal{F}_{01}(Z) + \mathcal{F}_{02}(Z) \tag{29}$$

is integrable, and has an analytical solution $Z^{(0)}(t; z_0, \tilde{\varepsilon})$.

Consider that $\tilde{\varepsilon}$ is sufficiently small, such that the difference between the solution of the full system and that of the approximated system $\|Z(t; Z_0, \tilde{\varepsilon}) - Z^{(0)}(t; Z_0, \tilde{\varepsilon})\|$ remains small enough for a finite time interval. The norm for vectors represents the maximum absolute value of its components.

LEMMA 2. *Let $Z^{(1)}(t; z_0, \tilde{\varepsilon})$ be a solution of*

$$\dot{Z}^{(1)}(t; z_0, \tilde{\varepsilon}) = (\mathbf{D}_Z\mathcal{F}_0)\,|_{Z^{(0)}} Z^{(1)} + \mathcal{F}_1(Z^{(0)}) \tag{30}$$

with an initial condition $Z^{(1)}(0;z_0,\tilde{\varepsilon})=\mathbf{0}$. *In a finite time interval* $t\in[0,t_0]$, *the solution of the full system* (28) *can be expressed as*

$$Z(t;z_0,\tilde{\varepsilon})=Z^{(0)}(t;z_0,\tilde{\varepsilon})+\tilde{\varepsilon}Z^{(1)}(t;z_0,\tilde{\varepsilon})+\tilde{\varepsilon}^2Z_R(t;z_0,\tilde{\varepsilon}), \tag{31}$$

where $\|Z(t;z_0,\varepsilon)-Z^{(0)}(t;z_0,\tilde{\varepsilon})\|$ *is of order* $\tilde{\varepsilon}$. *The maximum absolute values for elements in* $(\mathbf{D}_Z Z^{(1)})|_{Z_0}$ *and* $(\mathbf{D}_Z Z_R)|_{Z_0}$ *are of zeroth order of* $\tilde{\varepsilon}$.

Following Lemma 2, one can get the formulas of Q_1, Q_2, Q_3 in $\mathscr{L}_b$ with the initial solution Z_0 in (27) for the full system after a finite time $T=T_0^*+\delta T=(m+\frac{1}{2}-k)\pi+\delta T$,

$$\begin{aligned}
Q_1(T)&=\left[\frac{\tilde{\varepsilon}^{-1}}{(\mathcal{L}^*+\delta\mathcal{L})^3}-1\right]T+i\pi+\tilde{\varepsilon}Q_1^{(1)}+O(\tilde{\varepsilon}^2)=k\pi+i\pi,\\
Q_2(T)&=\delta_{P2}\sin(T_0^*+\delta T)+\tilde{\varepsilon}Q_2^{(1)}+O(\tilde{\varepsilon}^2)=0,\\
Q_3(T)&=\frac{\tilde{\varepsilon}^{-1}}{(\mathcal{L}^*+\delta\mathcal{L})^3}T+j\pi+\tilde{\varepsilon}Q_3^{(1)}+O(\tilde{\varepsilon}^2)=j\pi+(m+\frac{1}{2})\pi,
\end{aligned} \tag{32}$$

the equations can be reduced, firstly let the third equation minus the first equation, secondly substitute

$$\tilde{\varepsilon}^{-1}=\varepsilon^{-3}=\frac{(m+\frac{1}{2})}{m+\frac{1}{2}-k}(\mathcal{L}^*)^3=\frac{(m+\frac{1}{2})\pi}{T_0^*}(\mathcal{L}^*)^3, \tag{33}$$

into the third equation, finally one has

$$\left\{\begin{array}{l}
\Psi_1=\delta T+\tilde{\varepsilon}(Q_3^{(1)}-Q_1^{(1)})+O(\tilde{\varepsilon}^2)=0,\\
\Psi_2=(-1)^{m-k}\delta_{P2}\cos\delta T+\tilde{\varepsilon}Q_2^{(1)}+O(\tilde{\varepsilon}^2)=0,\\
\Psi_3=\frac{1+\frac{\delta T}{T_0^*}}{\left(1+\frac{\delta\mathcal{L}}{\mathcal{L}^*}\right)^3}-1+\frac{1}{(m+\frac{1}{2})\pi}\left[\tilde{\varepsilon}Q_3^{(1)}+O(\tilde{\varepsilon}^2)\right]=0.
\end{array}\right. \tag{34}$$

The three equations combine a vector $\Psi(\mathbf{X},\tilde{\varepsilon})$, where $\mathbf{X}=(\delta T,\delta_{P2},\delta\mathcal{L})^T$. The Jacobian matrix derived from the partial derivatives of the above three equations over $\mathbf{X}$ is non-degenerated, and one has

$$\frac{\partial\Psi(\mathbf{X},0)}{\partial\mathbf{X}}=\begin{pmatrix}
1 & 0 & 0\\
(-1)^{m-k+1}\sin\delta T\cdot\delta_{P2} & (-1)^{m-k}\cos\delta T & 0\\
\frac{1}{T_0^*}\frac{1}{\left(1+\frac{\delta\mathcal{L}}{\mathcal{L}^*}\right)^3} & 0 & -\frac{3\left(1+\frac{\delta T}{T_0^*}\right)}{\left(1+\frac{\delta\mathcal{L}}{\mathcal{L}^*}\right)^4\mathcal{L}^*}
\end{pmatrix}. \tag{35}$$

There is a corollary of Arenstorf's theorem given by Cors et.al. [22],

LEMMA 3 (Cors, Pinyol & Soler). *Let* $\mathbb{U}$ *be an open domain in* $\mathbb{R}^n$, $\mathbb{I}\subset\mathbb{R}$ *an open neighbourhood of the origin and* $\mathbf{f}:\mathbb{U}\times\mathbb{I}\to\mathbb{R}^n$ *with* $\mathbf{f}(\mathbf{0},0)=\mathbf{0}$, *differentiable with respect to* $x\in\mathbb{U}$, *and* $\mathbf{f}_x(\mathbf{0},0)$ *non-singular. Assume that there exist* $c_1>0,c_2>0$ *such that for* $x\in\mathbb{U}$, $\varepsilon\in\mathbb{I}$, *1.* $\|\mathbf{f}_x(x,\varepsilon)-\mathbf{f}_x(\mathbf{0},0)\| \leq c_1(\|x\|+\varepsilon)$,

2. $\|\mathbf{f}(\mathbf{0},\varepsilon)\| \leq c_2\varepsilon$.

Then there exists a function $x(\varepsilon) \in \mathbb{U}$*, defined for* $\varepsilon \in \mathbb{I}' \subset \mathbb{I}$*, such that* $\mathbf{f}(x,\varepsilon) = 0$ *and* $x(0) = 0$.

Suppoe $\tilde{\varepsilon}$ is in a neighborhood of zero and positive, one has

$$\begin{aligned} &\|\Psi_{\mathbf{X}}(\mathbf{X},\tilde{\varepsilon})-\Psi_{\mathbf{X}}(\mathbf{0},0)\| \\ &\leq \|\Psi_{\mathbf{X}}(\mathbf{X},\tilde{\varepsilon})-\Psi_{\mathbf{X}}(\mathbf{X},0)\|+\|\Psi_{\mathbf{X}}(\mathbf{X},0)-\Psi_{\mathbf{X}}(\mathbf{0},0)\| \\ &\leq C_1\tilde{\varepsilon}+C_2\max(\|\delta T\|,\|\delta_{P2}\|,\|\delta\mathcal{L}\|) \leq C_3(\|X\|+\tilde{\varepsilon}), \end{aligned} \tag{36}$$

and also

$$0 < \|\Psi(\mathbf{0},\tilde{\varepsilon})\| < C_4\tilde{\varepsilon}, \tag{37}$$

where C_j $(j = 1,2,3,4)$ are constants greater than zero. One can choose the values of δ_{P1} and δ_{P2}, under the condition of $\delta\mathcal{L} = \delta_{P1}+\delta_{P2}$. After the continuation of $\mathbf{X}$, the averaged initial value Z_0 can be transformed back to the original full system. The conclusion of this paper is

THEOREM 1. *For the spatial Hill's lunar problem with the second primary oblate, there exists a class of doubly-symmetric and near-circular periodic solutions around the oblate primary. These orbits are symmetric with respect to the line joining two primaries, and to a plane. This plane contains that line connecting two primaries, and this plane is perpendicular to the primaries' motion plane.*

E. Discussion

This paper completes a proof on the existence of a class of doubly-symmetric and spatial near-circular periodic solutions in the Hill's lunar problem with the second primary oblate. The method is almost the same as X.B.Xu(2019)[14]. New questions are arised after finishing this paper. For exampler, the stability and the global bifurcations of these orbits are still unknown, one can apply the averaging transforms, the shooting method and the Poincaré cross section method to give some inspirations. It will be interesting to calculate these solutions with the background of astronomy in the future.

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Xingbo Xu,
Faculty of Mathematics and Physics, Institution of Huaiyin Technology
No. 1 of Meicheng Road, 223002 HuaiAn City, China
e-mail: `xbxu@hyit.edu.cn`

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