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ON POLYNOMIAL SOLUTIONS OF THE DIOPHANTINE EQUATION $(X+Y-1)^2 = WXY$

Abstract. In this paper we consider a particular class of polynomials arising from the solutions of the Diophantine equation $(x+y-1)^2 = wxy$. We highlight some interesting aspects, describing their relationship with many iportant integer sequences and pointing out their connection with Dickson and Chebyshev polynomials. We also study their coefficients finding a new identity involving Catalan numbers and proving that they are a Riordan array.

1. A class of polynomials related to integer sequences, Dickson and Chebyshev polynomials

In [1], the authors solved the Diophantine equation

$$(x+y-1)^2 = wxy, \tag{1}$$

where w is a given positive integer and x, y are unknown numbers, whose values are to be sought in the set of positive integers.
In particular, (x,y) is a solution of the Diophantine equation (1) if and only if $(x,y) = (u_{m+1}(w), u_m(w))$, for a given $m \in \mathbb{N}$, where $(u_n(w))_{n=0}^{+\infty}$ is the following linear recurrent sequence:

$$\begin{cases} u_0(w) = 0, \quad u_1(w) = 1, \quad u_2(w) = w \\ u_n(w) = (w-1)u_{n-1}(w) - (w-1)u_{n-2}(w) + u_{n-3}(w) \quad \forall n \geq 3. \end{cases} \tag{2}$$

This polynomial sequence is very interesting. Indeed, for several values of w, the polynomial sequence $(u_n(w))$ coincides with some well–known and studied integer sequences. For example, for $w = 4$, $(u_n(4)) = n^2$, that is the sequence A000290 in OEIS [7]. When $w = 5$, $(u_n(5))$ is the sequence of the alternate Lucas numbers minus 2 (see sequence A004146 in OEIS). If $w = 9$, $(u_n(9)) = F_{2n}^2$, where (F_n) is the sequence of the Fibonacci numbers. For $w = 4, \ldots, 20$, the sequence $(u_n(w))$ appears in OEIS [7]. In Table 1, we summarize sequences $u_n(w)$ for different values of w.

In the following, we prove that polynomials $u_n(w)$ are related to some well–known and studied polynomials like Chebyshev polynomials of the first and second kind, respectively $T_n(x)$ and $U_n(x)$ (see, e.g., [5]), and Dickson polynomials $D_n(x)$ and $E_n(x) = U_n\left(\frac{x}{2}\right)$ (see, e.g., [3]).
Here we define $T_n(x)$ and $U_n(x)$ as the n–th element of the linear recurrent sequence $(T_n(x))_{n=0}^{+\infty}$ and $(U_n(x))_{n=0}^{+\infty}$ with characteristic polynomial $t^2 - 2xt + 1$ and initial conditions $T_0(x) = 1$, $T_1(x) = x$ and $U_0(x) = 1$, $U_1(x) = 2x$, respectively.

w	$(u_n(w))_{n=0}^{+\infty}$	OEIS reference
4	$0,1,4,9,16,25,\ldots$	A000290=$(n^2)_{n=0}^{+\infty}$,
5	$0,1,5,16,45,121,\ldots$	A004146=Alternate Lucas numbers - 2
6	$0,1,6,25,96,361,\ldots$	A092184
7	$0,1,7,36,175,841,\ldots$	A054493 (shifted by one)
8	$0,1,8,49,288,1681,\ldots$	A001108
9	$0,1,9,64,441,3025,\ldots$	A049684=F_{2n}^2 (F_n Fibonacci numbers)
10	$0,1,10,81,640,5041,\ldots,$	A095004 (shifted by one)
11	$0,1,11,100,891,7921,\ldots,$	A098296
12	$0,1,12,121,1200,11881,\ldots$	A098297
13	$0,1,13,144,1573,17161,\ldots$	A098298
14	$0,1,14,169,2016,24025,\ldots$	A098299
15	$0,1,15,196,2535,32761,\ldots$	A098300
16	$0,1,16,225,3136,43681,\ldots$	A098301
17	$0,1,17,256,3825,57121,\ldots$	A098302
18	$0,1,18,289,4608,73441,\ldots$	A098303
19	$0,1,19,324,5491,93025,\ldots$	A098304
20	$0,1,20,361,6480,116281,\ldots$	A049683=$(L_{6n}-2)/16$ (L_n Lucas numbers)

Table 1.1: Sequence $u_n(w)$ for different values of w

We recall that Dickson polynomials are defined as follows:

$$D_n(x)=\sum_{i=0}^{\lfloor n/2\rfloor}\frac{n}{n-i}\binom{n-i}{i}(-1)^i x^{n-2i}$$

and

$$E_n(x)=\sum_{i=0}^{\lfloor n/2\rfloor}\binom{n-i}{i}(-1)^i x^{n-2i}.$$

We also recall that for Dickson polynomials the following identities hold

$$D_n\left(x+x^{-1}\right)=x^n+x^{-n},\quad E_n\left(x+x^{-1}\right)=\frac{x^{n+1}-x^{-(n+1)}}{x-x^{-1}} \tag{3}$$

THEOREM 1. *We have*

$$u_n(w)=\frac{D_n(w-2)-2}{w-4}=2\frac{T_n(\frac{w-2}{2})-1}{w-4},\quad \forall n\geq 0 \tag{4}$$

and in particular for all $n\geq 1$

$$u_{2n}(w)=wE_{n-1}^2(w-2)=wU_{n-1}^2\left(\frac{w-2}{2}\right) \tag{5}$$

$$\begin{aligned}u_{2n-1}(w)&=(E_{n-1}(w-2)+E_{n-2}(w-2))^2=\\&=\left(U_{n-1}\left(\frac{w-2}{2}\right)+U_{n-2}\left(\frac{w-2}{2}\right)\right)^2\end{aligned} \tag{6}$$

Proof. The recurrence relation described in (2) clearly shows that the characteristic polynomial of $(u_n(w))$ is

$$x^3-(w-1)x^2+(w-1)x-1=(x-1)(x^2-(w-2)x+1)$$

whose zeros are $x_1 = 1$ and $x_{2,3} = \frac{w-2\pm\sqrt{w^2-4w}}{2}$. If we set $x_2 = \zeta$ we easily observe that $x_3 = \zeta^{-1}$ so that $\zeta+\zeta^{-1} = w-2$ and $\zeta-\zeta^{-1} = \sqrt{w^2-4w}$. Moreover, using the initial conditions in (2), with standard tecniques we find the following closed form for every element of $(u_n(w))$

$$u_n(w) = \frac{\zeta^n+\zeta^{-n}-2}{w-4} = \frac{\zeta^n+\zeta^{-n}-2}{\zeta+\zeta^{-1}-2} \tag{7}$$

. Thanks to the first identity in (3) it is straightforward to observe that

$$u_n(w) = \frac{D_n(\zeta+\zeta^{-1})-2}{w-4} = \frac{D_n(w-2)-2}{w-4}. \tag{8}$$

Since $x^2-(w-2)x+1$ is the characteristic polynomial of the sequence $(T_n\left(\frac{w-2}{2}\right))$, with roots $x_2 = \zeta$ and $x_3 = \zeta^{-1}$, and the initial conditions are $T_0\left(\frac{w-2}{2}\right) = 1$, $T_1\left(\frac{w-2}{2}\right) = \frac{w-2}{2}$ we obtain

$$T_n\left(\frac{w-2}{2}\right) = \frac{\zeta^n+\zeta^{-n}}{2} = \frac{D_n(\zeta+\zeta^{-1})}{2} = \frac{D_n(w-2)}{2} \tag{9}$$

Thus substituting (9) in (8) we prove equality (4). Now considering the equality (7) and the second identity in (3) we have

$$u_{2n}(w) = \frac{\zeta^{2n}+\zeta^{-2n}-2}{\zeta+\zeta^{-1}-2} = \frac{(\zeta^n-\zeta^{-n})^2}{(\zeta-\zeta^{-1})^2}\,\frac{(\zeta-\zeta^{-1})^2}{\zeta+\zeta^{-1}-2} = w(E_{n-1}(w-2))^2,$$

which proves (5), and

$$u_{2n-1}(w) = \frac{\zeta^{2n-1}+\zeta^{-2n+1}-2}{\zeta+\zeta^{-1}-2} = \frac{(\zeta^{2n-1}+\zeta^{-2n+1}-2)(\zeta+\zeta^{-1}+2)}{(\zeta-\zeta^{-1})^2} \tag{10}$$

where we use the identity

$$(\zeta-\zeta^{-1})^2 = w(w-4) = (\zeta+\zeta^{-1}+2)(\zeta+\zeta^{-1}-2).$$

An easy calculation shows that

$$(\zeta^{2n-1}+\zeta^{-2n+1}-2)(\zeta+\zeta^{-1}+2) = \left(\zeta^n-\zeta^{-n}+\zeta^{n-1}-\zeta^{-(n-1)}\right)^2$$

and substituting in (10) we find

$$\begin{aligned} u_{2n-1}(w) &= \frac{\left(\zeta^n-\zeta^{-n}+\zeta^{n-1}-\zeta^{-(n-1)}\right)^2}{(\zeta-\zeta^{-1})^2} = \\ &= \left(\frac{\zeta^n-\zeta^{-n}}{\zeta-\zeta^{-1}}+\frac{\zeta^{n-1}-\zeta^{-(n-1)}}{\zeta-\zeta^{-1}}\right)^2 = \\ &= \left(E_{n-1}(w-2)+E_{n-2}(w-2)\right)^2, \end{aligned}$$

proving (6). □

As a consequence of (4) we highlight the following relation, where we posed $\frac{w-2}{2} = x$

$$T_n(x) = 2D_n(2x) = u_n(2x+2)\cdot(x-1)+1 \tag{11}$$

The coefficients of polynomials $u_n(w)$ are particularly interesting and we explicitly determine them in the following

THEOREM 2. *For any integer $n \geq 1$, we have*

$$u_n(w) = \sum_{k=0}^{n} d_n(k)w^k,$$

where

$$d_n(k) = \sum_{i=0}^{n-k-1}(-1)^i\binom{i+2k}{2k}, \quad \forall 0 \leq k < n$$

and $d_n(n) = 0$.

Proof. The theorem can be proved by induction. For $n = 1$, we have $u_1(w) = 1$ and $d_1(0)w^0 + d_1(1)w = 1$. Similarly, it is straightforward to check the theorem when $n = 2$ and $n = 3$.
Now, let us suppose that the thesis holds for any integer less or equal than n, for a given integer n. We have

$$u_{n+1}(w) = (w-1)u_n(w) - (w-1)u_{n-1}(w) + u_{n-2}(w) =$$

$$= (w-1)\sum_{k=0}^{n} d_n(k)w^k - (w-1)\sum_{k=0}^{n-1} d_{n-1}(k)w^k + \sum_{k=0}^{n-2} d_{n-2}(k)w^k.$$

Observing that

$$d_n(k) = d_{n-1}(k) + (-1)^{n-k-1}\binom{n+k-1}{2k}$$

we obtain

$$\begin{aligned}
u_{n+1}(w) &= (w-1)\sum_{k=0}^{n} d_n(k)w^k - (w-1)\sum_{k=0}^{n-1}\left(d_n(k) - (-1)^{n-k-1}\binom{n+k-1}{2k}\right)w^k + \\
&\quad + \sum_{k=0}^{n-2} d_{n-2}(k)w^k = \\
&= (w-1)\sum_{k=0}^{n-1}(-1)^{n-k-1}\binom{n+k-1}{2k}w^k + \sum_{k=0}^{n-2}\left(d_{n+1}(k) - (-1)^{n-k}\binom{n+k}{2k} + \right. \\
&\quad \left. -(-1)^{n-k-1}\binom{n+k-1}{2k} - (-1)^{n-k-2}\binom{n+k-2}{2k}\right)w^k.
\end{aligned}$$

Thus we have to prove that

$$(12)\quad (w-1)\sum_{k=0}^{n-1}(-1)^{n-k-1}\binom{n+k-1}{2k}w^k$$
$$+\sum_{k=0}^{n-2}\left((-1)^{n-k-1}\binom{n+k}{2k}-(-1)^{n-k-1}\binom{n+k-1}{2k}-(-1)^{n-k-2}\binom{n+k-2}{2k}\right)w^k$$
$$-w^n+2(n-1)w^{n-1}=0$$

in order to prove that

$$u_{n+1}(w)=\sum_{k=0}^{n+1}d_{n+1}(k)w^k.$$

The left member of equation (12) is equal to

$$\sum_{k=0}^{n-3}(-1)^{n-k-1}\binom{n+k-1}{2k}w^{k+1}-\sum_{k=0}^{n-2}(-1)^{n-k-1}\binom{n+k-1}{2k}w^k+$$
$$+\sum_{k=0}^{n-2}\left((-1)^{n-k-1}\binom{n+k}{2k}-(-1)^{n-k-1}\binom{n+k-1}{2k}-(-1)^{n-k-2}\binom{n+k-2}{2k}\right)w^k=$$
$$=\sum_{k=1}^{n-2}(-1)^{n-k}\left(\binom{n+k-2}{2k-2}+2\binom{n+k-1}{2k}-\binom{n+k}{2k}-\binom{n+k-2}{2k}\right)w^k$$

and using the property of binomial coefficients

$$\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$$

it is easy to check that

$$\binom{n+k-2}{2k-2}+2\binom{n+k-1}{2k}-\binom{n+k}{2k}-\binom{n+k-2}{2k}=0.$$

□

Thanks to previous theorems and relation (11) we find the following expression for Chebyshev polynomials

$$T_n(x)=1+(x-1)\sum_{k=0}^{n}d_n(k)(2x+2)^k,\quad \forall n\geq 1,$$

and an analogous one for Dickson polynomials

$$D_n(x)=\frac{1}{4}\left(2+(x-2)\sum_{k=0}^{n}d_n(k)(x+2)^k\right),\quad \forall n\geq 1.$$

In the following section, we see that coefficients $d_n(k)$ allow us to determine a new identity for Catalan numbers and they can be used to obtain a Riordan array.

2. Catalan numbers and Riordan array

Catalan numbers are very famous and interesting, deeply studied for their significance in combinatorics. In the beautiful book of Stanley [8] many combinatorial interpretations and identities involving Catalan numbers can be found. We whish to point out another new identity involving Catalan numbers and the coefficients $d_n(k)$ studied in the previous section.

THEOREM 3. *For any positive integer n, we have*

$$\sum_{k=0}^{n} d_n(k)C_k = 1,$$

where $(C_k)_{k=0}^{+\infty}$ *is the sequence of the Catalan numbers (A000108 in OEIS)*

Proof. Since

$$\int_{-1}^{1} \frac{T_n(x)}{\sqrt{1-x^2}}dx = 0,$$

by Theorem 1, we have

$$\int_{-1}^{1} \frac{u_n(2x+2)(x-1)+1}{\sqrt{1-x^2}}dx = 0.$$

Posing $y = 2x+2$, we obtain

$$\int_{0}^{4} \left(\frac{u_n(y)(y-4)+1}{2}\right) \frac{1}{\sqrt{y(4-y)}}dy = 0$$

and consequenlty

$$\int_{0}^{4} \frac{u_n(y)(y-4)}{2\sqrt{y(4-y)}}dy = -\pi,$$

$$\sum_{k=0}^{n} \int_{0}^{4} \frac{d_n(k)y^k(4-y)}{\sqrt{y(4-y)}}dy = 2\pi.$$

Moreover, it is well–known that

$$\int_{0}^{4} \frac{y^k(4-y)}{\sqrt{y(4-y)}} = 2\pi C_k,$$

thus

$$\sum_{k=0}^{n} d_n(k)C_k = 1.$$

□

Catalan numbers can be arranged in order to define a Riordan array. We recall that a Riordan array is an infinite lower triangular matrix, where the k–th column is a sequence having ordinary generating function of the form $f(x)g(x)^k$, see [6]. Catalan numbers are used to generate a particular Riordan array defined by $f(x) = \dfrac{1-\sqrt{1-4x}}{2x}$ and $g(x) = \dfrac{1-\sqrt{1-4x}}{2}$, see [4]. Thus, considering the previous relation between Catalan numbers and the coefficients of polynomials $u_n(w)$, we can suppose that also $d_n(k)$ may generate a Riordan array. Indeed, in the following theorem, we prove that the sequence $(d_n(k))_{n=0}^{+\infty}$ define a Riordan array where $f(x) = \dfrac{x}{1-x^2}$ and $g(x) = \dfrac{x}{(1+x)^2}$.

THEOREM 4. *Given an integer k the ordinary generating function of the sequence $(d_n(k))_{n=0}^{+\infty}$ is*

$$\frac{x}{1-x^2} \cdot \frac{x^k}{(1+x)^{2k}}$$

Proof. The ordinary generating function of the sequence $(d_n(k))_{n=0}^{+\infty}$ is

$$\sum_{n=0}^{+\infty} d_n(k)x^n = \sum_{n=k+1}^{+\infty} \sum_{i=0}^{n-k-1} (-1)^i \binom{i+2k}{2k} x^n,$$

where in the right member the first sum starts from $k+1$, since for $n < k+1$ the coefficients $d_n(k)$ are not defined. If we pose $n-k-1 = m$, the ordinary generating function becomes

$$\sum_{m=0}^{+\infty} \sum_{i=0}^{m} (-1)^i \binom{i+2k}{2k} x^{m+k+1} = x^{k+1} \sum_{m=0}^{+\infty} \sum_{i=0}^{m} (-1)^i \binom{i+2k}{2k} x^m =$$

$$= x^{k+1} \sum_{i=0}^{+\infty} (-1)^i \binom{i+2k}{2k} x^i \sum_{m=i}^{+\infty} x^{m-i} = x^{k+1} \sum_{i=0}^{+\infty} \binom{i+2k}{2k} (-x)^i \sum_{h=0}^{+\infty} x^h.$$

Considering that

$$\frac{1}{(1-z)^{n+1}} = \sum_{i=0}^{+\infty} \binom{i+n}{n} z^i,$$

(see, e.g., [2] pag. 199) we finally have that the ordinary generating function is

$$\frac{x^{k+1}}{1-x} \cdot \frac{1}{(1-(-x))^{2k+1}} = \frac{x}{1-x^2} \cdot \frac{x^k}{(1+x)^{2k}}.$$

$\square$

Thus the following matrix is a Riordan array

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & 0 & \cdots \\ 0 & 4 & -4 & 1 & 0 & \cdots \\ 1 & -6 & 11 & -6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where the k–th column is the sequence $(d_n(k))$.

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