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RECENT RESULTS ON RAMANUJAN EXPANSIONS WITH APPLICATIONS TO CORRELATIONS

Abstract. This is a survey on some very recent results about Ramanujan expansions and their applications to the representation of shifted convolution sums of two arithmetical functions.

1. Some basic properties of the Ramanujan expansions

The so-called *Ramanujan sum* is [22]

$$c_q(n) \stackrel{def}{=} \sum_{\substack{j\le q\\(j,q)=1}} \cos(2\pi jn/q), \quad q\in\mathbb{N}, n\in\mathbb{Z}.$$

Hereafter, $d=(a,b)$ means that d is the greatest common divisor of a and b.

Note that $|c_q(n)|\le c_q(0)=\varphi(q)\stackrel{def}{=}\#\{j\in\mathbb{N}: j\le q \text{ and } (j,q)=1\}$ for all $q\in\mathbb{N}, n\in\mathbb{Z}$. Moreover, the arithmetic function $n\in\mathbb{N}\mapsto c_q(n)$ is periodic with period q. Throughout the paper, sometimes without further references, we apply other properties of the Ramanujan sums, quoted from [11], [13], [21] and summarized in the next proposition.

PROPOSITION 1. *Let μ be the Möbius function [24] and let $\omega(q)$ denote the number of the prime factors of $q\in\mathbb{N}$.*

① $$c_q(n)=\sum_{\substack{d|q\\d|n}} d\mu(q/d)=\varphi(q)\frac{\mu\big(q/(q,n)\big)}{\varphi\big(q/(q,n)\big)}, \quad \textit{for all } q\in\mathbb{N}, n\in\mathbb{Z}.$$

② $|c_q(n)|\le (q,n)$, *for all* $q,n\in\mathbb{N}$.

③ $\sum_{d|q} c_d(n)=q\mathbf{1}_{q|n}$, *for all* $q\in\mathbb{N}, n\in\mathbb{Z}$,
where $\mathbf{1}_{q|n}$ *is the characteristic function of* $\{n\in\mathbb{Z}:\ n\equiv 0\ (q)\}$.

④ *(Delange inequality)* $$\sum_{d|q}|c_d(n)|\le n\sum_{d|q}\mu(d)^2=n2^{\omega(q)}, \quad \textit{for all } q,n\in\mathbb{N}.$$

⑤ *If* $\ell,q\in\mathbb{N}$, $k\in\mathbb{Z}$, *then* $$\lim_{x\to\infty}\frac{1}{x}\sum_{n\le x}c_\ell(n)c_q(n+k)=\begin{cases} c_\ell(k) & \textit{if } \ell=q,\\ 0 & \textit{otherwise.}\end{cases}$$

REMARK 1. We will write $n\equiv m\ (q)$ to abbreviate $n\equiv m \pmod q$. We denote the characteristic function of any set $U\cap\mathbb{Z}$, with $U\subseteq\mathbb{R}$, by $\mathbf{1}_U$. Equivalently, such a function is denoted by $\mathbf{1}_\wp$, as in ③, provided that $\wp$ is a characteristic property of $U\cap\mathbb{Z}$. Note that the first equality in ① shows that $c_q(n)\in\mathbb{Z}$ for all $q\in\mathbb{N}, n\in\mathbb{Z}$.

DEFINITION 1. *Let* $\mathcal{A} \stackrel{def}{=} \{f : \mathbb{N} \to \mathbb{C}\}$ *be the set of all the arithmetic functions. The* **Ramanujan series** *associated to* $g \in \mathcal{A}$ *is the series*

$$\mathcal{R}_g(n) \stackrel{def}{=} \sum_{q=1}^{\infty} g(q)c_q(n), \quad \text{for all } n \in \mathbb{N}.$$

We say that $f \in \mathcal{A}$ *admits a* **Ramanujan expansion** *if there exists* $g \in \mathcal{A}$ *such that*

$$f = \mathcal{R}_g.$$

For this, $g(q)$ *is the so-called* q*-th* **Ramanujan coefficient** *of the expansion* $f = \mathcal{R}_g$.

All classical expansions have this form (compare [22], [21], [23]).
However, we include the possibility that $g(q)$ also depends on n. As an example, take the following expansion of $0-$function, $\mathbf{0}(n) \stackrel{def}{=} 0$, $\forall n \in \mathbb{N}$ (recall $c_1(n) = 1$, $\forall n \in \mathbb{Z}$) :

$$\mathbf{0}(n) = 2c_1(n) + \left(2 \cdot \mathbf{1}_{n \not\equiv 0(3)} - \mathbf{1}_{n \equiv 0(3)}\right) c_3(n), \quad \forall n \in \mathbb{N}. \tag{1}$$

In case $g(q)$ doesn't depends on n, as n varies in $\mathbb{N}$, we call $f = \mathcal{R}_g$ a **pure** expansion.

Given any $f \in \mathcal{A}$ (except $f = \mathbf{0}$, see Remark 4), we don't know if it has a pure expansion or not. (As far as we know no such a result is in the literature.)

However, all $f \in \mathcal{A}$ have at least one $n-$pointwise convergent $f(n) = \mathcal{R}_g(n)$, as $n \in \mathbb{N}$ (and it's finite, see Remark 3).

Consequently, we introduce

$$\langle f \rangle \stackrel{def}{=} \{g \in \mathcal{A} : f = \mathcal{R}_g\} \quad \text{the } \textit{Ramanujan } \textbf{cloud} \text{ of } f,$$

which is always non-empty, and its "pure part" (that can be empty)

$$\langle f \rangle_* \stackrel{def}{=} \{g \in \langle f \rangle : \mathcal{R}_g \text{ is pure}\}.$$

Assume that $\langle f \rangle_* \neq \emptyset$. We call the expansion $f = \mathcal{R}_g$ **completely uniform** if it's pure and the convergence of $f(n) = \mathcal{R}_g(n)$ is uniform w.r.t. n, as n varies in $\mathbb{N}$. We write

$$\langle f \rangle_{**} \stackrel{def}{=} \{g \in \langle f \rangle : \mathcal{R}_g \text{ is completely uniform}\}.$$

Let us write $f =_{\#} \mathcal{R}_g$ to mean that $\mathcal{R}_g$ is a finite sum. Assuming that f and g are not the identically zero functions, such a finite expansion, if pure, can be written in the form

$$f(n) = \mathcal{R}_g(n) = \sum_{q \leq Q} g(q)c_q(n), \qquad \forall n \in \mathbb{N},$$

where $Q \stackrel{def}{=} \max\{q \in \mathbb{N} : g(q) \neq 0\}$ is the so-called **length** of $\mathcal{R}_g(n)$. (Compare Remark 3 about finite expansions.) We indicate $\langle f \rangle_{\#} \stackrel{def}{=} \{g \in \langle f \rangle : \mathcal{R}_g \text{ is finite}\}$.

Henceforth, $\mathcal{R}$*-expansion* and $\mathcal{R}$*-coefficient* abbreviate *Ramanujan expansion* and *Ramanujan coefficient*, respectively. Sometimes, we'll abbreviate $\mathcal{R}-$cloud for the Ramanujan cloud.

REMARK 2. The reader is cautioned that some authors refer to Ramanujan series as *Fourier-Ramanujan* series ([15], [23]). Moreover, in the literature *$\mathcal{R}$-expansion* is often synonymous of Ramanujan series. Here we explicitly point out that by definition a $\mathcal{R}$-expansion is a convergent Ramanujan series taken as a representation of its sum (compare Remark 4 for its non-uniqueness). Further, it should be plain that in the present context Ramanujan sum cannot be synonymous of finite $\mathcal{R}$-expansion.

REMARK 3. A celebrated theorem of Hildebrand [18] ensures that for every $f \in \mathcal{A}$ and $n \in \mathbb{N}$ there exist $Q(n) \in \mathbb{N}$ and $h(q,n) \in \mathbb{C}$, with $1 \le q \le Q(n)$, such that

$$f(n) = \sum_{q=1}^{Q(n)} h(q,n)c_q(n) \qquad \forall n \in \mathbb{N}. \tag{2}$$

See also [23] for the proof, where the coefficients $h(q,n)$ are recursively defined. In other words, Hildebrand's result yields that $\langle f \rangle_{\#} \neq \emptyset$ for all $f \in \mathcal{A}$, implying that all $\mathcal{R}-$clouds are non-empty; however, it leaves open the possibilty that some $\langle f \rangle_* = \emptyset$. Note that for every $f \in \mathcal{A}$ there are always expansions of the form (2), where the dependence of $Q(n)$ and the coefficients $h(q,n)$ on n is effective. Indeed, besides the trivial choices of $Q(n) = 1$ and $h(1,n) = f(n)$, the expansion (2) holds also by taking

$$Q(n) = n, \quad h(q,n) = \sum_{\substack{d \le n \\ d \equiv 0 (q)}} \frac{(f * \mu)(d)}{d},$$

where $*$ denotes the Dirichlet product [24]. (See the last line of the proof of the Wintner-Delange formula in Proposition 2 below.) The latter case yields the so-called *standard finite $\mathcal{R}$-expansion* of f, $\forall f \in \mathcal{A}$. This, in turn, proves (2) immediately.

REMARK 4. Since the first appearance of the Ramanujan series, it was clear at once that the $\mathcal{R}$-expansion of a given arithmetical function is very far from being unique. Indeed, besides the trivial fact that the identically zero function $\mathbf{0}$ belongs to $\langle \mathbf{0} \rangle$, non-trivial $\mathcal{R}$-expansions of $\mathbf{0}$ were found by Ramanujan himself [22] and Hardy [16], respectively as

$$\mathbf{0}(n) = \sum_{q=1}^{\infty} R_0(q)c_q(n), \qquad \text{where} \quad R_0(q) \stackrel{def}{=} \frac{1}{q}, \tag{3}$$

$$\mathbf{0}(n) = \sum_{q=1}^{\infty} H_0(q)c_q(n), \qquad \text{where} \quad H_0(q) \stackrel{def}{=} \frac{1}{\varphi(q)}. \tag{4}$$

Further samples of Ramanujan expansions are found in [23], [19], [21]. Moreover, it is plain that $\alpha\langle \mathbf{0} \rangle \subseteq \langle \mathbf{0} \rangle$ for all $\alpha \in \mathbb{C}$. Furthermore, for any $g \in \langle f \rangle$ one has

$$g + \langle \mathbf{0} \rangle \stackrel{def}{=} \{h \in \mathcal{A} : \ h = g + k \text{ for some } k \in \langle \mathbf{0} \rangle\} \subseteq \langle f \rangle.$$

Together with the aforementioned Hildebrand's result, this implies that the set $\langle f\rangle$ is infinite for any $f\in\mathcal{A}$. Namely, all $\mathcal{R}-$clouds contain infinitely many expansions each. Further, they are convex sets : $\alpha g_1+(1-\alpha)g_2\in\langle f\rangle$, $\forall g_1,g_2\in\langle f\rangle$ and $\forall\alpha\in\mathbb{C}$.
Some of the recent results concern the problem of the unique representation of the expansion $f=\mathcal{R}_g$, namely the search for suitable requirements on g that would yield uniqueness of such expansion. These results are discussed in §3.

REMARK 5. The convergence of a $\mathcal{R}$-expansion needs not be absolute. Indeed, since $|c_q(n)|=\mu(q)^2$ for $(q,n)=1$ (see Proposition 1), then for any fixed integer n we can write

$$\sum_{q=1}^{\infty}\frac{|c_q(n)|}{q}\geq\sum_{\substack{q=1\\(q,n)=1}}^{\infty}\frac{\mu(q)^2}{q}\geq\sum_p\frac{1}{p}-\sum_{p|n}\frac{1}{p}.$$

Hereafter, the letter p, with or without subscripts, is devoted to prime numbers. Thus, the absolute divergence of the series (3) follows from the well-known divergence of the series of prime numbers reciprocals.

2. The Wintner coefficients and the Carmichael coefficients

DEFINITION 2. *The Eratosthenes transform of f is $f'\in\mathcal{A}$ such that $f=f'*\mathbf{1}$, i.e.*

$$f(n)=\sum_{d|n}f'(d),\quad\forall n\in\mathbb{N}.$$

In particular, if for every $n\in\mathbb{N}$ and for some $Q\in\mathbb{N}$ independent of n one has

$$f(n)=\sum_{\substack{d\leq Q\\d|n}}f'(d),$$

then f is said to be a truncated divisor sum of range Q. We also say that f' is the Q-truncated Eratosthenes transform of f. The set of the truncated divisor sums of range Q is denoted by $\mathcal{A}_Q$.

REMARK 6. Henceforth, *E-transform* means *Eratosthenes transform*. Note that we have already abbreviated Wintner's terminology, where f' is used to be called the *Eratosthenes-Möbius transform* of f (see [23], [25]), being plain that $f'=f*\mu$ by the Möbius inversion formula [24]. We also refer to f as the *inverse* E-transform of f'. Finally, by definition the truncated divisor sum of range Q *associated* to $f=f'*\mathbf{1}$ is

$$f_Q(n)\stackrel{def}{=}\sum_{\substack{d\leq Q\\d|n}}f'(d),\quad\forall n\in\mathbb{N}.$$

In other words, the E-transform of f_Q is f' in $[1,Q]\cap\mathbb{N}$ and 0 otherwise. In §3, the set $\mathcal{A}_Q$ is characterized in terms of some peculiar $\mathcal{R}$-expansions (see Theorem 3).

DEFINITION 3. *Let f' be the E-transform of $f \in \mathcal{A}$.*
If $\sum_{\substack{d=1 \\ d\equiv 0\,(q)}}^{\infty} \frac{f'(d)}{d}$ *converges, then its sum* $\mathcal{W}_f(q)$ *is the q-th* **Wintner coefficient** *of* f.

If $\mathcal{M}(f\cdot c_q) \overset{def}{=} \lim_{x\to\infty} \frac{1}{x}\sum_{n\le x} f(n)c_q(n)$ *exists and is finite, then* $\mathcal{C}_f(q) \overset{def}{=} \frac{\mathcal{M}(f\cdot c_q)}{\varphi(q)}$ *is the q-th* **Carmichael coefficient** *of* f.

REMARK 7. If it exists and is finite, then $\mathcal{M}(f\cdot c_q)$ is the so-called *mean value* of $f\cdot c_q$. Namely, $\mathcal{M}(f) = \mathcal{M}(f\cdot c_1) = \mathcal{C}_f(1) = \lim_{x\to\infty} \frac{1}{x}\sum_{n\le x} f(n)$ is the mean value of f.

The next proposition summarizes three classical results. In the first part we spot *Wintner's criterion* [23], a sufficient condition for the mere existence of both Wintner and Carmichael coefficients, which turn out to be equal. The second part of the proposition is the Wintner-Delange theorem [13], that provides with a sufficient condition for a given $f \in \mathcal{A}$ to be such that $\langle f\rangle_* \neq \emptyset$. In particular, such a theorem reveals that the Wintner coefficients (or equivalently the Carmichael ones in view of Wintner's criterion) are instances of $\mathcal{R}$-coefficients for f. The third part is Lucht's theorem [19] that gives a deep link between the $\mathcal{R}$-expansion of a function and its E-transform. In §3 we present a new result yielding the converse of Lucht's theorem. Such a result is a key argument for the problem of the unique $\mathcal{R}$-expansion (see Remark 4). In what follows, for any $f,g \in \mathcal{A}$ with g real and non-negative, the notation $f(n) \ll g(n)$, equivalent to $f(n) = O(g(n))$, means that there exist $n_0 \in \mathbb{N}$ and a real number $C > 0$ such that $|f(n)| \le Cg(n)$ for all $n > n_0$. The implicit constant C might depend on other variables, in which case they are displayed as subscripts in the symbols $\ll$ or O.

PROPOSITION 2. *Let f' be the E-transform of $f \in \mathcal{A}$.*

① Wintner's criterion. *If* $\sum_{d=1}^{\infty} \frac{f'(d)}{d}$ *converges absolutely, then* $\mathcal{W}_f(q)$ *and* $\mathcal{C}_f(q)$ *exist for all* $q \in \mathbb{N}$. *Moreover, one has* $\mathcal{W}_f = \mathcal{C}_f$.

② The Wintner-Delange formula. *If* $\sum_{d=1}^{\infty} 2^{\omega(d)}\frac{f'(d)}{d}$ *converges absolutely, then the function* $\mathcal{W}_f = \mathcal{C}_f$ *belongs to* $\langle f\rangle_*$ *:*

$$f(n) = \sum_{q=1}^{\infty} \mathcal{W}_f(q)c_q(n) = \sum_{q=1}^{\infty} \mathcal{C}_f(q)c_q(n), \quad \forall n \in \mathbb{N}.$$

③ Lucht's theorem. *If* $\sum_{\substack{q=1 \\ q\equiv 0\,(d)}}^{\infty} g(q)\mu(q/d)$ *converges for every* $d \in \mathbb{N}$, *then* $g \in \langle f\rangle$, *where f is the inverse E-transform of* $f'(d) \overset{def}{=} d \sum_{\substack{q=1 \\ q\equiv 0(d)}}^{\infty} g(q)\mu(q/d)$, *i.e.*

$$f(n) = \sum_{d|n} f'(d) = \sum_{d|n} d \sum_{\substack{q=1 \\ q\equiv 0(d)}}^{\infty} g(q)\mu(q/d) = \mathcal{R}_g(n) \quad \forall n \in \mathbb{N}.$$

Proof. ① Clearly, the existence of $\mathcal{W}_f(q)$ for all $q \in \mathbb{N}$ is a straightforward consequence of the hypothesis. Thus, we have to show that for any fixed $q \in \mathbb{N}$ one has

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} f(n)c_q(n) = \varphi(q)\mathcal{W}_f(q).$$

To this end, after recalling that $[\beta]$ and $\|\beta\|$ denote respectively the integer part of $\beta \in \mathbb{R}$ and the distance of β from the nearest integer, let us write

$$\begin{aligned}
\sum_{n\le x} f(n)c_q(n) &= \sum_{n\le x} c_q(n) \sum_{d|n} f'(d) = \sum_{d\le x} f'(d) \sum_{m\le x/d} c_q(dm) \\
&= \sum_{d\le x} f'(d) \sum_{\substack{j\le q \\ (j,q)=1}} \sum_{m\le x/d} e_q(jdm) \\
&= \varphi(q) \sum_{\substack{d\le x \\ d\equiv 0(q)}} f'(d)\left[\frac{x}{d}\right] + O\Bigg(\sum_{\substack{d\le x \\ d\not\equiv 0(q)}} |f'(d)| \sum_{\substack{j\le q \\ (j,q)=1}} \left\|\frac{jd}{q}\right\|^{-1} \Bigg),
\end{aligned}$$

where we have applied the well-known inequality (see [12], Ch.26)

$$\sum_{m\le x} e(m\beta) \ll \min(x, \|\beta\|^{-1}), \quad \forall x \ge 1, \forall \beta \in \mathbb{R}.$$

Since the O-term vanishes for $q=1$, we can assume $q>1$ henceforth. We see that

$$\begin{aligned}
\frac{1}{x}\sum_{n\le x} f(n)c_q(n) = \varphi(q) \sum_{\substack{d\le x \\ d\equiv 0(q)}} \frac{f'(d)}{d} \; &+ \; O\Bigg(\frac{\varphi(q)}{x} \sum_{d\le x} |f'(d)|\Bigg) \\
&+ \; O\Bigg(\frac{1}{x} \sum_{\substack{d\le x \\ d\not\equiv 0(q)}} |f'(d)| \sum_{\substack{j\le q \\ (j,q)=1}} \left\|\frac{jd}{q}\right\|^{-1}\Bigg).
\end{aligned}$$

Now, let us write

$$\sum_{\substack{d\le x \\ d\not\equiv 0(q)}} |f'(d)| \sum_{\substack{j\le q \\ (j,q)=1}} \left\|\frac{jd}{q}\right\|^{-1} \le \sum_{\substack{r<q \\ r|q}} \sum_{\substack{d\le x \\ (d,q)=r}} |f'(d)| \sum_{\substack{j\le q \\ j\not\equiv 0(q/r)}} \left\|\frac{jd/r}{q/r}\right\|^{-1}$$

and note that, since $(d,q)=r$ yields $(d/r, q/r)=1$, it turns out that (compare [20],§3.2)

$$\sum_{\substack{j\le q \\ j\not\equiv 0(q/r)}} \left\|\frac{jd/r}{q/r}\right\|^{-1} \le r \sum_{j'<q/r} \left\|\frac{j'}{q/r}\right\|^{-1} \ll q \sum_{j'<q/r} \frac{1}{j'} \ll q\log q, \quad \forall r|q, r\ne q.$$

Thus, we get

$$\frac{1}{x}\sum_{n\le x} f(n)c_q(n) = \varphi(q) \sum_{\substack{d\le x\\ d\equiv 0\,(q)}} \frac{f'(d)}{d} + O_q\left(\frac{1}{x}\sum_{d\le x}|f'(d)|\right). \tag{5}$$

Hence, the desired conclusion follows once it is shown that the O-term goes to zero as $x\to\infty$. To this end, by partial summation [24] we write

$$\sum_{d\le x}|f'(d)| = \sum_{d\le x}\frac{|f'(d)|}{d} + \int_1^x\left(\sum_{d\le x}\frac{|f'(d)|}{d} - \sum_{d\le t}\frac{|f'(d)|}{d}\right)\mathrm{d}t.$$

Note that by hypothesis, for any fixed real number $\varepsilon>0$, there exists $x_\varepsilon<x$ such that

$$\left|\sum_{d\le x}\frac{|f'(d)|}{d} - \sum_{d\le t}\frac{|f'(d)|}{d}\right| < \varepsilon,\ \forall t\in(x_\varepsilon, x).$$

Consequently, for all $x>x_\varepsilon$ one has

$$\begin{aligned}\frac{1}{x}\sum_{d\le x}|f'(d)| &< \frac{1}{x}\sum_{d\le x}\frac{|f'(d)|}{d} + \frac{1}{x}\int_1^{x_\varepsilon}\left|\sum_{d\le x}\frac{|f'(d)|}{d} - \sum_{d\le t}\frac{|f'(d)|}{d}\right|\mathrm{d}t + \varepsilon\\ &\le \frac{1+2x_\varepsilon}{x}\sum_{d=1}^{\infty}\frac{|f'(d)|}{d} + \varepsilon.\end{aligned}$$

② It is plain that the hypothesis and ① yield that $\mathcal{W}_f = \mathcal{C}_f$. From ④ of Prop. 1 we get

$$\begin{aligned}\sum_{q\le x}|\mathcal{W}_f(q)c_q(n)| &\le \sum_{q\le x}|c_q(n)|\sum_{d\equiv 0\,(q)}\frac{|f'(d)|}{d} = \sum_{d=1}^{\infty}\frac{|f'(d)|}{d}\sum_{\substack{q|d\\ q\le x}}|c_q(n)|\\ &\le n\sum_{d=1}^{\infty}\frac{2^{\omega(d)}}{d}\,\left|f'(d)\right|,\end{aligned}$$

where the double series on d and q converges absolutely because the latter series converges by hypothesis. (In particular, $\mathcal{R}_{\mathcal{W}_f}(n)$ is absolutely convergent for any fixed n.) Hence, we can exchange d and q summations, and apply ③ of Proposition 1 so that

$$\mathcal{R}_{\mathcal{W}_f}(n) = \sum_{q=1}^{\infty}\mathcal{W}_f(q)c_q(n) = \sum_{d=1}^{\infty}\frac{f'(d)}{d}\sum_{q|d}c_q(n) = \sum_{d=1}^{\infty}f'(d)\mathbf{1}_{d|n} = f(n),\ \ \forall n\in\mathbb{N}.$$

③ For $x\ge n$, from ① of Proposition 1 we get

$$\sum_{q\le x}g(q)c_q(n) = \sum_{d|n}d\sum_{\substack{q\le x\\ q\equiv 0\,(d)}}g(q)\mu(q/d),$$

yielding the conclusion immediately. The proposition is completely proved. □

REMARK 8. We underline that the absolute convergence of $\mathcal{R}_{\mathcal{W}_f}$ alone does not suffice to conclude that $f = \mathcal{R}_{\mathcal{W}_f}$. For example, if $\mathcal{R}_{\mathcal{W}_f}(n)$ is a finite sum, i.e. there exists $Q \in \mathbb{N}$ such that $\mathcal{W}_f(q) = 0$ for all $q > Q$, then obviously its convergence is absolute. However, the argument used to prove that $f = \mathcal{R}_{\mathcal{W}_f}$ in ② is no longer helpful because ③ of Proposition 1 cannot apply to

$$\mathcal{R}_{\mathcal{W}_f}(n) = \sum_{d=1}^{\infty} \frac{f'(d)}{d} \sum_{\substack{q \le Q \\ q|d}} c_q(n).$$

The reader should compare this case with Remark 14 after Theorem 3 below.

REMARK 9. We give many small new results, now.
Note that $|c_q(n)| \le \varphi(q)$ yields

$$\frac{1}{\varphi(q)} \left| \frac{1}{x} \sum_{n \le x} f(n) c_q(n) \right| \le \frac{1}{x} \sum_{n \le x} |f(n)|.$$

Hence, if $\mathcal{C}_f(q)$ and the mean value of $|f|$ exist, then $|\mathcal{C}_f(q)| \le \mathcal{M}(|f|) = \mathcal{C}_{|f|}(1)$. As a consequence, if $\mathcal{C}_f(q)$ exists for all $q \in \mathbb{N}$, then $\mathcal{M}(|f|) = 0$ implies $\mathcal{C}_f = \mathbf{0}$. In particular, this means that *a non-negative real function $f \ne \mathbf{0}$ with a null mean value does not admit its Carmichael coefficients as $\mathcal{R}$-coefficients, i.e. $\mathcal{C}_f \notin \langle f \rangle$*. Samples of such functions are the characteristic functions of subsets of $\mathbb{N}$ with zero density. Indeed, recalling that the density of $B \subseteq \mathbb{N}$ is

$$\delta(B) \stackrel{def}{=} \lim_{x \to \infty} \frac{\#\{n \in B : n \le x\}}{x} \in [0,1]$$

(provided that such a limit exists), this can be equivalently written as

$$\delta(B) = \lim_{x \to \infty} \frac{1}{x} \sum_{n \le x} \mathbf{1}_B(n) = \mathcal{C}_B(1),$$

where the first Carmichael coefficient of $\mathbf{1}_B$ is shortly denoted $\mathcal{C}_B(1)$. In particular, the set of prime numbers has zero density (esp., from the prime number theorem).
Further, it is well-known [T] that the inverse E-transform of the Liouville function, i.e.,

$$\lambda(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_t^{\alpha_t}) \stackrel{def}{=} (-1)^{\alpha_1 + \alpha_2 + \dots + \alpha_t},$$

is the characteristic function of the set S of the square numbers, that has plainly zero density, i.e. $\mathcal{C}_S = \mathbf{0}$; a theorem of Landau and von Mangoldt states that $\sum_{d=1}^{\infty} \frac{\lambda(d)}{d} = 0$ is equivalent to the prime number theorem. Thus, being λ completely multiplicative, we see that $\sum_{\substack{d=1 \\ d \equiv 0(q)}}^{\infty} \frac{\lambda(d)}{d} = \frac{\lambda(q)}{q} \sum_{d=1}^{\infty} \frac{\lambda(d)}{d} = 0$ for all $q \in \mathbb{N}$, i.e. $\mathcal{W}_S = \mathbf{0} = \mathcal{C}_S$.

On the other hand, $|\mathbf{1}'_S| = |\lambda| = \mathbf{1}$ doesn't satisfy Wintner's criterion hypothesis.

More in general, if $f \in \mathcal{A}$ is such that f' is completely multiplicative (c.m.), then

$$\mathcal{W}_f(q) = \sum_{d\equiv 0\,(q)} \frac{f'(d)}{d} = \frac{f'(q)}{q}\sum_{m=1}^{\infty}\frac{f'(m)}{m} = \frac{f'(q)}{q}\mathcal{W}_f(1), \ \ \forall q \in \mathbb{N}.$$

Consequently,

$$f' \ \text{ c.m.}, \ \ \mathcal{W}_f(1) = 0 \implies \mathcal{W}_f = \mathbf{0}$$

and also

$$f' \ \text{ c.m.}, \ \ \mathcal{W}_f(1) \neq 0 \ \text{ and } \ \mathcal{W}_f(q) = 0, \forall q > Q \implies f'(q) = 0, \forall q > Q.$$

Furthermore, it is easily seen that (notice that here f' is not necessarily c.m.)

$$f' \geq 0 \ \text{ and } \ \mathcal{W}_f(q) = 0, \forall q > Q \implies f'(q) = 0, \ \ \forall q > Q.$$

These properties suggest the following

Conjecture: *If $f \in \mathcal{A}$ is such that $\mathcal{W}_f(1) \neq 0$, then*

$$\mathcal{W}_f(q) = 0, \ \ \forall q > Q \implies f'(q) = 0, \ \ \forall q > Q.$$

In §4 it is shown how such a conjecture might replace the Delange hypothesis on the series $\sum_{d=1}^{\infty} 2^{\omega(d)} f'(d)/d$ within Proposition 2 in pursuing the Wintner-Delange formula for the shifted convolution sums.

REMARK 10. Formula (5) reveals that if $\mathcal{C}_{|f'|}(1) = 0$, i.e.

$$\sum_{d\leq x} |f'(d)| = o(x), \quad \text{as } x \to \infty, \tag{6}$$

then $\mathcal{C}_f(q)$ exists if and only if $\mathcal{W}_f(q)$ does. Further, if this is the case, then $\mathcal{C}_f(q) = \mathcal{W}_f(q)$. From the proof of Wintner's criterion it transpires that the absolute convergence of $\sum_{d=1}^{\infty}\frac{f'(d)}{d}$ implies (6), which alone however does not yield the existence of the Wintner coefficients; on the other hand, by taking $f'(d) = 1/\log(d+1)$ it is easily seen that the converse of such an implication is not true. Moreover, by taking f' as the Liouville function λ, it is plain that (6) does not hold, while the characteristic function of square numbers $\mathbf{1}_S = \lambda * \mathbf{1} = f' * \mathbf{1} = f$, say, satisfies the hypotheses of the next proposition, that is a result of Delange (see the theorem and remark 1.5 in [14]).

PROPOSITION 3. *Let $f \in \mathcal{A}$ and $q \in \mathbb{N}$ be such that $\sum_{n\leq x} |f(n)| = O(x)$ and $\mathcal{C}_f(d)$ exists for all $d|q$. Then $\mathcal{C}_f(q) = \mathcal{W}_f(q)$.*

In particular, by taking $q=1$, this result yields that if $\sum_{n\le x}|f(n)|=O(x)$ and there exists the mean value $\mathcal{M}(f)=\mathcal{C}_f(1)$, then

$$\mathcal{M}(f)=\sum_{d=1}^{\infty}\frac{f'(d)}{d}.$$

3. Uniqueness for Ramanujan coefficients

Here we quote from [2] the next theorem, that provides with a kind of converse of Lucht's theorem (see ③ of Proposition 2).

THEOREM 1. *Let $f\in\mathcal{A}$ be such that $\langle f\rangle_*\neq\emptyset$.*
① *For any given $g\in\langle f\rangle_*$ the E-transform of f is*

$$f':d\in\mathbb{N}\to f'(d)=d\sum_{\substack{q=1\\ q\equiv 0\,(d)}}^{\infty}g(q)\mu(q/d).$$

② *If $g\in\langle f\rangle_*$ is such that*

$$\sum_{q=1}^{\infty}2^{\omega(q)}g(q)\ \textit{converges absolutely}, \tag{7}$$

then $g=\mathcal{W}_f$.

Proof. ① We can exchange the sums in

$$\sum_{d|n}d\sum_{\substack{q=1\\ q\equiv 0\,(d)}}^{\infty}g(q)\mu(q/d)$$

because g does not depend on n by hypothesis. Thus, from ① of Proposition 1 for $x\ge n$ we get that

$$\sum_{d|n}d\sum_{\substack{q\le x\\ q\equiv 0\,(d)}}g(q)\mu(q/d)=\sum_{q\le x}g(q)c_q(n).$$

As $x\to\infty$, it follows that

$$\sum_{d|n}d\sum_{\substack{q=1\\ q\equiv 0\,(d)}}^{\infty}g(q)\mu(q/d)=\mathcal{R}_g(n)=f(n),$$

yielding that the *E*-transform of f is the claimed f'.
② From ① one has that

$$\mathcal{W}_f(q)=\sum_{\substack{d=1\\ d\equiv 0\,(q)}}^{\infty}\frac{f'(d)}{d}=\sum_{\substack{d=1\\ d\equiv 0\,(q)}}^{\infty}\sum_{k=1}^{\infty}\mu(k)g(dk)\quad\forall q\in\mathbb{N}.$$

Now, by using the well-known property [24]

$$\sum_{k|n}\mu(k)=\begin{cases}1 & \text{if } n=1,\\ 0 & \text{otherwise,}\end{cases}$$

for any $q\in\mathbb{N}$ we can write

$$g(q)=\sum_{n=1}^{\infty}g(qn)\sum_{k|n}\mu(k),$$

that converges unconditionally because of (7). Indeed, since $\omega(qn)\geq\omega(n)$ yields $2^{\omega(qn)}\geq 2^{\omega(n)}$, one has

$$\begin{aligned}\sum_{n=1}^{\infty}|g(qn)|\sum_{k|n}\mu^2(k) &= \sum_{n=1}^{\infty}2^{\omega(n)}|g(qn)|\leq\sum_{n=1}^{\infty}2^{\omega(qn)}|g(qn)|\\ &= \sum_{\substack{m=1\\ m\equiv 0(q)}}^{\infty}2^{\omega(m)}|g(m)|\leq\sum_{m=1}^{\infty}2^{\omega(m)}|g(m)|.\end{aligned}$$

Therefore, we can exchange the sums to get

$$\begin{aligned}g(q)=\sum_{n=1}^{\infty}g(qn)\sum_{k|n}\mu(k) &= \sum_{k=1}^{\infty}\mu(k)\sum_{m=1}^{\infty}g(qmk)=\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\mu(k)g(qmk)\\ &= \sum_{\substack{d=1\\ d\equiv 0(q)}}^{\infty}\sum_{k=1}^{\infty}\mu(k)g(dk)=\mathcal{W}_f(q).\end{aligned}$$

The theorem is completely proved. □

REMARK 11. Since *any* $g\in\langle f\rangle_*$ determines the E-transform of f, the previous theorem establishes the uniqueness of g *modulo* $\langle\mathbf{0}\rangle_*$.

We refer to ② of the previous theorem as the *Wintner-Delange uniqueness formula* and (7) is called the *"Dual" Delange condition*. In fact, from Theorem 1 it follows that $\{g\in\langle f\rangle_*: g \text{ satisfies } (7)\}$ is either the empty set or $\{\mathcal{W}_f\}$.

REMARK 12. While it is plain that $\langle\mathbf{0}\rangle_{**}\neq\emptyset$ (for $\mathbf{0}\in\langle\mathbf{0}\rangle_{**}$), it might be possible that $\langle f\rangle_{**}=\emptyset$ for some $f\in\mathcal{A}$. Indeed, recall that there is the possibility that the larger set $\langle f\rangle_*$ might be empty because the (finite) $\mathcal{R}$-expansions ensured by Hildebrand's theorem are not necessarily pure. The next theorem, quoted from a lemma of [11], gives another positive answer to the uniqueness question posed in Remark 4. (First one coming from Theorem 1.) In particular, it implies $\langle\mathbf{0}\rangle_{**}=\{\mathbf{0}\}$.

THEOREM 2. *For any $f\in\mathcal{A}$, either $\langle f\rangle_{**}$ is the empty set or $\{\mathcal{C}_f\}$.*

Proof. We have to show that if $g\in\langle f\rangle_{**}$, then

$$g(\ell)=\frac{1}{\varphi(\ell)}\lim_{x\to\infty}\frac{1}{x}\sum_{h\leq x}f(h)c_\ell(h)\quad\forall\ell\in\mathbb{N}.$$

Let $\ell \in \mathbb{N}$ be fixed. Note that, from the uniform convergence of the $\mathcal{R}$-expansion $f = \mathcal{R}_g$, it follows that for every $\varepsilon > 0$ there exists $Q = Q(\varepsilon, \ell) > \ell$ such that

$$\Big| \sum_{q>Q} g(q) c_q(h) \Big| < \frac{\varepsilon}{\mathbf{d}(\ell)}, \quad \forall h \in \mathbb{N},$$

where $\mathbf{d}(\ell) \stackrel{def}{=} \sum_{t|\ell} 1$ is the number of positive divisors of ℓ. Since the expansion $f = \mathcal{R}_g$ is also pure, this entails

$$\frac{1}{x} \sum_{h \le x} f(h) c_\ell(h) = \sum_{q \le Q} g(q) \frac{1}{x} \sum_{h \le x} c_\ell(h) c_q(h) + \frac{1}{x} \sum_{h \le x} c_\ell(h) \sum_{q>Q} g(q) c_q(h).$$

Recalling ④ of Proposition 1, from which in particular one has that

$$\lim_{x \to \infty} \frac{1}{x} \sum_{h \le x} c_\ell(h)^2 = \varphi(\ell),$$

and applying $|c_\ell(h)| \le (\ell, h)$ (see ② of Proposition 1), we can write

$$\begin{aligned}
\Bigg| \frac{1}{\varphi(\ell)} \lim_{x \to \infty} \frac{1}{x} \sum_{h \le x} f(h) c_\ell(h) - \frac{1}{\varphi(\ell)} \sum_{q \le Q} g(q) \lim_{x \to \infty} \frac{1}{x} \sum_{h \le x} c_\ell(h) c_q(h) \Bigg| &= \\
\Bigg| \frac{1}{\varphi(\ell)} \lim_{x \to \infty} \frac{1}{x} \sum_{h \le x} f(h) c_\ell(h) - g(\ell) \Bigg| &\le \\
\frac{\varepsilon}{\varphi(\ell) \mathbf{d}(\ell)} \lim_{x \to \infty} \frac{1}{x} \sum_{h \le x} (\ell, h).&
\end{aligned}$$

Therefore, the conclusion follows once it is proved that

$$\lim_{x \to \infty} \frac{1}{x} \sum_{h \le x} (\ell, h) = \sum_{d|\ell} \frac{\varphi(d)}{d}.$$

To this end, we write

$$\begin{aligned}
\frac{1}{x} \sum_{h \le x} (\ell, h) &= \frac{1}{x} \sum_{t|\ell} t \sum_{\substack{h' \le \frac{x}{t} \\ (h', \frac{\ell}{t}) = 1}} 1 = \frac{1}{x} \sum_{t|\ell} t \sum_{d | \frac{\ell}{t}} \mu(d) \left[\frac{x}{dt} \right] \\
&= \sum_{t|\ell} \sum_{d | \frac{\ell}{t}} \frac{\mu(d)}{d} + O\left(\frac{1}{x} \sum_{t|\ell} t \mathbf{d}(\ell/t) \right) \\
&= \sum_{t|\ell} \frac{\varphi(\ell/t)}{\ell/t} + o(1) = \sum_{d|\ell} \frac{\varphi(d)}{d} + o(1).
\end{aligned}$$

The theorem is completely proved. □

REMARK 13. To emphasize the fact that the $\mathcal{R}$-coefficients of f are uniquely determined, we set

$$\widehat{f} \stackrel{def}{=} C_f;$$

or also, in the hypotheses of Theorem 1,

$$\widehat{f} \stackrel{def}{=} \mathcal{W}_f.$$

More generally, we write $\widehat{f} \stackrel{def}{=} g$ even if $\langle f \rangle = g + \langle \mathbf{0} \rangle$ or $\langle f \rangle_* = g + \langle \mathbf{0} \rangle_*$.

The next theorem yields the uniqueness of the $\mathcal{R}$-coefficients for pure and finite $\mathcal{R}$-expansions (see the following remark).

THEOREM 3. *$f \in \mathcal{A}_Q \Leftrightarrow \exists g \in \langle f \rangle_* \cap \langle f \rangle_\#$ such that $f =_\# \mathcal{R}_g$ has length at most Q.*

Proof. Let f' be the Q-truncated E-transform of f. By ③ of Proposition 1 we see that

$$f(n) = \sum_{\substack{d \le Q \\ d|n}} f'(d) = \sum_{d \le Q} \frac{f'(d)}{d} \sum_{q|d} c_q(n) = \sum_{q \le Q} \mathcal{W}_f(q) c_q(n),$$

where

$$\mathcal{W}_f(q) \stackrel{def}{=} \begin{cases} \displaystyle\sum_{\substack{d \le Q \\ d \equiv 0\,(q)}} \frac{f'(d)}{d} & \text{if } q \le Q, \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

is a *Q-truncated* Wintner coefficient, say. It is plain that $\mathcal{W}_f \in \langle f \rangle_* \cap \langle f \rangle_\#$.
Vice versa, let $g \in \langle f \rangle_* \cap \langle f \rangle_\#$ be such that the length of the expansion $f(n) =_\# \mathcal{R}_g(n)$ is at most Q for all $n \in \mathbb{N}$. By applying ① of Proposition 1 we write

$$f(n) = \sum_{q \le Q} g(q) c_q(n) = \sum_{d|n} d \sum_{\substack{q \le Q \\ q \equiv 0\,(d)}} g(q) \mu(q/d) = \sum_{\substack{d \le Q \\ d|n}} f'(d),$$

where we have set

$$f'(d) \stackrel{def}{=} \begin{cases} d \displaystyle\sum_{\substack{q \le Q \\ q \equiv 0\,(d)}} g(q) \mu(q/d) & \text{if } d \le Q, \\ 0 & \text{otherwise.} \end{cases}$$

The theorem is completely proved. □

REMARK 14. Theorems 2 and 3 imply that $\langle f \rangle_* \cap \langle f \rangle_\# = \{C_f\} = \{\mathcal{W}_f\}$ with $\mathcal{W}_f$ defined as in (8). In particular, note that

$$\frac{Q}{2} < q \le Q \Longrightarrow \widehat{f}(q) = \mathcal{W}_f(q) = \frac{f'(q)}{q}$$

(for $q > Q/2$, the conditions $d \le Q$, $q|d$ hold simultaneously if and only if $d = q$). Also, see that if we assume the conjecture in Remark 9 for $f \in \mathcal{A}$ with $\mathcal{W}_f(1) \neq 0$, from Theorems 2 and 3 we get

$$\mathcal{W}_f(q) = 0 \; \forall q > Q \Longrightarrow f \in \mathcal{A}_Q \Longrightarrow f =_{\#} \mathcal{R}_{\widehat{f}},$$

where it turns out that $\widehat{f} = \mathcal{C}_f = \mathcal{W}_f$ is given by (8).
Finally, we underline the fact that the above proof provides with an explicit method to express a truncated divisor sum as a finite $\mathcal{R}$-expansion, and vice versa.

4. Ramanujan expansions of shifted convolution sums

DEFINITION 4. *The* **correlation** *(or shifted convolution sum) of $f,g \in \mathcal{A}$ is*

$$C_{f,g}(N,a) \stackrel{def}{=} \sum_{n \le N} f(n)g(n+a).$$

Since without loss of generality one can assume that $f(N)g(N+a) \neq 0$, the number N is the length of such a correlation. Here $a \in \mathbb{N}$ is the **shift***.*

From (2) it follows that

$$C_{f,g}(N,a) = \sum_q h(a,q)c_q(a), \;\; \forall a \in \mathbb{N}, \tag{9}$$

for some $h(a,q) = h(a,q,f,g,N) \in \mathbb{C}$. We refer to (9) as the *shift $\mathcal{R}$-expansion* of the correlation $C_{f,g}$. On the other hand, denoting by f', g' the E-transforms of f,g, respectively, we see that

$$C_{f,g}(N,a) = \sum_{n \le N} \sum_{d|n} f'(d) \sum_{q|n+a} g'(q), \tag{10}$$

where observe that the conditions $n \le N$ and $d|n$ yield $d \le N$ in the second sum, while $n \le N$ and $q|n+a$ yield $q \le N+a$ in the third sum. In other words, within their correlation of length N, the functions f and g can be replaced respectively by the truncated divisor sums associated to f and g, of range respectively N and $N+a$, i.e.

$$f_N(n) \stackrel{def}{=} \sum_{\substack{d \le N \\ d|n}} f'(d), \qquad g_{N+a}(n+a) \stackrel{def}{=} \sum_{\substack{q \le N+a \\ q|n+a}} g'(q).$$

These functions admit pure (w.r.t n and $n+a$, respectively) and finite $\mathcal{R}$-expansions because of Theorem 3 (see also Remark 14). However, by plugging such expansions into (10) we can only get a finite expansion as

$$C_{f,g}(N,a) = \sum_{d \le N} \sum_{q \le N+a} \widehat{f_N}(d)\widehat{g_{N+a}}(q) \sum_{n \le N} c_d(n)c_q(n+a).$$

Evidently, the latter cannot be considered as a shift $\mathcal{R}$-expansion of the form (9). On the other hand, by combining ④ of Proposition 1 with the Wintner-Delange formula of Proposition 2, the $\mathcal{R}$-expansions of the above truncated divisor sums can help in finding a pure and finite $\mathcal{R}$-expansion, which well approximates $C_{f,g}(N,a)$, provided that this correlation is *fair* and both functions f and g satisfy the *Ramanujan Conjecture* (see [10], [11]), accordingly to the following definitions.

DEFINITION 5. *The correlation $C_{f,g}(N,a)$ of $f,g\in\mathcal{A}$ is fair if it depends on a only because of the argument $n+a$ of g.*

DEFINITION 6. *We say that $f\in\mathcal{A}$ satisfies the Ramanujan Conjecture (or equivalently that f is essentially bounded) if $f(n)\ll_\varepsilon n^\varepsilon$ for any real number $\varepsilon>0$. In this case, we also write $f\lll 1$. We denote the set of the essentially bounded arithmetic functions by $\mathcal{A}^\varepsilon$.*

For example, the correlation $C_{f,g}(N,a)$ is not fair if the support of f or g depends on a (see [11] for a specific example). In particular, note that the expression (10) is not fair in that the support of g' does depend on a. If $f,g\in\mathcal{A}^\varepsilon$ (consequently, also $f',g'\in\mathcal{A}^\varepsilon$), then somehow we can get rid of such a nuisance by writing

$$\begin{aligned} C_{f,g}(N,a) &= \sum_{n\le N}\sum_{d|n} f'(d)\sum_{\substack{q\le N\\ q|n+a}} g'(q)+\sum_{n\le N}\sum_{d|n} f'(d)\sum_{\substack{N<q\le N+a\\ q|n+a}} g'(q)\\ &= \sum_{n\le N}\sum_{d|n} f'(d)\sum_{\substack{q\le N\\ q|n+a}} g'(q)+\sum_{N<q\le N+a} g'(q)\sum_{\substack{n\le N\\ n\equiv -a\,(q)}}\sum_{d|n} f'(d). \end{aligned}$$

Since for $q>N$ one has that $\#\{n\le N: n\equiv -a\ (q)\}\le 1$, the second sum on the right hand side is $\ll aN^\varepsilon \max_{N<q\le N+a}|g'(q)|\max_{n\le N}|f'(n)|$. In all, we have for $f,g\in\mathcal{A}^\varepsilon$ that

$$C_{f,g}(N,a)=C_{f,g_N}(N,a)+O_\varepsilon(N^\varepsilon(N+a)^\varepsilon a).$$

Thus, we are reduced to deal with the correlation of the truncated divisor sums f_N, g_N of the same range N. For this reason we'll assume that $g\in\mathcal{A}_N$ (and, concerning $C_{f,g}(N,a)$, the hypothesis $f\in\mathcal{A}$ is equivalent to $f\in\mathcal{A}_N$). Using the finite $\mathcal{R}$-expansion of length at most N for g (see Theorem 3), namely $g(m)=\sum_{q\le N}\widehat{g}(q)c_q(m)$,

$$C_{f,g}(N,a)=\sum_{q\le N}\widehat{g}(q)\sum_{n\le N} f(n)c_q(n+a),\quad \forall a\in\mathbb{N}. \tag{11}$$

Note that $C_{f,g}(N,a)$ is fair, provided that neither $\widehat{g}$ nor f depends on a. By using ⑤ of Proposition 1 we calculate its Carmichael coefficients (we set $\mathcal{C}_C=\mathcal{C}_{C_{f,g}}$ for brevity):

$$\begin{aligned} \mathcal{C}_C(N,\ell) &= \frac{1}{\varphi(\ell)}\sum_{q\le N}\widehat{g}(q)\sum_{n\le N} f(n)\lim_{x\to\infty}\frac{1}{x}\sum_{a\le x} c_q(n+a)c_\ell(a)\\ &= \begin{cases} \dfrac{\widehat{g}(\ell)}{\varphi(\ell)}\displaystyle\sum_{n\le N} f(n)c_\ell(n) & \text{if } \ell\le N,\\ 0 & \text{if } \ell>N. \end{cases} \end{aligned} \tag{12}$$

Assuming that the E-transform $C'_{f,g}$ satisfies the Delange hypothesis (see Prop. 2), i.e.

$$\sum_{d} \frac{2^{\omega(d)} C'_{f,g}(N,d)}{d} \quad \text{converges absolutely,} \tag{13}$$

Proposition 2, Theorems 2 and 3 yield the so-called *Ramanujan exact explicit formula* (REEF)

$$C_{f,g}(N,a) = \sum_{q \le N} \mathcal{C}_C(N,q) c_q(a) = \sum_{q \le N} \left(\frac{\widehat{g}(q)}{\varphi(q)} \sum_{n \le N} f(n) c_q(n) \right) c_q(a), \ \ \forall a \in \mathbb{N}.$$

Without having (13) at our disposal we can proceed as it follows. Let us write

$$C_{f,g}(N,a) = \sum_{\substack{d \le N \\ d|a}} C'_{f,g}(N,d) + \sum_{\substack{d > N \\ d|a}} C'_{f,g}(N,d) = \sum\nolimits_I(a) + \sum\nolimits_{II}(a), \text{ say,}$$

where clearly $\sum_{II}(a) = 0$, unless $a > N$. Since $\sum_I(a)$ is a truncated divisor sum, from Theorem 3 we get

$$C_{f,g}(N,a) = \sum_{q \le N} \mathcal{W}_C(N,q) c_q(a) + \sum\nolimits_{II}(a)$$

with

$$\mathcal{W}_C(N,q) \stackrel{def}{=} \begin{cases} \displaystyle\sum_{\substack{h \le N \\ h \equiv 0 (q)}} \frac{C'_{f,g}(N,h)}{h} & \text{if } q \le N, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, calculating the Carmichael coefficients of

$$\sum_{q \le N} \mathcal{W}_C(N,q) c_q(a) = \begin{cases} C_{f,g}(N,a) - \sum_{II}(a) & \text{if } a > N, \\ C_{f,g}(N,a) & \text{if } a \le N, \end{cases}$$

from (12) it follows that (recall the discussion on the *uniqueness* in Remark 14)

$$\begin{aligned} \mathcal{W}_C(N,q) = \ & \frac{1}{\varphi(q)} \lim_{x \to \infty} \frac{1}{x} \sum_{m \le N} C_{f,g}(N,m) c_q(m) + \\ & \frac{1}{\varphi(q)} \lim_{x \to \infty} \frac{1}{x} \sum_{N < m \le x} \Big(C_{f,g}(N,m) - \sum_{\substack{d > N \\ d|m}} C'_{f,g}(N,d) \Big) c_q(m) \\ = \ & \mathcal{C}_C(N,q) - \mathcal{L}(N,q), \ \ \forall q \in \mathbb{N}, \end{aligned}$$

where

$$\mathcal{L}(N,q) = \mathcal{L}(f,g,N,q) \stackrel{def}{=} \frac{1}{\varphi(q)} \lim_{x \to \infty} \frac{1}{x} \sum_{N < m \le x} c_q(m) \sum_{\substack{d > N \\ d|m}} C'_{f,g}(N,d), \ \ \forall q \in \mathbb{N}.$$

In particular, $\mathcal{C}_C(N,q) = \mathcal{L}(N,q)$, $\forall q > N$, because $\mathcal{W}_C(N,q) = 0$, $\forall q > N$.

Therefore, under the only hypothesis that the correlation $C_{f,g}$ is fair, we obtain

$$\begin{aligned} C_{f,g}(N,a) &= \sum_{q\le N}\big(\mathcal{C}_C(N,q)-\mathcal{L}(N,q)\big)c_q(a)+\sum\nolimits_{II}(a) \\ &= \sum_{q\le N}\left(\frac{\widehat{g}(q)}{\varphi(q)}\sum_{n\le N} f(n)c_q(n)-\mathcal{L}(N,q)\right)c_q(a)+\sum\nolimits_{II}(a),\ \ \forall a\in\mathbb{N}, \end{aligned}$$

where

$$\sum\nolimits_{II}(a)\overset{def}{=}\begin{cases}\sum\limits_{\substack{d>N\\ d|a}} C'_{f,g}(N,d) & \text{if } a>N,\\ 0 & \text{if } a\le N.\end{cases}$$

Hence, the following theorem and corollary are proved.

THEOREM 4. *If $f\in\mathcal{A}$ and $g\in\mathcal{A}_N$ are such that $C_{f,g}(N,a)$ is fair, then*

$$C_{f,g}(N,a)=\sum_{q\le N}\left(\frac{\widehat{g}(q)}{\varphi(q)}\sum_{n\le N} f(n)c_q(n)-\mathcal{L}(N,q)\right)c_q(a)+\sum\nolimits_{II}(a),$$

where $\sum_{II}(a)$ is defined above,

$$\mathcal{L}(N,q)=\mathcal{L}(f,g,N,q)\overset{def}{=}\frac{1}{\varphi(q)}\lim_{x\to\infty}\frac{1}{x}\sum_{N<m\le x} c_q(m)\sum_{\substack{d>N\\ d|m}} C'_{f,g}(N,d),$$

and $C'_{f,g}$ is the E-transform of $C_{f,g}$.

In particular, if $\sum_{d=1}^{\infty}\frac{2^{\omega(d)}}{d}|C'_{f,g}(N,d)|$ converges, then for all $a\in\mathbb{N}$ one has the REEF:

$$C_{f,g}(N,a)=\sum_{q\le N}\widehat{C_{f,g}}(N,q)c_q(a),\ \ \textit{with}\ \ \widehat{C_{f,g}}(N,q)=\frac{\widehat{g}(q)}{\varphi(q)}\sum_{n\le N} f(n)c_q(n).$$

COROLLARY 1. *Let $f,g\in\mathcal{A}^{\varepsilon}$. If $C_{f,g_N}(N,a)$ is fair and such that the series $\sum_{d=1}^{\infty}\frac{2^{\omega(d)}}{d}|C'_{f,g_N}(N,d)|$ converges, then for all $a\in\mathbb{N}$ one has*

$$C_{f,g}(N,a)=\sum_{q\le N}\left(\frac{\widehat{g_N}(q)}{\varphi(q)}\sum_{n\le N} f(n)c_q(n)\right)c_q(a)+O_{\varepsilon}(N^{\varepsilon}(N+a)^{\varepsilon}a).$$

REMARK 15. Given $f\in\mathcal{A}$ and $g\in\mathcal{A}_N$, assuming that $\sum_{d=1}^{\infty}\frac{C'_{f,g}(N,d)}{d}$ converges absolutely, from Wintner's criterion it follows that $C_{f,g}(N,a)$ has both Carmichael and Wintner coefficients with $\mathcal{C}_C(N,q)=\mathcal{W}_C(N,q)$ for all $q\in\mathbb{N}$. In particular, (12) yields that $\mathcal{W}_C(N,q)=0$ for all $q>N$. From this, if we further assume the conjecture formulated in Remark 9, we get that $C'_{f,g}(N,d)=0$ for all $d>N$.

Besides the consequence that the above series reduces to the finite sum of length at most N, such a conjecture yields the REEF without the Delange hypothesis (13). In other words, such a conjecture is an alternative way to get the REEF of Theorem 4.

REMARK 16. Assume that $f \in \mathcal{A}$ and $g \in \mathcal{A}_Q$, with $Q \leq N$, are such that $C_{f,g}(N,a)$ is fair. From (11) it follows that $C_{f,g}(N,a)$ is periodic with respect to a, which implies that it is a bounded arithmetic function of a. Together with (12), this reveals that $C_{f,g}(N,a)$ satisfies the hypotheses of Proposition 3, so that its Carmichael coefficients coincide with its Wintner ones.

Now, let us quote here the main result of [11].

THEOREM 5. *Let $f \in \mathcal{A}^\varepsilon$ and $g \in \mathcal{A}_N \cap \mathcal{A}^\varepsilon$ be such that $C_{f,g}(N,a)$ is fair for all $a \in \mathbb{N}$ and admits the shift $\mathcal{R}$-expansion (9). The following propositions are equivalent:*

① *The shift $\mathcal{R}$-expansion (9) is completely uniform, i.e. it is pure, with $h(a,q) = h(q)$, and converges uniformly with respect to a.*

② *The coefficients of (9) are the Carmichael coefficients of $C_{f,g}(N,a)$, i.e.*

$$h(a,q) = h(q) = \frac{1}{\varphi(q)} \lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} C_{f,g}(N,n)c_q(n).$$

③ *The coefficients of (9) are the REEF coefficients of $C_{f,g}(N,a)$, i.e.*

$$h(a,q) = h(q) = \widehat{C_{f,g}}(N,q) = \frac{\widehat{g}(q)}{\varphi(q)} \sum_{n\leq N} f(n)c_q(n).$$

④ *The shift $\mathcal{R}$-expansion (9) is finite and pure.*

We underline the latter equivalence between the condition $\langle C_{f,g}\rangle_* \cap \langle C_{f,g}\rangle_\# \neq \emptyset$ and the REEF. It highlights the fundamental role of Theorem 3 in this new approach to the $\mathcal{R}$-expansions of the correlations. Moreover, Corollary 1 has to be compared to the following result that can be proved in a similar fashion of Corollary 1 in [11].

THEOREM 6. *Let $g \in \mathcal{A}^\varepsilon$ and $f \in \mathcal{A}_D \cap \mathcal{A}^\varepsilon$, with $D < N^{1-\delta}$ for some $\delta \in (0,1)$. If $C_{f,g_N}(N,a)$ is fair and such that $\sum_{d=1}^{\infty} \frac{2^{\omega(d)}}{d} |C'_{f,g_N}(N,d)|$ converges, then uniformly for all $a \in \mathbb{N}$ one has*

$$C_{f,g}(N,a) = \mathfrak{S}_{f,g}(a)N + O\big(N^{1-\delta}\big) + O_\varepsilon\left(N^\varepsilon (N+a)^\varepsilon a\right),$$

where $\mathfrak{S}_{f,g}$ is the so-called **singular sum** *defined as*

$$\mathfrak{S}_{f,g}(a) \stackrel{def}{=} \sum_{q\leq N} \hat{f}(q)\hat{g}(q)c_q(a), \quad \forall a \in \mathbb{N}.$$

The elements of $\mathcal{A}_D \cap \mathcal{A}^\varepsilon$ are known as *sieve functions* (of range D). We refer the reader to [4]-[9] for further deepenings about such a class of functions.

We conclude the present section by quoting a result from [1] on the convolution sum of the von Mangoldt function Λ. Indeed, in [1] Corollary 1 is applied by taking $f = g = \Lambda$, that clearly belongs to $\mathcal{A}^{\varepsilon}$. From the well-known property [24]

$$\Lambda(n) = -\sum_{d|n} \mu(d) \log d,$$

it follows that its E-transform is $\Lambda'(n) = -\mu(n)\log n$. Further, for the N-truncated divisor sum of Λ we have (see Theorem 3)

$$\Lambda_N(n) = -\sum_{\substack{d \le N \\ d|n}} \mu(d) \log d = \sum_{q \le N} \widehat{\Lambda_N}(q) c_q(n),$$

with

$$\widehat{\Lambda_N}(q) \stackrel{def}{=} -\sum_{\substack{d \le N \\ d \equiv 0 \, (q)}} \frac{\mu(d) \log d}{d} \ll \frac{L^2}{q},$$

where we have set $L \stackrel{def}{=} \log N$. Therefore, since $C_{\Lambda,\Lambda_N}(N,a)$ is fair, by assuming that $\sum_{d=1}^{\infty} \frac{2^{\omega(d)}}{d} C'_{\Lambda,\Lambda_N}(N,d)$ converges absolutely, Corollary 1 yields

$$C_{\Lambda,\Lambda}(N,a) = \sum_{q \le N} \left(\frac{\widehat{\Lambda_N}(q)}{\varphi(q)} \sum_{n \le N} \Lambda(n) c_q(n) \right) c_q(a) + O_{\varepsilon}(N^{\varepsilon}(N+a)^{\varepsilon} a).$$

The result established in [1] shows that the Delange hypotesis for $C_{\Lambda,\Lambda_N}(N,2k)$ yields the Hardy-Littlewood conjecture for the $2k$-twin primes [17]. Here it is stated as a further corollary of Theorem 4.

COROLLARY 2. *Let $k \in \mathbb{N}$ be such that $0 < k < N^{1-\delta}$, with $\delta \in (0,1/2)$ fixed. If $\sum_{d=1}^{\infty} \frac{2^{\omega(d)}}{d} |C'_{\Lambda,\Lambda_N}(N,d)|$ converges, then*

$$C_{\Lambda,\Lambda}(N,2k) = \mathfrak{S}_{\Lambda,\Lambda}(2k) N + O(N e^{-c\sqrt{\log N}}),$$

where $c > 0$ is an absolute constant and

$$\mathfrak{S}_{\Lambda,\Lambda}(2k) \stackrel{def}{=} \sum_{q=1}^{\infty} \frac{\mu^2(q)}{\varphi^2(q)} c_q(2k) = 2 \prod_{p|k} \left(1 + \frac{1}{p-1}\right) \prod_{(p,2k)=1} \left(1 - \frac{1}{(p-1)^2}\right).$$

5. Ramanujan expansions and smooth numbers

In the present section we resume some results of [3], where it is showed that all essentially bounded functions, with E-transform supported on smooth numbers, admit a unique $\mathcal{R}$-expansion with coefficients satisfying the "Dual" Delange condition (7).

DEFINITION 7. *Let $Q \geq 2$ be an integer.*
The set of the Q-smooth positive integers is $\mathcal{S} = \mathcal{S}(Q) \stackrel{def}{=} \{n \in \mathbb{N} : \ p|n \Rightarrow p \leq Q\} \cup \{1\}$.
The set of the Q-sifted positive integers is $\mathcal{T} = \mathcal{T}(Q) \stackrel{def}{=} \{n \in \mathbb{N} : \ p \leq Q \Rightarrow p \not| n\}$.

REMARK 17. Note that $\mathcal{S} \cap \mathcal{T} = \{1\}$ and $(n,m) = 1$ for all $n \in \mathcal{S}, m \in \mathcal{T}$. Further, any $n \in \mathcal{S}$ can be written as $n = p_1^{v_1} \cdots p_r^{v_r}$, for some integers $v_j \geq 0$, $j = 1, \ldots, r = \pi(Q)$, where $\pi(Q) = \#\{p \leq Q : \ p \text{ prime}\}$ and $2 = p_1, p_2, \ldots, p_r$ are all the consecutive prime numbers $\leq Q$. Thus, for any real number $x > 1$ and for all $\varepsilon > 0$ one has

$$\#\mathcal{S} \cap [1,x] \leq \sum_{\substack{n \in \mathcal{S} \\ n \leq x}} \frac{x^\varepsilon}{n^\varepsilon} \ll x^\varepsilon \sum_{v_1=0}^{\infty} \cdots \sum_{v_r=0}^{\infty} \frac{1}{p_1^{\varepsilon v_1}} \cdots \frac{1}{p_r^{\varepsilon v_r}} = x^\varepsilon \prod_{p \leq Q} \frac{1}{1 - p^{-\varepsilon}} \ll_{\varepsilon,Q} x^\varepsilon.$$

Similarly, if $\varepsilon \in (0,1)$, we see that

$$\sum_{m \in \mathcal{S}} \frac{1}{m^{1-\varepsilon}} \ll_{\varepsilon,Q} 1.$$

Moreover, from the Legendre formula applied to $\#\mathcal{T} \cap [1,x] = \#\{n \leq x : \ (n, P_Q) = 1\}$, where $P_Q = \prod_{p \leq Q} p$, it follows that

$$\#\mathcal{T} \cap [1,x] = \sum_{d|P_Q} \mu(d) \left[\frac{x}{d}\right] = x \sum_{d|P_Q} \frac{\mu(d)}{d} + O\Big(\sum_{d|P_Q} |\mu(d)|\Big) = x \prod_{p \leq Q} \left(1 - \frac{1}{p}\right) + O_Q(1).$$

DEFINITION 8. *Let $Q \geq 2$ be an integer. The Q-smooth restriction of $f \in \mathcal{A}$ is the arithmetic function defined as*

$$f_{\mathcal{S}}(n) \stackrel{def}{=} \sum_{\substack{d|n \\ d \in \mathcal{S}}} f'(d), \ \ \forall n \in \mathbb{N},$$

where f' is the E-trasform of f.

REMARK 18. It is plain that $f_{\mathcal{S}}$ is the inverse E-trasform of $f' \cdot \mathbf{1}_{\mathcal{S}}$, where $\mathbf{1}_{\mathcal{S}}$ is the characteristic function of $\mathcal{S}$. Also note that $f_{\mathcal{S}}(n) = f(n)$ for all $n \in \mathcal{S}$. Further, one has

$$f_{\mathcal{S}}(n) = \sum_{t \in \mathcal{S}_n} f(t), \ \text{ where } \mathcal{S}_n \stackrel{def}{=} \{t \in \mathcal{S} : \ t|n \text{ and } n/t \in \mathcal{T}\}.$$

Indeed, this is trivially true for $n = 1$.
If $n \geq 2$, recall that $f' = f * \mu$ and use the (complete) multiplicativity of $\mathbf{1}_{\mathcal{S}}$, to get it:

$$\begin{aligned} f_{\mathcal{S}}(n) &= \sum_{\substack{d|n \\ d \in \mathcal{S}}} \sum_{t|d} f(t) \mu\left(\frac{d}{t}\right) = \sum_{\substack{t \in \mathcal{S} \\ t|n}} f(t) \sum_{\substack{k \in \mathcal{S} \\ k|\frac{n}{t}}} \mu(k) \\ &= \sum_{\substack{t \in \mathcal{S} \\ t|n}} f(t) \sum_{k|\frac{n}{t}} \mu(k) \mathbf{1}_{\mathcal{S}}(k) = \sum_{\substack{t \in \mathcal{S} \\ t|n}} f(t) \prod_{p|\frac{n}{t}} \big(1 - \mathbf{1}_{\mathcal{S}}(p)\big) = \sum_{\substack{t \in \mathcal{S} \\ t|n}} f(t) \mathbf{1}_{\mathcal{T}}(n/t). \end{aligned}$$

LEMMA 1. *Given any integer $Q \ge 2$, let us consider the set $\mathcal{S} = \mathcal{S}(Q)$ of the Q-smooth positive integers. For any $f \in \mathcal{A}^{\varepsilon}$, with $f' = f * \mu$, one has*

① $$\sum_{t\in\mathcal{S}} \frac{|f(t)c_q(t)|}{t} \ll_{\varepsilon,q,Q} 1 \quad \textit{for all} \quad q \in \mathbb{N}$$

② $$\sum_{t\in\mathcal{S}} \frac{|f'(t)|}{t} \ll_{\varepsilon,Q} 1 \qquad \textit{and} \qquad \sum_{t\in\mathcal{S}} \frac{2^{\omega(t)}|f'(t)|}{t} \ll_{\varepsilon,Q} 1.$$

Proof. Without loss of generality, we can assume that $\varepsilon \in (0,1)$.
① Using ② of Prop. 1 and the inequality $\sum_{m\in\mathcal{S}} m^{\varepsilon-1} \ll_{\varepsilon,Q} 1$ (see Remark 17), we get

$$\sum_{t\in\mathcal{S}} \frac{|f(t)c_q(t)|}{t} \ll_{\varepsilon} \sum_{t\in\mathcal{S}} (q,t)t^{\varepsilon-1} \ll_{\varepsilon} \sum_{\substack{d\in\mathcal{S}\\ d|q}} d \sum_{\substack{t\in\mathcal{S}\\ t\equiv 0(d)}} t^{\varepsilon-1} \ll_{\varepsilon} \sum_{\substack{d\in\mathcal{S}\\ d|q}} d^{\varepsilon} \sum_{m\in\mathcal{S}} m^{\varepsilon-1} \ll_{\varepsilon,q,Q} 1.$$

② Recalling that $f' \in \mathcal{A}^{\varepsilon}$ and arguing as before, we see that

$$\sum_{t\in\mathcal{S}} \frac{|f'(t)|}{t} \ll_{\varepsilon} \sum_{t\in\mathcal{S}} t^{\varepsilon-1} \ll_{\varepsilon,Q} 1.$$

Since $2^{\omega(t)} \le 2^{\pi(Q)}$ for all $t \in \mathcal{S}$, the second inequality follows from the first one. □

THEOREM 7. *Let $Q \ge 2$ be an integer. For any $f \in \mathcal{A}^{\varepsilon}$, let us consider the Q-smooth restriction $f_{\mathcal{S}}$, where $\mathcal{S} = \mathcal{S}(Q)$ is the set of the Q-smooth positive integers. The Carmichael coefficients of $f_{\mathcal{S}}$ and the Wintner ones coincide, both given by*

$$\widehat{f_{\mathcal{S}}}(q) = \begin{cases} \dfrac{\tilde{f}(q,\mathcal{S})}{\varphi(q)} \displaystyle\prod_{p\le Q}\left(1-\frac{1}{p}\right) = \sum_{\substack{d\in\mathcal{S}\\ d\equiv 0\,(q)}} \frac{f'(d)}{d} & \textit{if } q\in\mathcal{S}, \\ 0 & \textit{otherwise,} \end{cases} \tag{14}$$

where we set

$$\tilde{f}(q,\mathcal{S}) \stackrel{def}{=} \sum_{t\in\mathcal{S}} \frac{f(t)c_q(t)}{t}.$$

Further, one has $\widehat{f_{\mathcal{S}}} \in \langle f_{\mathcal{S}} \rangle$, i.e.

$$f_{\mathcal{S}}(a) = \sum_{q\in\mathcal{S}} \widehat{f_{\mathcal{S}}}(q)c_q(a), \quad \forall a \in \mathbb{N}, \tag{15}$$

and $\widehat{f_{\mathcal{S}}}$ satisfies the "Dual" Delange condition (7).

Proof. Without loss of generality, we can assume that $\varepsilon \in (0,1/2)$. First, note that $\tilde{f}(q,\mathcal{S})$ is well-defined for all $q \in \mathbb{N}$ because of ① in Lemma 1. Then, recalling that $f' \cdot \mathbf{1}_{\mathcal{S}}$ is the E-trasform of $f_{\mathcal{S}}$, the second inequality in ② of Lemma 1 implies that the hypothesis of the Wintner-Delange formula holds for $f_{\mathcal{S}}$.

Therefore, it follows from ② of Proposition 2 that Carmichael coefficients of $f_{\mathcal{S}}$ equal Wintner ones and they are $\mathcal{R}$-coefficients for $f_{\mathcal{S}}$. Being such coefficients uniquely determined (see the next remark), we denote the qth coefficient by $\widehat{f_{\mathcal{S}}}(q)$. In particular, since it is plain that the conditions $d \in \mathcal{S}$ and $q|d$ imply that $q \in \mathcal{S}$, the qth Wintner coefficient of $f_{\mathcal{S}}$ is

$$\widehat{f_{\mathcal{S}}}(q) = \begin{cases} \displaystyle\sum_{d \equiv 0\,(q)} \frac{(f' \cdot \mathbf{1}_{\mathcal{S}})(d)}{d} = \sum_{\substack{d \in \mathcal{S} \\ d \equiv 0\,(q)}} \frac{f'(d)}{d} & \text{if } q \in \mathcal{S}, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, recalling that $\sum_{m \in \mathcal{S}} m^{\varepsilon - 1} \ll_{\varepsilon,Q} 1$ (see Remark 17), we see that

$$\begin{aligned} \sum_{q=1}^{\infty} 2^{\omega(q)} |\widehat{f_{\mathcal{S}}}(q)| &= \sum_{q \in \mathcal{S}} 2^{\omega(q)} \left| \sum_{\substack{d \in \mathcal{S} \\ d \equiv 0\,(q)}} \frac{f'(d)}{d} \right| \\ &\le 2^{\pi(Q)} \sum_{q \in \mathcal{S}} \sum_{\substack{d \in \mathcal{S} \\ d \equiv 0\,(q)}} \frac{|f'(d)|}{d} \ll_{\varepsilon,Q} \sum_{q \in \mathcal{S}} q^{\varepsilon - 1} \sum_{k \in \mathcal{S}} k^{\varepsilon - 1} \ll_{\varepsilon,Q} 1, \end{aligned}$$

that is $\widehat{f_{\mathcal{S}}}$ satisfies (7). Thus, it remains to prove that for every $q \in \mathcal{S}$ the qth Carmichael coefficient of $f_{\mathcal{S}}$ is

$$\frac{1}{\varphi(q)} \prod_{p \le Q} \left(1 - \frac{1}{p}\right) \sum_{t \in \mathcal{S}} \frac{f(t) c_q(t)}{t}.$$

From Remark 18, $f_{\mathcal{S}}(a) = \sum_{t \in \mathcal{S}_a} f(t)$ and $\mathcal{S}_a \overset{def}{=} \{t \in \mathcal{S} :\ t|a \text{ and } a/t \in \mathcal{T}\}$, whence

$$\sum_{a \le x} f_{\mathcal{S}}(a) c_q(a) = \sum_{\substack{t \in \mathcal{S} \\ t \le x}} f(t) \sum_{\substack{k \in \mathcal{T} \\ k \le x/t}} c_q(tk) = \sum_{\substack{t \in \mathcal{S} \\ t \le x}} f(t) c_q(t) \# \mathcal{T} \cap [1, x/t],$$

where we exchange sums and $c_q(tk) = c_q(t)$ follows from the conditions $q \in \mathcal{S}$, $k \in \mathcal{T}$, which yield $(q,k) = 1$. Now, since (see Remark 17)

$$\# \mathcal{T} \cap [1, x/t] = \frac{x}{t} \prod_{p \le Q} \left(1 - \frac{1}{p}\right) + O_Q(1),$$

for every $q \in \mathcal{S}$ the qth Carmichael coefficient of $f_{\mathcal{S}}$ is given by

$$\begin{aligned} \mathcal{C}_{f_{\mathcal{S}}}(q) &= \frac{1}{\varphi(q)} \lim_{x \to \infty} \frac{1}{x} \sum_{a \le x} f_{\mathcal{S}}(a) c_q(a) \\ &= \frac{1}{\varphi(q)} \prod_{p \le Q} \left(1 - \frac{1}{p}\right) \sum_{t \in \mathcal{S}} \frac{f(t) c_q(t)}{t} + \frac{1}{\varphi(q)} \lim_{x \to \infty} \sum_{\substack{t \in \mathcal{S} \\ t \le x}} f(t) c_q(t) O_Q(1/x). \end{aligned}$$

The conclusion follows once we show that the latter limit is 0. Recalling that $f \in \mathcal{A}^{\varepsilon}$, $\#\mathcal{S} \cap [1,x] \ll_{\varepsilon,Q} x^{\varepsilon}$ (see Remark 17), and applying ② of Proposition 1, we see that

$$\sum_{\substack{t\in\mathcal{S}\\ t\le x}} f(t)c_q(t)O_Q(1/x) \ll_{\varepsilon,Q} x^{\varepsilon-1} \sum_{\substack{t\in\mathcal{S}\\ t\le x}} (q,t) \ll_{\varepsilon,Q} x^{\varepsilon-1} \sum_{d|q} d\#\mathcal{S}\cap[1,x/d] \ll_{\varepsilon,q,Q} x^{2\varepsilon-1}.$$

The theorem is completely proved. □

REMARK 19. The previous theorem and Theorem 1 yield that if $g \in \langle f_{\mathcal{S}} \rangle_*$ satisfies the "Dual" Delange condition (7), then $g = \widehat{f}_{\mathcal{S}}$. In other words, the $\mathcal{R}$-coefficients of a smooth restriction of an essentially bounded function are uniquely determined. But even more important, since $f_{\mathcal{S}}(a) = f(a)$ for all $a \in \mathcal{S}$, from (15) we obtain the $\mathcal{R}$-expansion for the restriction of f to $\mathcal{S}$:

$$f(a) = \sum_{q=1}^{\infty} \widehat{f}_{\mathcal{S}}(q)c_q(a), \quad \forall a \in \mathcal{S}, \tag{16}$$

with the coefficients $\widehat{f}_{\mathcal{S}}(q)$ defined by (14). In particular, given $f \in \mathcal{A}$ and $g \in \mathcal{A}_Q$, with $Q \le N$, a fair correlation $C_{f,g}(N,a)$ for all $a \in \mathbb{N}$, being bounded (see Remark 16), satisfies the hypotheses of the previous theorem. Thus, from (16) we get the unique $\mathcal{R}$-expansion

$$\begin{aligned} C_{f,g}(N,a) &= \sum_{q\in\mathcal{S}} \sum_{\substack{d\in\mathcal{S}\\ d\equiv 0\,(q)}} \frac{C'_{f,g}(N,d)}{d} c_q(a) \\ &= \prod_{p\le Q}\left(1-\frac{1}{p}\right) \sum_{q\in\mathcal{S}} \frac{c_q(a)}{\varphi(q)} \sum_{t\in\mathcal{S}} \frac{C_{f,g}(N,t)c_q(t)}{t}, \quad \forall a \in \mathcal{S}. \end{aligned}$$

(It is easily see that such an expansion holds also if $f,g \in \mathcal{A}^{\varepsilon}$.) On the other hand, in view of Theorem 4, the conditions $f \in \mathcal{A}$, $g \in \mathcal{A}_Q$, with $Q \le N$, and $C_{f,g}(N,a)$ fair, do not suffice to get the REEF for such a correlation on the Q-smooth positive integers, i.e.

$$C_{f,g}(N,a) = \sum_{\ell\le Q} \left(\frac{\widehat{g}(\ell)}{\varphi(\ell)} \sum_{n\le N} f(n)c_\ell(n) \right) c_\ell(a), \ \ \forall a \in \mathcal{S}.$$

In [3] it is provided the following counterexample. For a fixed $q_0 \in [3,Q]\cap\mathbb{N}$, let us take $n_0 \in [1,N]\cap\mathbb{N}$ with $n_0 \equiv -1\,(q_0)$ and define $f,g \in \mathcal{A}$ as $f(n) = \mathbf{1}_{\{n_0\}}(n)$, $g(n) = c_{q_0}(n)$, $\forall n \in \mathbb{N}$.

It is easily seen that $g \in \mathcal{A}_Q$ and $C_{f,g}(N,a)$ is fair. However, it turns out that

$$C_{f,g}(N,1) = \varphi(q_0) \ne \frac{\mu(q_0)^2}{\varphi(q_0)} = \sum_{\ell\le Q} \left(\frac{\widehat{g}(\ell)}{\varphi(\ell)} \sum_{n\le N} f(n)c_\ell(n) \right) c_\ell(a).$$

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