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LEAST COMMON MULTIPLE OF POLYNOMIAL SEQUENCES

Abstract. We collect some results and problems about the quantity

$$L_f(n) := \operatorname{lcm}(f(1), f(2), \ldots, f(n)),$$

where f is a polynomial with integer coefficients and lcm denotes the least common multiple.

1. Introduction

For each positive integer n, let us define

$$L(n) := \operatorname{lcm}(1, 2, \ldots, n),$$

that is, the lowest common multiple of the first n positive integers. It is not difficult to show that

$$\log L(n) = \psi(n) := \sum_{p \leq n} \log p,$$

where ψ denotes the first Chebyshev function, and p runs over all primes numbers not exceeding n. Hence, bounds for $L(n)$ are directly related to bounds for $\psi(n)$ and, consequently, to estimates for the prime counting function $\pi(n)$. In particular, since the Prime Number Theorem is equivalent to $\psi(n) \sim n$ as $n \to +\infty$, we have

$$\log L(n) \sim n.$$

In 1936 Gelfond and Shnirelman, proposed a new elementary and clever method for deriving a lower bound for the prime counting function $\pi(x)$ (see Gelfond's editorial remarks in the 1944 edition of Chebyshev's Collected Works [15, pag. 287–288]). In 1982 the Gelfond-Shnirelman method was rediscovered and developed by Nair [16, 17]. Their method was actually based on estimating $L(n)$, and in its simplest form [16] it gives

$$n \log 2 \leq \log L(n) \leq n \log 4,$$

for every $n \geq 9$, which in turn implies

$$(\log 2 + o(1)) \frac{n}{\log n} \leq \pi(n) \leq (\log 4 + o(1)) \frac{n}{\log n},$$

after some manipulations. Later, it was proved [18] that the Gelfond-Shnirelman-Nair method can give lower bound in the form

$$\pi(n) \geq C \, \frac{n}{\log n},$$

only for constants C less than 0.87, which is quite far from what is expected by the Prime Number Theorem. (A possible way around this problem has been considered in [13, 14, 19].)

Moving from this initial connection with estimates for $\pi(n)$ and the Prime Number Theorem, several authors have considered bounds and asymptotic for the following generalization of $L(n)$ to polynomials. For every polynomial $f \in \mathbb{Z}[x]$, let us define

$$L_f(n) := \operatorname{lcm}(f(1), f(2), \ldots, f(n)).$$

In the next section we collect some results on $L_f(n)$.

2. Products of linear polynomials

Stenger [12] used the Prime Number Theorem for arithmetic progressions to show the following asymptotic estimate for linear polynomials:

THEOREM 1. *For any linear polynomial $f(x) = ax + b \in \mathbb{Z}[x]$, we have*

$$\log L_f(n) \sim n \frac{q}{\varphi(q)} \sum_{\substack{1 \le r \le q \\ (q,r)=1}} \frac{1}{r},$$

as $n \to +\infty$, where $q = a/(a,b)$ and φ denotes the Euler's totient function.

Hong, Qian, and Tan [6] extended this result to polynomials f which are the product of linear polynomials, showing that an asymptotic of the form $\log L_f(n) \sim A_f n$ holds as $n \to +\infty$, where $A_f > 0$ is a constant depending only on f.

Moreover, effective lower bounds for $L_f(n)$ when f is a linear polynomial have been proved by Hong and Feng [3], Hong and Kominers [4], Hong, Tan and Wu [7], Hong and Yang [8], and Oon [9],

3. Quadratic polynomials

Cilleruelo [2, Theorem 1] considered irreducible quadratic polynomials and proved the following result:

THEOREM 2. *For any irreducible quadratic polynomial with integer coefficients $f(x) = ax^2 + bx + c$, we have*

$$\log L_f(n) = n \log n + B_f n + o(n),$$

where

$$B_f := \gamma - 1 - 2\log 2 - \sum_p \frac{(d/p)\log p}{p-1} + \frac{1}{\varphi(q)} \sum_{\substack{1\le r\le q\\(r,q)=1}} \log\left(1+\frac{r}{q}\right)$$
$$+\log a + \sum_{p\mid 2aD} \log p \left(\frac{1+(d/p)}{p-1} - \sum_{k\ge 1} \frac{s(f,p^k)}{p^k}\right),$$

and γ *is the Euler–Mascheroni constant,* $D = b^2 - 4ac = d\ell^2$*, where* d *is a fundamental discriminant,* (d/p) *is the Kronecker symbol,* $q = a/(a,b)$ *and* $s(f,p^k)$ *is the number of solutions of* $f(x) \equiv 0 \pmod{p^k}$.

Rué, Šarka, and Zumalacárregui [11, Theorem 1.1] provided a more precise error term for the particular polynomial $f(x) = x^2+1$,

THEOREM 3. *Let* $f(x) = x^2 + 1$*. For any* $\theta < 4/9$ *we have*

$$\log L_f(n) = n\log n + B_f n + O_\theta\left(\frac{n}{(\log n)^\theta}\right).$$

4. Higher degree polynomials

Regarding general irreducible polynomials, Cilleruelo [2] formulated the following conjecture.

CONJECTURE 1. If $f(x) \in \mathbb{Z}[x]$ is an irreducible polynomial of degree $d \ge 2$, then

$$\log L_f(n) \sim (d-1)n\log n,$$

as $n \to +\infty$.

Except for the result of Theorem 2, no other case of Conjecture 1 is known to date. It can be proved (see [10, p. 2]) that for any irreducible f of degree $d \ge 3$, we have

$$n\log n \ll \log L_f(n) \le (1+o(1))(d-1)n\log n.$$

Also, Rudnick and Zehavi [10, Theorem 1.2] proved the following result, which established Conjecture 1 for almost all shifts of a fixed polynomial, in a range of n depending on the range of shifts.

THEOREM 4. *Let* $f(x) \in \mathbb{Z}[x]$ *be a monic polynomial of degree* $d \ge 3$*. Then, as* $T \to +\infty$*, we have that for all* $a \in \mathbb{Z}$ *with* $|a| \le T$*, but a set of cardinality* $o(T)$*, it holds*

$$\log L_{f(x)-a}(n) \sim (d-1)\, n \log n$$

uniformly for $T^{1/(d-1)} < n < T/\log T$.

Regarding lower bounds for $L_f(n)$, Hong and Qian [5, Lemma 3.1] proved the following:

THEOREM 5. *Let $f(x) \in \mathbb{Z}[x]$ be a polynomial of degree $d \geq 1$ and with leading coefficient a_d. Then for all integers $1 \leq m \leq n$, we have*

$$\operatorname{lcm}(f(m), f(m+1), \ldots, f(n)) \geq \frac{1}{(n-m)!} \prod_{k=m}^{n} \left| \frac{f(k)}{a_d} \right|^{1/d}.$$

Shparlinski [1] suggested to study a bivariate version of $L_f(n)$, posing the following problem:

PROBLEM 1. *Given a polynomial $f \in \mathbb{Z}[x,y]$, obtain an asymptotic formula for*

$$\log \operatorname{lcm}\{f(m,n) : 1 \leq m, n \leq N\}$$

with a power saving in the error term.

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