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ON THE MAXIMAL FINITE IWASAWA SUBMODULE IN $\mathbb{Z}_P$-EXTENSIONS AND CAPITULATION OF IDEALS

Abstract. For $\mathbb{Z}_p$-extensions of a number fields the properties of stabilization and capitulation of ideal classes are of great interest and are also related to very important aspects and problems such as Greenberg's conjecture. In [3] these properties are deeply investigated from the point of view of the maximum finite submodule of the Iwasawa module and new invariants and parameters are introduced to give precise characterizations of these phenomena. In this article we will discuss some bounds that control the increment of the index that measures the capitulation delay in the tower and moreover we will prove how some results on the capitulation kernels in [3] have to be considered optimal. Finally, we will also give some further applications and examples that emphasize the cases of false (or failed) stabilization in this context.

1. Introduction.

Iwasawa's theory in the last half century has been one of the richest areas of research in number theory. In this paper we will consider some basic objects of the theory such as $\mathbb{Z}_p$-extensions and the Iwasawa module with particular emphasis on its maximum finite submodule, in relation to the problems of capitulation and stabilization, typical of this context.

Let p be a prime number, k a number field and K/k a $\mathbb{Z}_p$-extension of k. Let moreover $L = L(K)$ be the maximal abelian unramified pro-p extension of K (in a fixed algebraic closure of $\mathbb{Q}$) and we also pose $\Gamma := \mathrm{Gal}(K/k)$ and $X(K) := \mathrm{Gal}(L(K)/K)$. We denote by k_n the n-th layer of K/k and by $A_n = A(k_n)$ the p-part of the ideal class group of k_n. For any $m \geq n \geq 0$ we write $N_{m,n} : A_m \to A_n$ and $i_{n,m} : A_n \to A_m$ for the natural maps induced by the norm and the inclusion of ideals, and we also consider the limits

$$(1) \qquad \varprojlim_n A_n \qquad \text{and} \qquad A = A(K) := \varinjlim_n A_n$$

obtained via the $N_{m,n}$ and the $i_{n,m}$ maps, respectively. By class field theory, the first limit in (1) is canonically isomorphic to $X(K) \simeq \varprojlim \mathrm{Gal}(L(k_n)/k_n)$, where $L(k_n)$ (or L_n for short) is the maximal abelian unramified p-extension of k_n. Let

$$\mathbb{Z}_p[\![\Gamma]\!] := \varprojlim_n \mathbb{Z}_p[\Gamma/\Gamma^{p^n}] \simeq \varprojlim_n \mathbb{Z}_p[\mathrm{Gal}(k_n/k)]$$

be the *Iwasawa algebra* (completed group ring) associated to K/k, which is isomorphic to the formal power series ring $\Lambda := \mathbb{Z}_p[[T]]$ via the noncanonical isomorphism

$$(2) \qquad \mathbb{Z}_p[\![\Gamma]\!] \xrightarrow{\simeq} \mathbb{Z}_p[[T]], \quad \gamma \mapsto T+1,$$

where γ is a topological generator of Γ. Since $\mathbb{Z}_p[\![\Gamma]\!]$ acts in a natural way, via conjugation, on $X(K)$, then it becomes a Λ-module through the isomorphism given in (2). With this structure, $X(K)$ (also denoted by $X(K/k)$ or simply X) is usually referred to as the *Iwasawa module* of K/k and, although to have been extensively studied by a lot of authors in the last sixty years, many problems of crucial importance remain open today (the interested reader can see [10] for the state-of-the-art and/or [16, 19, 22, 27] for some introductory references on the matter).

A classical problem concerns *capitulation* of ideals and ideal classes going up along the intermediate fields of the tower K/k. More precisely, denoting by $H_{n,m}$, $n \leq m$, the kernel of $i_{n,m} : A_n \to A_m$, we say that $[\mathfrak{a}] \in A_n$ *capitulates* in A_m if $[\mathfrak{a}] \in H_{n,m}$ (or, equivalently, $\mathfrak{a}$ becomes principal in k_m). From the beginning of the theory the groups $H_{n,m}$ and $H_n := \bigcup_{m \geq n} H_{n,m} = \mathrm{Ker}(i_n : A_n \to A)$ have been related, for example, to the finiteness of the module $X(K)$ (see, e.g., [9, 11, 13, 14, 20, 23]), but the phenomenon remains of a wild nature in general. In [3] the authors provide a description of the $H_{n,m}$ (named *relative capitulation kernels*) and of the H_n (named *absolute capitulation kernels*) in terms of the maximal finite submodule $D = D(K/k)$ of $X(K/k)$, obtaining isomorphisms with quotients of suitable submodules of D, finding some formulas for their order, and investigating their properties of stabilization both for orders and for p-ranks. For these purposes, in particular, two new functions $h(n)$ and $\rho(n)$ are introduced: the former is more technical and is linked to the vanishing of some submodules of D (see Definition 4 for a precise definition), while the second measures when the last ideal in A_n capitulates going up along the tower K/k_n.

In this paper we want to deepen the researches concerning the functions $h(n)$ and $\rho(n)$ and, in particular, we want to find bounds that allow to control their growth and therefore the so-called *capitulation delay*, especially in the lower levels that are potentially much more irregular. We also construct some examples in which we compute the explicit values of the mentioned invariants and parameters and then, generalizing such constructions, we give a method useful to show how some central results of [3] can be said to be optimal and some of the previous bounds sharp. Finally, even some emerging evidences of false, or failed, stabilization are discussed.

As regards the organization of the paper, it can be divided into two parts comprising two sections each. In Section 2 we give an overview of few known classical results about stabilization and capitulation in $\mathbb{Z}_p$-extensions, instead in Section 3 we will describe some of the main developments contained in the recent work [3] and necessary for the sequel. The second part, consisting of Section 4 and 5, contains the original results we referred to above. In particular, Section 4 deals with the bounds for $h(m) - h(n)$ and $\rho(m) - \rho(n)$ $(m \geq n)$, instead Section 5 draws the method for the optimality of results and sharpness of the bounds.

Lastly a notational remark: by convenience we include also zero in the set $\mathbb{N}$ of natural numbers.

2. A brief overview of classical results.

The literature object of this section is placed from the beginning of the theory until the 1990s. We divide it into two subsections for stabilization and capitulation, respectively.

2.1. Stabilization.

The term "stabilization" has the expected obvious meaning: given a sequence of finite $\mathbb{Z}_p$-modules or p-groups $\{M_n\}_{n\in\mathbb{N}}$, we say that their orders stabilize at an index $q \in \mathbb{N}$ if $|M_n| = |M_q|$ for all $n \geq q$. In the same way, we say that their p-ranks[*] stabilize at $q' \in \mathbb{N}$ if $\mathrm{rk}_p(M_n) = \mathrm{rk}_p(M_{q'})$ for all $n \geq q'$.

Stabilization is quite natural in Iwasawa theory even if there are not many results in this direction. We have stabilization theorems for $\{|A_n|\}_{n\in\mathbb{N}}$ and for $\{\mathrm{rk}_p(A_n)\}_{n\in\mathbb{N}}$ but not much else. Usually Iwasawa modules tend to stabilize at the very first level in which they do not grow (i.e., if we have no growth from n to $n+1$, we are not going to have any growth at all from n on).

Let $n_0 = n_0(K/k)$ be the minimal $n \geq 0$ such that every prime which ramifies in the extension K/k_n is totally ramified. References for stabilization and for the following theorems are [1, 8, 18].

THEOREM 1. ([8, Theorem 1(1)]) *If* $|A_n| = |A_{n+1}|$ *for some* $n \geq n_0$*, then* $A_m \simeq A_n \simeq X$ *for all* $m \geq n$*.*

THEOREM 2. ([8, Theorem 1(2)]) *If* $\mathrm{rk}_p(A_n) = \mathrm{rk}_p(A_{n+1})$ *for some* $n \geq n_0$*, then* $\mathrm{rk}_p(A_m) = \mathrm{rk}_p(A_n) = \mathrm{rk}_p(X)$ *for all* $m \geq n$ *(and hence* $\mu(K/k) = 0$*).*

2.2. Capitulation.

For any finitely generated Λ-module X we have an exact sequence of Λ-modules

$$0 \to D(X) \to X \xrightarrow{\varphi} E(X) \to B(X) \to 0 \tag{3}$$

where φ is a *pseudo-isomorphism*, $E(X)$ an elementary Λ-module and $D(X)$, $B(X)$ are finite (see [3, Section 2] or [5, Chapter VII], [22, Chapter V], [27, Chapter 13] for more details). $D(X)$ is the maximal finite submodule of X and when $X = X(K/k)$ for some $\mathbb{Z}_p$-extension K/k, we also call $D(K/k)$ the *maximal finite Iwasawa module* of K/k, and we shall often simply write D to denote it in the following. We moreover recall that X is said *pseudo-null* if $E(X) = 0$ in (3), or equivalently if $X = D(X)$.

The capitulation kernels $H_{n,m}$ and H_n are very important in Iwasawa theory, for example because of the following proposition which links them to Greenberg's conjecture which predicts the finiteness, or equivalently the pseudo-nullity, of $X(k_{cyc}/k)$ whenever k is a totally real number field (and k_{cyc} its cyclotomic $\mathbb{Z}_p$-extension).

[*]For any finitely generated $\mathbb{Z}_p$-module A, we obviously define its *p-rank* as $\mathrm{rk}_p(A) := \dim_{\mathbb{F}_p}(A/pA)$.

PROPOSITION 1. ([11, Proposition 2]) *We have that* $\lambda(K/k) = \mu(K/k) = 0$ *if and only if* $H_n = A_n$ *for every* $n \geq 0$*.*

Let $s = s(K/k)$ be the number of ramified prime ideals in K/k_{n_0}. In the following theorem the statement is not exactly the original one appearing in [11], but it can be easily derived from it because the proof only uses the hypothesis $s(K/k) = 1$.

THEOREM 3. ([11, Theorem 1]) *Let* $n_0 = 0$ *and* $s(K/k) = 1$ *(i.e., there is only one prime in* k *which ramifies in* K*), then* X *is pseudo-null if and only if* $H_0 = A_0$*.*

A very important remark is that, in the previous theorem, the hypothesis $n_0 = 0$ can be easily suppressed. This has been showed by J. Minardi and we write down it separately.

COROLLARY 1. ([21, Proposition 1.B]) *Assume* $s = 1$*, then* X *is pseudo-null if and only if* $H_0 = A_0$*.*

COROLLARY 2. ([21, pag. 6, Corollary]) *If there is only one prime* $\mathfrak{p}$ *in* k *dividing* p *and the class of some power of* $\mathfrak{p}$ *generates the whole* A_0*, then* $X(K/k)$ *is pseudo-null for every* $\mathbb{Z}_p$*-extension* K/k*.*

The statement (a) of the next theorem provides a stronger result and it was proved by T. Fukuda in 1994 in a very elegant way. Indeed Theorem 4 (a) gives the layer k_m for which $X \simeq A_m$, but there is a price to pay: with this kind of proof the hypothesis $n_0 = 0$ acquires a crucial role and it cannot be removed anymore.

THEOREM 4. ([8, Theorem 2]) *Let* $s(K/k) = 1$ *and* $n_0(K/k) = 0$*.*

(a) If $H_{0,n} = A_0$ *for some* $n \geq 1$*, then* $|A_m| = |A_n| = |X|$ *for all* $m \geq n$*.*

(b) If $|A_{n+1}| = |A_n|$ *for some* $n \geq 0$ *and the exponent of* A_n *is* p^t*, then* $H_{n,n+t} = A_n$*.*

We conclude this section recalling

THEOREM 5. ([23, Theorem]) *Let* $n_0(K/k) = 0$*, then* $X(K/k)$ *is pseudo-null if and only if* $\ker N_{1,0} \subseteq H_1$*.*

Not much more was known about the $H_{n,m}$'s before [3], in particular regarding their orders and their stabilization properties. For example, Iwasawa himself proved that $H_{n,m}$ is bounded by $|D| \cdot |B(X)| \cdot |A_{n_0}|$ independently from n and m (see e.g. [16] or [22]), M. Ozaki showed in [23] a relation between the H_n's and the submodule D of X, and M. Grandet and J.F. Jaulent considered capitulation in the special case $\mu = 0$ in [9]. Other documents related to these themes are [2, 13, 14, 18, 20, 21, 24, 26], while [7], although referring to a very different context, also draws inspiration from the vertical structures of Iwasawa's theory.

In the next section we will briefly summarize the description of the capitulation kernels provided in [3] by going deeper into the study of the module D.

3. Recent developments.

The results obtained in [3] can be grouped into two families: the first concerns the behaviour of the capitulation kernels (see Subsection 3.1), the second instead provides a series of new equivalent conditions, which often involve capitulation kernels, for the vanishing of the Iwasawa invariants μ and λ, and therefore for the finiteness of $X(K/k)$ (see Subsection 3.2).

3.1. The behaviour of the capitulation kernels.

Focusing the attention on the sequence of the absolute capitulation kernels

$$H_{n_0},\ H_{n_0+1},\ H_{n_0+2},\dots \tag{4}$$

and on the chain of the relative capitulation ones (at the level n)

$$H_{n,n+1} \subseteq H_{n,n+2} \subseteq H_{n,n+3} \subseteq \dots, \tag{5}$$

it is natural to ask, first of all, for growing and stabilization both for the sequence of the orders and for the sequence of the p-ranks arising from (4) and (5). To answer to these questions, [3] begins with the introduction of some Λ-submodules of D as in Definition 2. But first we need a further piece of notation because, recalling the isomorphism in (2) which will be read as an identification in the following, there are some elements in $\Lambda = \mathbb{Z}_p[[T]]$ which play an important role in the study of the class groups A_n.

DEFINITION 1. *For every $m \geq n \geq 0$, we set*

- $\omega_n := \gamma^{p^n} - 1 = (1+T)^{p^n} - 1,$
- $\nu_{n,m} := 1 + \gamma^{p^n} + \gamma^{2p^n} + \dots + \gamma^{p^m - p^n}$
  $= \frac{\omega_m}{\omega_n} = \frac{(1+T)^{p^m}-1}{(1+T)^{p^n}-1} = 1 + (1+T)^{p^n} + \dots + ((1+T)^{p^n})^{p^{m-n}-1}.$

For simplicity we write ν_n in place of $\nu_{0,n}$: hence $\nu_n = 1 + \gamma + \gamma^2 + \dots + \gamma^{p^n - 1} = \frac{\omega_n}{\omega_0} = \frac{(1+T)^{p^n}-1}{T}$ and $\nu_{n,m} = \frac{\nu_m}{\nu_n}$ as well.

DEFINITION 2. *We put*

(a) for all $m \geq n \geq 0$, $D_{n,m} := \nu_{n,m} D$;

(b) for all $n \geq n_0$, $D_n := D \cap Y_n$.

It is important, also for future use, to notice that the D_n have a good behaviour with respect to the usual Iwasawa relations, in the sense of the following

LEMMA 1. *For all $m \geq n \geq n_0$, we have* $\nu_{n,m} D_n = D_m = D_{n,m} \cap Y_m$.

For a proof see [3, Lemma 3.2]. Then, two new invariants for K/k are introduced as follows.

DEFINITION 3. *We set*

(a) $r = r(K/k) := \min\{z \geq n_0 : D_z = 0\}$, *and*

(b) $\widetilde{r} = \widetilde{r}(K/k) := \min\{z \geq n_0 : D_z \subseteq pD\}$.

Using Lemma 1 and Nakayama's Lemma, it is immediate to see that $r(K/k)$ and $\widetilde{r}(K/k)$ are always finite (nonnegative) integers.

Now we provide an isomorphism for the kernels $H_{n,m}$ in terms of the finite module D which leads to the formulas of Corollary 3 (a) and (b). For the proofs of the following results see [3, Section 3].

THEOREM 6. *For all $m \geq n \geq n_0$ there are the following isomorphisms*

$$H_{n,m} \simeq \operatorname{Ker}\{\nu_{n,m} : D/D_n \longrightarrow D/D_m\} \tag{6}$$

and

$$H_n \simeq D + Y_n/Y_n \simeq D/D_n. \tag{7}$$

Moreover, if $m \geq n \geq r(K/k)$, $H_{n,m} \simeq D[p^{m-n}]$ (where $D[p^{m-n}]$ is the submodule of the p^{m-n}-torsion elements of D).

COROLLARY 3. *For all $m \geq n \geq n_0$ we have*

(a) $|H_{n,m}| = \dfrac{|D| \cdot |D_m|}{|D_n| \cdot |D_{n,m}|}$;

(b) $|H_{n,m}| = |D + Y_n/D_{n,m} + Y_m| \cdot \dfrac{|A_n|}{|A_m|}$;

(c) if $D \neq 0$ and $n \geq n_0$, then $i_n : A_n \to A$ is injective if and only if $n = n_0$ and D is contained in Y_{n_0}.

COROLLARY 4. *For any $\mathbb{Z}_p$-extension K/k, the following are equivalent:*

(a) X does not contain any nontrivial finite submodule;

(b) $H_{n_0+1} = 0$;

(c) $i_{n,m} : A_n \to A_m$ are injective for all $m \geq n \geq n_0$.

An example for (a) is provided by the minus part of the Iwasawa module for the $\mathbb{Z}_p$-cyclotomic extension of a CM field (see [27, Propositions 13.26 and 13.28]); similar results can be derived from [23]. The following corollary instead generalizes [8, Proposition].

COROLLARY 5. *Let K/k be a $\mathbb{Z}_p$-extension, assume that $A_n \neq 0$ and $i_{n,m}$ is injective for some $m > n \geq n_0$. Then $|A_m| \geq p^{m-n}|A_n|$.*

As customary for Iwasawa modules, the H_n's verify some stabilization results.

THEOREM 7. *Assume $n \geq n_0$:*

(a) if $|H_n|$=$|H_{n+1}|$, then $H_m \simeq H_n \simeq D$ for all $m \geq n$. In particular

$$|H_{n_0}| < |H_{n_0+1}| < \ldots < |H_r| = |H_{r+1}| = \ldots = |D|; \tag{8}$$

(b) if $\mathrm{rk}_p(H_n) = \mathrm{rk}_p(H_{n+1})$, then $\mathrm{rk}_p(H_m) = \mathrm{rk}_p(H_n) = \mathrm{rk}_p(D)$ for all $m \geq n$. In particular

$$\mathrm{rk}_p(H_{n_0}) < \mathrm{rk}_p(H_{n_0+1}) < \ldots < \mathrm{rk}_p(H_{\widetilde{r}}) = \mathrm{rk}_p(H_{\widetilde{r}+1}) = \ldots = \mathrm{rk}_p(D). \tag{9}$$

From Theorem 7, we have $r = \min\{z \geq n_0 : H_z = H_{z+1}\}$ and $\widetilde{r} = \min\{z \geq n_0 : \mathrm{rk}_p(H_z) = \mathrm{rk}_p(H_{z+1})\}$, so these two invariants indicate also the stabilization of orders and p-ranks of the H_n's. To study instead the stabilization of the $H_{n,m}$'s and the delay of capitulation, we need to define an *intrinsic parameter* $h(n)$ and an *extrinsic* one $\rho(n)$, as follows.

DEFINITION 4. *For any $n \geq 0$ we set*

(a) $h(n) := \min\{z \geq n : D_{n,z} = 0\}$;

(b) $\rho(n) := \min\{z \geq n : H_{n,z} = H_z\}$;

(c) $\widetilde{\rho}(n) := \min\{z \geq n : \mathrm{rk}_p(H_{n,z}) = \mathrm{rk}_p(H_n)\}$.

REMARK 1.

(i) If $n \geq n_0$, the previous results imply $\rho(n) = \min\{z \geq n : D_{n,z} = D_z\}$.

(ii) If $n \geq n_0$, then $n \leq \widetilde{\rho}(n) \leq \rho(n) \leq h(n)$.

PROPOSITION 2. *Let $\delta, \varepsilon \in \mathbb{N}$ such that $|D| = p^{\delta}$ and p^{ε} is the exponent of D (i.e., the minimum positive integer t for which $tD = 0$). Then*

(a) for every $n \geq 0$, we have $h(n) - n \leq \delta$ and, for every $n \geq \delta - 1$, $h(n) - n = \varepsilon$;

(b) $h(n) - n = \varepsilon$ holds, also, for all $n \geqslant r$.

In the future we will continue to use δ and ε with the same meaning as in the previous proposition. Now we observe how the relative capitulation kernels $H_{n,m}$ have a rather irregular and therefore more interesting behaviour, as shown by the following

THEOREM 8.

(a) If $n_0 \le n < r$, then we have

$$
\begin{aligned}
1 = |H_{n,n}| \le |H_{n,n+1}| \le |H_{n,n+2}| \le \dots \le |H_{n,r}| \\
= |H_{n,r}| < |H_{n,r+1}| < |H_{n,r+2}| < \dots < |H_{n,h(n)}| \\
= |H_{n,h(n)}| = |H_{n,h(n)+1}| = |H_{n,h(n)+2}| = \dots = |D/D_n|.
\end{aligned} \tag{10}
$$

(b) If $n \ge r$, then $|H_{n,m}| = |D|/|D_{n,m}|$ for all $m \ge n$ and

$$1 = |H_{n,n}| < |H_{n,n+1}| < \dots < |H_{n,h(n)}| = |H_{n,h(n)+1}| = \dots = |D|.$$

The three-line layout used in (10) with the last elements of the first two lines repeated on the next line, should help to better visualize what happens. If, for instance, $h(n) = r$, then the middle row disappears and we could have the stabilization of the order of the $H_{n,m}$'s even before arriving at the invariant $r(K/k)$. If, instead, $h(n) \neq r$ and the first line of (10) shows signs of equality, then we are faced with a "false stabilization" (which is something unusual and therefore very interesting in Iwasawa theory) as we will also see in Example 1 of Section 5.[†]

We close this subsection by collecting, for future use, some useful facts and consequences of what we have seen so far in the following

PROPOSITION 3.

(a) *If $n \ge n_0$ and $h(n) \neq r$, then $\rho(n) = h(n)$. Moreover if $n \ge r$, then $\rho(n) = h(n) = n + \varepsilon$.*

(b) *For all $n \ge n_0$, $\rho(n) - n \le \delta$.*

(c) *Let $D \neq 0$ and $r \ge \delta$. If $r > n_0 + 1$ or $r = n_0 + 1$ and $D \not\subseteq Y_{n_0}$, then $\rho(r-1) = r - 1 + \varepsilon$.*

We finally notice that for other cases of false stabilization in a rather different context (non-abelian Iwasawa theory), the interested reader can see [4, Subsection 2.1].

3.2. Equivalent conditions to $\mu(K/k) = 0$ and/or $\lambda(K/k) = 0$.

As recalled in Section 2, the following conditions, found by different authors in the last 30 years of the last century, are equivalent to the finiteness of the Iwasawa module $X(K/k)$:

(i) $A_n = A_{n+1}$ for some $n \ge n_0$;

(ii) $H_n = A_n$ for all $n \ge 0$;

[†] See moreover the discussion immediately after the proof of Proposition 5.

(iii) $H_n = A_n$ for some $n \geq n_0 + 1$;

(iv) $\mathrm{Ker}(N_{n,n-1}) \subseteq H_n$ for some $n \geq n_0 + 1$.

Note that condition (iv) is equivalent at all to Ozaki's Theorem 5[‡] and condition (iii) can be also viewed as a particular case of (iv) itself. Theorem 9 gives instead some new conditions: see [3, Section 4] for the proof and for other similar results.

THEOREM 9. *The conditions below are equivalent to $\mu(K/k) = \lambda(K/k) = 0$:*

(a) $\mathrm{Im}(i_{n,m}) = \mathrm{Im}(i_{n-1,m})$ *for some $m \geq n \geq n_0 + 1$;*

(b) $\mathrm{Ker}(N_{m,n}) = \mathrm{Ker}(N_{m,n-1})$ *for some $m \geq n \geq n_0 + 1$;*

(c) $\mathrm{rk}_p(H_n) = \mathrm{rk}_p(A_n) = \mathrm{rk}_p(A_{n+1})$ *for some $n \geq n_0$.*

Note in particular as condition (c) is rather unexpected. It seems in fact the first statement that relates, or better, interprets the finiteness of the Iwasawa module $X(K/k)$ as equality not of orders like in Theorem 1, but of p-ranks.

The sequence of p-ranks $\{\mathrm{rk}_p(A_n)\}_n$ is classically linked only to the μ-invariant (e.g., it is bounded if and only if $\mu = 0$, see [27, Proposition 13.23]) and has the property of stabilization expressed in Theorem 2. The following theorem, instead, provides new insights in this direction: see [3, Theorem 4.4] for the proof and also [3, Subsections 4.1–4.3] for other properties regarding p-ranks.

THEOREM 10. *The following conditions are equivalent to $\mu(K/k) = 0$:*

(a) $\mathrm{rk}_p(\mathrm{Ker}(N_{m,n})) = \mathrm{rk}_p(\mathrm{Ker}(N_{m+1,n}))$ *for some $m \geq n \geq n_0$;*

(b) $\mathrm{rk}_p(\mathrm{Ker}(N_{m,n})) = \mathrm{rk}_p(\mathrm{Ker}(N_{m,n-1}))$ *for some $m \geq n \geq n_0 + 1$;*

(c) $\mathrm{rk}_p(\mathrm{Coker}(i_{n,m})) = \mathrm{rk}_p(\mathrm{Coker}(i_{n,m+1}))$ *for some $m \geq n \geq n_0$;*

(d) $\mathrm{rk}_p(\mathrm{Coker}(i_{n,m})) = \mathrm{rk}_p(\mathrm{Coker}(i_{n-1,m}))$ *for some $m \geq n \geq n_0 + 1$;*

(e) $\mathrm{rk}_p(\mathrm{Coker}(i_{n,m})) = \mathrm{rk}_p(A_m)$ *for some $m \geq n \geq n_0 + 1$.*

4. Bounds for $\rho(m) - \rho(n)$.

By definition we have that the last ideal in H_n capitulates exactly in $A_{\rho(n)}$, hence $\rho(n) - n$ measures how much complete capitulation is delayed in the tower. We have already given estimates for $\rho(n) - n$ in Proposition 3, and now we are going to provide bounds for the rate of growth of the sequence $\{\rho(n)\}_{n \in \mathbb{N}}$, i.e., for $\rho(m) - \rho(n)$ when $m \geq n$. By Proposition 3 (a) we know that $\rho(n) = n + \varepsilon$ for any $n \geq r$, hence for any $m \geq n \geq r$ one has $\rho(m) - \rho(n) = m - n$. But since in Iwasawa theory explicit computations are usually

[‡] It is enough, in fact, to consider the extension K/k_n.

possible only at very low levels,[§] it is more important to find bounds that hold in general and in particular for the layers n between $n_0(K/k)$ and $r(K/k)$. Hence, considering such indices, to avoid trivialities we assume $X \neq 0$ (while $D = 0$ is permitted even if no proofs would be needed in that case).

We begin with some estimates on the growth of the parameter $h(n)$ defined for all $n \geq 0$. The easiest one follows from $h(m) \leq m + \delta = m + \log_p(|D|)$ (Proposition 2 (a)) which yields

$$h(m) - h(n) \ \leq \ \log_p(|D|) + m - n \tag{11}$$

(if $D \neq 0$ one can add -1 on the right side). The following results improve this bound and lead to an estimate for $\rho(m) - \rho(n)$. They can all be formulated (and proved) in terms of the D_n's (which provide sharper bounds), but in the main statements we prefer to use the A_n's which are more convenient for computations in explicit examples.

PROPOSITION 4. *For all $m > n \geq 0$ we have*

$$h(m) - h(n) \leq \log_p\left(\frac{|D|}{|D_{n,m}|}\right) + \max\{0, m - h(n)\}. \tag{12}$$

Proof. Using Lemma 1, if $h(n) \geq m$ then $\nu_{n,m} D_{m,h(n)} = \nu_{n,m}\nu_{m,h(n)} D = \nu_{n,h(n)} D = 0$. Hence $|D_{m,h(n)}| \leq \frac{|D|}{|D_{n,m}|}$. Now note that $\nu_{h(n),h(m)} D_{m,h(n)} = D_{m,h(m)} = 0$, so, by Nakayama's Lemma and the minimality of $h(m)$, we have $|D_{m,h(n)}| \geq p^{h(m)-h(n)}$. Therefore

$$h(m) - h(n) \leq \log_p\left(\frac{|D|}{|D_{n,m}|}\right).$$

But if $m > h(n)$, then $D_{n,m} = 0$ and we write $h(m) - h(n) = h(m) - m + (m - h(n))$. Thus we also have

$$h(m) - h(n) \leq \log_p(|D|) + m - h(n) = \log_p\left(\frac{|D|}{|D_{n,m}|}\right) + m - h(n),$$

and the thesis is proved. □

Note that if $h(n) \geq m$, then the last term in (12) (i.e., $\max\{0, m - h(n)\}$) disappears: this certainly happens, for example, when $m = n + 1$.

THEOREM 11. *For all $m \geq n \geq n_0$ we have*

(a) $h(m) - h(n) \ \leq \ \log_p\left(\frac{|A_m|}{|A_n|}\right) + \log_p(|H_{n,m}|) + m - n;$

(b) $h(m) - h(n) \ \leq \ \log_p\left(\frac{|A_m|}{|A_n|}\right) + \sum_{i=1}^{m-n} \log_p(|H_{n+i-1,n+i}|).$

[§] See, for example, the methods used in [18] to determine the Iwasawa module for the cyclotomic $\mathbb{Z}_3$-extension of a real quadratic field of conductor less than 10^4 (and not congruent to 1 mod 3): all the machinery uses the very first layers of an extension and results of stabilization similar to those seen in Subsection 2.1.

Proof. For the first inequality we use Proposition 4 and Corollary 3 (a), which yield

$$h(m)-h(n) \leq \log_p\left(\frac{|D|}{|D_{n,m}|}\right)+m-n = \log_p\left(\frac{|D_n|}{|D_m|}\right)+\log_p(|H_{n,m}|)+m-n.$$

Moreover note that we can embed D_n/D_m into Y_n/Y_m and $|Y_n/Y_m| = |A_m|/|A_n|$.

For the inequality (b), if $D \neq 0$, take any $i \in \{1,\ldots,m-n\}$ and use (12) to get

$$h(n+i)-h(n+i-1) \leq \log_p\left(\frac{|D|}{|D_{n+i-1,n+i}|}\right)$$

(note that if $D \neq 0$ then $h(n+i-1) \geq n+i$). Summing up, one finds

$$h(m)-h(n) \leq \sum_{i=1}^{m-n}\log_p\left(\frac{|D|}{|D_{n+i-1,n+i}|}\right) = \log_p\left(\frac{|D|^{m-n}}{\prod_{i=1}^{m-n}|D_{n+i-1,n+i}|}\right).$$

Using again Corollary 3 (a), we have

$$\begin{aligned}\log_p\left(\frac{|D|^{m-n}}{\prod_{i=1}^{m-n}|D_{n+i-1,n+i}|}\right) &= \log_p\left(\prod_{i=1}^{m-n}|H_{n+i-1,n+i}|\cdot\frac{|D_{n+i-1}|}{|D_{n+i}|}\right)\\ &= \log_p\left(\frac{|D_n|}{|D_m|}\right)+\log_p\left(\prod_{i=1}^{m-n}|H_{n+i-1,n+i}|\right)\\ &= \log_p\left(\frac{|D_n|}{|D_m|}\right)+\sum_{i=1}^{m-n}\log_p(|H_{n+i-1,n+i}|),\end{aligned}$$

and since we have already seen that $|D_n/D_m| \leq |A_m|/|A_n|$, then the proof of (b) is complete. ☐

Now we use the previous results to achieve the bounds for $\rho(m)-\rho(n)$ when $m \geq n \geq n_0$. The easiest one, coming from (11), is

$$\rho(m)-\rho(n) \leq \delta+m-n, \tag{13}$$

and it is easy to realize that this bound is actually reached in very special cases.

COROLLARY 6. *For all $m \geq n \geq n_0$, we have*

(a) $\rho(m)-\rho(n) \leq \log_p\left(\frac{|D|}{|D_{n,m}|}\right)+\max\{0,m-h(n)\}+(h(n)-\rho(n))$;

(b) $\rho(m)-\rho(n) \leq \log_p\left(\frac{|A_m|}{|A_n|}\right)+\log_p(|H_{n,m}|)+m-n+\max\{0,r-\rho(n)\}$;

(c) $\rho(m)-\rho(n) \leq \log_p\left(\frac{|A_m|}{|A_n|}\right)+\sum_{i=1}^{m-n}\log_p(|H_{n+i-1,n+i}|)+\max\{0,r-\rho(n)\}$.

Proof. Just note that $\rho(m)-\rho(n)\le h(m)-\rho(n)=h(m)-h(n)+h(n)-\rho(n)$ and use the bounds of Proposition 4 and Theorem 11. □

The purpose of Corollary 6 is to find bounds that are as independent as possible from the knowledge of the module D, as done in (b) and (c). Between these two last bounds, there is not one always better than the other, but it is more convenient to use one or the other depending on the case (this, for instance, can be easily seen with techniques similar to those of the next Section 5). On the contrary, instead, the comparison between other bounds is often clear as the following remark, by way of example, shows.

REMARK 2. The bound given in Corollary 6 (a) is always better or equal to the one given in (13). The proof is an easy computation for which we have only to distinguish the following four cases: (1) $n\ge r$, (2) $n<r$, $m\le h(n)$ and $m\ge r$, (3) $n<r$ and $m\ge h(n)$, (4) $m<r$. We prove (1) as an example: we have $\log_p\left(\frac{|D|}{|D_{n,m}|}\right)\le\delta$, $h(n)=\rho(n)=n+\varepsilon$ and $\max\{0,m-h(n)\}=\max\{0,m-n-\varepsilon\}$. Then

$$\begin{aligned}\log_p\left(\frac{|D|}{|D_{n,m}|}\right)&+\max\{0,m-h(n)\}+h(n)-\rho(n)\\ &\le\begin{cases}\delta+m-h(n) & \text{if } m\ge h(n)\\ \delta & \text{if } m<h(n)\end{cases}\\ &\le\delta+m-n.\end{aligned}$$

The proofs of the other cases are similar.

5. Optimal results.

In this section we first give two examples in which we compute explicitly the values of the parameters and the invariants defined in the previous sections. Then, improving the methodologies used to construct such examples, we will show in what sense some of the results previously obtained can be considered optimal. We will therefore consider, by way of example, one of the most articulate statements like that of Theorem 8, and we will demonstrate how it is optimal, that is, not improvable from the point of view of the algebra of Λ-modules and, assuming Assumption 1, also in an absolute sense.

In the next examples we will use a result proved by M. Ozaki in [25]: for every prime p and every finite Λ-module D there exists a totally real field k whose cyclotomic $\mathbb{Z}_p$-extension has Iwasawa module isomorphic to D (see in particular [25, Theorem 1]).

EXAMPLE 1. Taking $D\simeq\Lambda/(p^u,T)$, by [25, Theorem 1] there exists at least a field k and a $\mathbb{Z}_p$-extension K/k that provides $X(K/k)\simeq D$. Moreover there exists u_0

such that $0 \le u_0 \le u$ and $D_0 = p^{u_0} D$. An easy calculation shows that

$$|H_{n,m}| = \begin{cases} 1 & \text{if } 0 \le n \le m \le u - u_0 \\ p^{m-u+u_0} & \text{if } n \le u - u_0 \text{ and } u - u_0 < m \le n + u \\ p^{m-n} & \text{if } n > u - u_0 \text{ and } n \le m \le n + u \\ p^u & \text{if } n > u - u_0 \text{ and } m > n + u \, . \end{cases}$$

Furthermore we can easily see that our invariants and parameters take the following values: $r(K/k) = u - u_0$, $\widetilde{r}(K/k) = 0$, and for any $n \ge 0$

$$h(n) = \rho(n) = n + u \quad \text{ and } \quad \widetilde{\rho}(n) = \max\{n+1,\ u - u_0 + 1\}.$$

In particular, if $n \le u - u_0$, the chain of inequalities in Theorem 8 (a) becomes

$$1 = |H_{n,n}| = \ldots = |H_{n,u-u_0}| < \ldots < |H_{n,n+u}| = |H_{n,n+u+1}| = \ldots = |H_n|,$$

and whenever $n < u - u_0 < n + u$, we face a phenomenon of false, or failed, stabilization.

EXAMPLE 2. Let $v \ge 1$ and $D \simeq \Lambda/(p, T^v)$. Thus, by using [25, Theorem 1], there exists a number field k whose cyclotomic $\mathbb{Z}_p$-extension provides $X(k_{cyc}/k) \simeq D$, $n_0(k_{cyc}/k) = 0$ and $Y_0(k_{cyc}/k) = 0$ (possibly starting with some large layer). By a short computation one can check that, for all $m \ge n \ge 0$, we have

$$|H_{n,m}| = \begin{cases} p^{p^m - p^n} & \text{if } m \le \lfloor \log_p(v + p^n - 1) \rfloor \\ p^v & \text{if } m > \lfloor \log_p(v + p^n - 1) \rfloor \end{cases}$$

and

$$\mathrm{rk}_p(H_{n,m}) = \begin{cases} p^m - p^n & \text{if } m \le \lfloor \log_p(v + p^n - 1) \rfloor \\ v & \text{if } m > \lfloor \log_p(v + p^n - 1) \rfloor \end{cases} ,$$

where $\lfloor a \rfloor$ is the floor of $a \in \mathbb{R}$. Our invariants and parameters take moreover the following values: $r(k_{cyc}/k) = 0$, $\widetilde{r}(k_{cyc}/k) = 0$, and for any $n \ge 0$

$$h(n) = \rho(n) = \lfloor \log_p(v + p^n - 1) \rfloor + 1 \quad \text{ and } \quad \widetilde{\rho}(n) = \lfloor \log_p(v + p^n - 1) \rfloor + 1.$$

As seen in the previous examples, for an explicit computation of all our parameters we need to know, in addition to X, the module Y_0 as well. There are easy cases in which $Y_0 = X$ or $Y_0 = TX$, depending also on the class group of k and on the number (and behaviour) of the primes of k above p, but in general Ozaki's results give no information about them. Going deeper in this direction we can show in what sense results like Theorem 8 can be considered optimal; it is also convenient to state the following assumption that can be considered as a conjecture generalizing Ozaki's results and similar others.

ASSUMPTION 1. Let Γ be a (multiplicative) topological group isomorphic to $\mathbb{Z}_p$ and let $D_0 \subseteq D$ be two finite $\mathbb{Z}_p[[\Gamma]]$-modules. Then there exists a number field k and a $\mathbb{Z}_p$-extension K/k with $n_0(K/k) = 0$, such that

– the Iwasawa module $X(K/k)$ is isomorphic to D and

– the submodule $Y_0(K/k)$ is isomorphic to D_0,

via the isomorphism induced by $\Gamma \simeq \mathrm{Gal}(K/k)$.

The following proposition provides a strategy (and the explicit modules) to closely analyze the thesis of Theorem 8.

PROPOSITION 5. *Assuming Assumption 1, then for any* $h' \geq r' \geq 0$ *and any finite sequence of nonnegative integers* $\{t_i\}_{1\leq i\leq h'}$ *such that*

$$t_{r'+1} \geq t_{r'+2} \geq \ldots \geq t_{h'} \geq 1, \tag{14}$$

there exist a number field k *and a* $\mathbb{Z}_p$*-extension* K/k *such that*

$$n_0(K/k) = 0, \;\; r(K/k) = r', \;\; h(0) = h', \;\; \frac{|H_{0,i}|}{|H_{0,i-1}|} = p^{t_i} \;\; \textit{for all } 1 \leq i \leq h'$$

and

$$|H_{0,j}| = p^{\sum_{i=1}^{h'} t_i} \;\; \textit{for every } j \geq h'.$$

Sketch of proof. Let

$$D \;=\; \bigoplus_{i=1}^{r'} (\Lambda/(p^i,T))^{t_i} \oplus \Lambda/(p^{r'},T) \oplus \bigoplus_{i=r'+1}^{h'-1} (\Lambda/(p^i,T))^{t_i-t_{i+1}} \oplus (\Lambda/(p^{h'},T))^{t_{h'}}$$

and

$$\begin{aligned} D_0 \;\;=\;\; & \bigoplus_{i=1}^{r'} ((p,T)/(p^i,T))^{t_i} \oplus \Lambda/(p^{r'},T) \\ & \oplus \bigoplus_{i=r'+1}^{h'-1} ((p^{i-r'},T)/(p^i,T))^{t_i-t_{i+1}} \oplus ((p^{h'-r'},T)/(p^{h'},T))^{t_{h'}}. \end{aligned}$$

There are four summands both in D and D_0: the first one influences the part $|H_{0,0}| \leq |H_{0,1}| \leq \ldots \leq |H_{0,r'}|$, the second one, which is the same for D and D_0, guarantees that $r(K/k) = r'$, the last two affect the part $|H_{0,r'}| < |H_{0,r'+1}| < \ldots < |H_{0,h'}|$. The checking of all the claims of the thesis is a matter of direct (but rather patient) calculation that we leave to the reader. □

Now, looking at Theorem 8 (a), the previous proposition implies that every possibility for the part $|H_{n,n}| \leq |H_{n,n+1}| \leq \ldots \leq |H_{n,r}|$ (in the case $n < r$) is realizable. In particular, if $|H_{n,q}|$ seems to stabilize at a certain index m for many subsequent layers $m+1,\ m+2,\ \ldots,\ m+t,$ that is, $|H_{n,m}| = |H_{n,m+1}| = \ldots = |H_{n,m+t}|$ for some $n \leq m < m+t \leq r$, then this does not guarantee a definitive stabilization, i.e., we can still have $|H_{n,m+t}| < |H_{n,m+t+1}|$.

We conclude by observing that, if $n \geq n_0$ and $h(n) = r+1$, then Proposition 5 could take the following form:

Every situation not explicitly prohibited by Theorem 8 is realizable.

The reason to require $h(n) = r+1$ is just technical and not substantial, and it lies precisely in the hypothesis (14). For instance, the proof of a similar proposition for the case $t_{r'+1} < t_{r'+2} < \ldots < t_{h'}$ can be easily realized,[¶] but it involves a more complicated (and not particularly enlightening) computation.

Lastly, we notice that similar techniques to those of Proposition 5 can be employed to show that many bounds, found in the previous section, are sharp as well and not improvable in the general case. For example, we observed that the bound (13) is reached in very special cases and we proved in Remark 2 that the bound of Corollary 6 (a) is always better or equal to the one in (13). Now, using the methods of Proposition 5 and analyzing a series of cases, we can show how the bound in Corollary 6 (a) itself can be reached case by case.

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