Rendiconti Sem. Mat. Univ. Pol. Torino
Vol. 77, 2 (2019), 99 – 159

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GEOMETRIC VARIATIONAL PROBLEMS OF NONLOCAL TYPE AND THEIR SINGULAR PERTURBATIONS

Abstract. These lecture notes grew out of a series of lectures given by the second author at the University of Torino and the Tata Institute in Bangalore, India. We survey here, in a rather detailed fashion, some geometric variational problems of nonlocal nature, having some applications in classical topics in geometric analysis. As a way to approximate the objects under consideration, we will consider singular perturbations of some nonlinear PDEs involving some powers of the laplacian, namely the fractional Ginzburg-Landau system and the fractional Allen-Cahn equation. Each of those exhibit nonlocal phenomena in the parameter we consider and new approaches have to be developed at the interface of Geometry, nonlinear Analysis and Geometric Measure Theory. We will also emphasize on some open issues, in relation with late major results in the field of minimal surfaces.

1. Classical and Nonlocal Minimal surfaces and their approximations

The main goal of this section is to introduce the notion of nonlocal minimal surfaces and clarify its relation with fractional Allen-Cahn equation. We will start by recalling the definition of classical minimal surfaces.

1.1. The Classical Minimal Surfaces

Consider the problem of finding a surface with minimal area enclosed by a given curve. Such a surface is called a minimal surface. The problem of finding a minimal surface is called Plateau's problem named after the famous physicist Joseph Plateau who first demonstrated that a minimal surface can be obtained by immersing a wire frame, representing the boundaries, into soapy water.

There are several ways to formulate Plateau's problem and hence defining minimal surfaces mathematically (for example see [39]) in $\mathbb{R}^n$. We will follow the approach of De Giorgi (See [48], [25]). In this approach we will see the minimal surface as boundaries of sets. First we define the perimeter of a set.

DEFINITION 1. *Let E be any measurable set in $\mathbb{R}^n$. Then the perimeter of E in an open set Ω is defined to be the total variation of $\nabla\chi_E$ in Ω i.e.*

$$P_\Omega(E) = \sup\left\{\left|\int_E \operatorname{div} g\, dx\right| : g \in C_0^1(\Omega;\mathbb{R}^n), ||g||_\infty \leqslant 1\right\}.$$

Also, $P(E)$ stands for the perimeter of E in $\mathbb{R}^n$.

A set E is said to be Caccioppoli set if it has locally finite perimeter. The Plateau

problem can be formulated by using the notion of perimeter in the following way

$$\text{Minimize } P(E) \text{ among all sets } E \text{ satisfying } E \setminus \Omega = L \setminus \Omega$$

for a given Caccioppoli set L. The boundary of any minimizer of the above problem is called a minimal surface in Ω with prescribed boundary $\partial L \cap \partial\Omega$. If Ω is a bounded open set in $\mathbb{R}^n$ then the existence of the minimal surface easily follows from compactness of BV functions in L^1(See [25] and [16]).

1.2. Allen-Cahn energy minimizer and Minimal Surfaces

To understand the connection between Allen-Cahn equation and minimal surfaces we need to recall few facts. First we assume the following structural assumptions on a potential $W : \mathbb{R} \to [0,\infty)$.

(H1) $W \in C^2\big(\mathbb{R};[0,\infty)\big)$.

(H2) $\{W = 0\} = \{\pm 1\}$ and $W''(\pm 1) > 0$.

(H3) There exist $p \in (1,\infty)$ and a constant $c_W > 0$ such that for all $t \in \mathbb{R}$,

$$\frac{1}{c_W}\left(|t|^{p-1} - 1\right) \leqslant |W'(t)| \leqslant c_W\left(|t|^{p-1} + 1\right).$$

Those assumptions are of course satisfied by the prototypical potential $W(t) = (1-t^2)^2/4$. Now assume Ω is a bounded open set in dimension $n \geqslant 2$. Then the Allen-Cahn energy in Ω associated to the double well potential W satisfying the structural assumptions $(H1)$, $(H2)$ and $(H3)$ is defined by

$$\int_\Omega \varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\,\mathrm{d}x, \qquad \varepsilon \in (0,1), \tag{1.1}$$

where $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$. Critical points of (1.1) satisfy the so-called elliptic Allen-Cahn (or scalar Ginzburg-Landau) equation

$$-\Delta u_\varepsilon - \frac{1}{\varepsilon^2}W'(u_\varepsilon) = 0 \quad \text{in } \Omega. \tag{1.2}$$

When ε is small, a control on the potential implies that $u_\varepsilon \simeq \pm 1$ away from a region whose volume is of order ε. Formally, the transition layer from the phase -1 to the phase $+1$ has a characteristic width of order ε. It should take place along an hypersurface which is expected to be a critical point of the area functional, i.e., a minimal surface. More precisely, the region $\{u_\varepsilon \simeq 1\}$, which is essentially delimited by this hypersurface and the container Ω, should be a stationary set in Ω of the perimeter functional, at least as $\varepsilon \to 0$. To make this heuristics mathematically precise we need the following notion of Γ-Convergence.

DEFINITION 2 (Γ-Convergence). *Let X be a metric space and J_n be a sequence of functional on X taking values in $[0,\infty)$. Let J be another functional on X taking values in $[0,\infty)$. Then we say J_n $\Gamma-$converge to J if the following is true.*

(i) For any sequence $x_n \in X$ satisfying $x_n \to x \in X$ as $n \to \infty$ we have

$$J(x) \leqslant \liminf_{n\to\infty} J_n(x_n).$$

(ii) For any $x \in X$ there exists a sequence $x_n \in X$ converging to x such that

$$J(x) \geqslant \limsup_{n\to\infty} J_n(x_n).$$

A very important property of $\Gamma-$convergence is that the minimizers converges to the minimizers. More precisely, if J_n $\Gamma-$converge to J and x_n is a minimizer of J_n, then every limit point of $\{x_n\}$ is a minimizer of J.

The connection between energy minimizing solutions (under their own boundary condition) of (1.2) and minimal surfaces was first found in [44] through one of the first examples of Γ-convergence. The result shows that if the energy is equibounded, then $u_\varepsilon \to u_*$ in $L^1(\Omega)$ as $\varepsilon \to 0$ for some function $u_* \in BV(\Omega;\{\pm 1\})$ (up to subsequences). The set $\{u_* = 1\}$ minimizes (locally) its perimeter in Ω, and up to a multiplicative constant, the energy converges (in the sense of Γ-Convergence) to the relative perimeter of $\{u_* = 1\}$ in Ω. The analogous analysis concerning global minimization of the energy under a volume constraint has been addressed in [43, 58].

The case of general critical points has been treated more recently in [30]. It presents a slightly different feature. Namely, if the energy is equibounded, then the energy density converges in the sense of measures as $\varepsilon \to 0$ to a stationary integral $(n-1)$-varifold, i.e., a generalized minimal hypersurface with integer multiplicity. The multiplicity of the limiting hypersurface comes from an eventual folding of the diffuse interface $\{|u_\varepsilon| \lesssim 1/2\}$ as $\varepsilon \to 0$. In such a case, the interface between the two regions $\{u_* = 1\}$ and $\{u_* = -1\}$ can be strictly smaller than the support of the limiting varifold. In fact, the boundary of the region $\{u_* = 1\}$ corresponds to the set of points where the varifold has odd multiplicity. In particular, the perimeter of $\{u_* = 1\}$ can be strictly smaller than the limit of the energy. This energy loss effect is in strong analogy with the lack of strong compactness as $\varepsilon \to 0$ of solutions of the (vectorial) Ginzburg-Landau system with a potential well $\{W = 0\}$ given by a smooth and compact manifold $\mathcal{M} \subseteq \mathbb{R}^d$, see [35, 36].

1.3. Non Local Minimal Surfaces

Motivated by motion by level set according to a nonlocal heat flow, one can introduce a notion of perimeter related to H^s semi-norms. Such a notion was first introduced in [10]. The non local s-perimeter of a measurable set E in an open set $\Omega \subseteq \mathbb{R}^n$ is denoted by $P_{2s}(E,\Omega)$ and is defined by

$$P_{2s}(E,\Omega) := \int_{E\cap\Omega}\int_{E^c\cap\Omega}\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{n+s}} + \int_{E\cap\Omega}\int_{E^c\setminus\Omega}\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{n+s}} + \int_{E\setminus\Omega}\int_{E^c\cap\Omega}\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{n+s}}\,. \tag{1.3}$$

To rule out some unintuitive situations we restrict ourself to the case $0<s<1$ and Ω bounded. For example when $s\leqslant 0$, $P_{2s}(E,B_1)<\infty$ for any $E\subseteq\mathbb{R}^n$. Here B_1 is the unit ball in $\mathbb{R}^n$. On the other hand if $s\geqslant 1$, $P_{2s}(\{x_n>0\},B_1)=\infty$. Using this newly defined s-perimeter a desirable definition of the non local minimal surface is the following.

DEFINITION 3. *For any bounded open set Ω, a Lebesgue measurable set $E\subseteq\mathbb{R}^n$ is said to be a nonlocal s-minimal set in Ω if*

$$P_{2s}(E,\Omega)\leqslant P_{2s}(F,\Omega)\qquad \forall F\subseteq\mathbb{R}^n,\ F\setminus\Omega=E\setminus\Omega\,. \tag{1.4}$$

The boundary $\partial E\cap\Omega$ are referred to as (minimizing) nonlocal (s-)minimal surface in Ω.

The existence of non local s-minimal surfaces is established in [10]. By the minimality condition (1.4), the first inner variation of the s-perimeter vanishes at E, i.e.,

$$\delta P_s(E_*,\Omega)[X] := \left[\frac{\mathrm{d}}{\mathrm{d}t}P_s\big(\phi_t(E_*),\Omega\big)\right]_{t=0} = 0 \tag{1.5}$$

for any vector field $X\in C^1(\mathbb{R}^n;\mathbb{R}^n)$ compactly supported in Ω, where $\{\phi_t\}_{t\in\mathbb{R}}$ denotes the flow generated by X. If the boundary $\partial E\cap\Omega$ of a set $E\subseteq\mathbb{R}^n$ is smooth enough (e.g. a C^2-hypersurface), the first variation of the s-perimeter at E can be computed explicitly (see e.g. [21, Section 6]), and it gives

$$\delta P_s(E,\Omega)[X] = \int_{\partial E\cap\Omega}\mathrm{H}^{(s)}_{\partial E}(x)\,X\cdot\nu_E\,\mathrm{d}\mathcal{H}^{n-1}\,, \tag{1.6}$$

where ν_E denotes the unit exterior normal field on ∂E, and $\mathrm{H}^{(s)}_{\partial E}$ is the so-called *nonlocal* (or fractional) *(s-)mean curvature* of ∂E, defined by

$$\mathrm{H}^{(s)}_{\partial E}(x) := \text{p.v.}\left(\int_{\mathbb{R}^n}\frac{\chi_{\mathbb{R}^n\setminus E}(y)-\chi_E(y)}{|x-y|^{n+s}}\,\mathrm{d}y\right),\quad x\in\partial E\,.$$

(See [1] for some relevant remarks on the geometry of the surface). Therefore, a set E_* whose boundary is a minimizing nonlocal s-minimal surface in Ω (i.e., such that (1.4) holds) satisfies in the weak sense the Euler-Lagrange equation

$$\mathrm{H}^{(s)}_{\partial E_*} = 0\quad\text{on }\partial E_*\cap\Omega\,. \tag{1.7}$$

The weak sense here being precisely relation (1.5). It has been proved in [10] that *minimizing* nonlocal $2s$-minimal surfaces also satisfies (1.7) in a stronger viscosity sense. This is one of the key ingredient in the regularity theory of [10]. It states that a minimizing nonlocal minimal surface is a $C^{1,\alpha}$-hypersurface away from a (relatively) closed subset of Hausdorff dimension less than $(n-2)$. Since then, the $C^{1,\alpha}$ regularity has been improved to C^∞ in [5], and the size of the singular set reduced to $(n-3)$ in [49]. Whether or not the singular set can be further reduced remains an open question (see [15, 22] in this direction).

1.4. Fractional Allen-Cahn energy minimizer and Non local Minimal Surface

To describe the connection between nonlocal $s-$minimal surface and the fractional Allen-Cahn energy minimizer, we consider one of the simplest fractional version of equation (1.2) where the Laplace operator is replaced by the fractional Laplacian $(-\Delta)^s$, i.e., the Fourier multiplier of symbol $(2\pi|\xi|)^{2s}$. In details, we are interested in the asymptotic behavior as $\varepsilon \to 0$ of weak solutions $v_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ of the fractional Allen-Cahn equation

$$(-\Delta)^s v_\varepsilon + \frac{1}{\varepsilon^{2s}} W'(v_\varepsilon) = 0 \quad \text{in } \Omega\,, \tag{1.8}$$

subject to an exterior Dirichlet condition of the form

$$v_\varepsilon = g_\varepsilon \qquad \text{on } \mathbb{R}^n \setminus \Omega\,, \tag{1.9}$$

where $g_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ is a given smooth and bounded function. For $s \in (0,1)$, the action of the integro-differential operator $(-\Delta)^s$ on a smooth bounded function $v : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$(-\Delta)^s v(x) := \text{p.v.}\left(\gamma_{n,s} \int_{\mathbb{R}^n} \frac{v(x)-v(y)}{|x-y|^{n+2s}}\,\mathrm{d}y\right) \quad \text{with } \gamma_{n,s} := s2^{2s}\pi^{-\frac{n}{2}} \frac{\Gamma\left(\frac{n+2s}{2}\right)}{\Gamma(1-s)}\,, \tag{1.10}$$

where the notation p.v. means that the integral is taken in the *Cauchy principal value* sense. In terms of distributions, the action of $(-\Delta)^s v$ on a test function $\varphi \in \mathscr{D}(\Omega)$ is defined by

$$\begin{aligned}\left\langle (-\Delta)^s v, \varphi \right\rangle_\Omega := \frac{\gamma_{n,s}}{2} \iint_{\Omega\times\Omega} \frac{\big(v(x)-v(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y \\ + \gamma_{n,s} \iint_{\Omega\times(\mathbb{R}^n\setminus\Omega)} \frac{\big(v(x)-v(y)\big)\varphi(x)}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y\,.\end{aligned} \tag{1.11}$$

This formula defines indeed a distribution on Ω whenever $v \in L^2_{\text{loc}}(\mathbb{R}^n)$ satisfies

$$\begin{aligned}\mathcal{E}(v,\Omega) := \frac{\gamma_{n,s}}{4} \iint_{\Omega\times\Omega} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y \\ + \frac{\gamma_{n,s}}{2} \iint_{\Omega\times(\mathbb{R}^n\setminus\Omega)} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y < \infty\,.\end{aligned} \tag{1.12}$$

More precisely, if (1.12) holds, then $(-\Delta)^s v$ belongs to $H^{-s}(\Omega)$. To include the Dirichlet condition (1.9), one considers the restricted class of functions given by a natural affine space built on the natural energy Sobolev space $H^s_{g_\varepsilon}(\Omega)$. Since $\mathcal{E}(\cdot,\Omega)$ is exactly the quadratic form induced by (1.11), the functional $\mathcal{E}(\cdot,\Omega)$ can be thought as *fractional Dirichlet energy* in Ω associated to $(-\Delta)^s$. Integrating the potential in (1.8), we obtain the *fractional Allen-Cahn energy* in Ω associated to equation (1.8), i.e.,

$$\mathcal{E}_\varepsilon(v,\Omega) := \mathcal{E}(v,\Omega) + \frac{1}{\varepsilon^{2s}}\int_\Omega W(v)\,\mathrm{d}x. \tag{1.13}$$

In this way, we define weak solutions of (1.8) as critical points of $\mathcal{E}_\varepsilon(\cdot,\Omega)$ with respect to perturbations supported in Ω.

Concerning minimizers of $\mathcal{E}_\varepsilon(\cdot,\Omega)$ over $H^s_{g_\varepsilon}(\Omega)$, their asymptotic behavior as $\varepsilon \to 0$ has been investigated quite recently in [50] through a Γ-convergence analysis. The result reveals a dichotomy between the two cases $s \geqslant 1/2$ and $s < 1/2$. In the case $s \geqslant 1/2$, the normalized energies

$$\widetilde{\mathcal{E}}_\varepsilon(\cdot,\Omega) := \begin{cases} \varepsilon^{2s-1}\mathcal{E}_\varepsilon(\cdot,\Omega) & \text{if } s \in (1/2,1)\,, \\ |\ln\varepsilon|^{-1}\mathcal{E}_\varepsilon(\cdot,\Omega) & \text{if } s = 1/2\,, \end{cases}$$

$\Gamma\big(L^1(\Omega)\big)$-converge as $\varepsilon \to 0$ to the functional $\widetilde{\mathcal{E}}_0(\cdot,\Omega)$ defined on $BV(\Omega;\{\pm 1\})$ by

$$\widetilde{\mathcal{E}}_0(v,\Omega) := \sigma\,\mathrm{Per}\big(\{v=1\},\Omega\big)\,,$$

where $\sigma = \sigma(W,n,s)$ is a positive constant, and $\mathrm{Per}(E,\Omega)$ denotes the distributional (relative) perimeter of the set E in Ω. In other words, for $s \geqslant 1/2$, fractional Allen-Cahn energies (and thus minimizers) behave essentially as in the classical case, and area-minimizing hypersurfaces arise in the limit $\varepsilon \to 0$. For $s \in (0,1/2)$, the variational convergence of $\mathcal{E}_\varepsilon(\cdot,\Omega)$ appears to be much simpler since H^s-regularity does not exclude (all) characteristic functions. In particular, there is no need in this case to normalize $\mathcal{E}_\varepsilon(\cdot,\Omega)$. Assuming that $g_\varepsilon \to g$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n\setminus\Omega)$ for some function g satisfying $|g| = 1$ a.e. in $\mathbb{R}^n\setminus\Omega$, the functionals $\mathcal{E}_\varepsilon(\cdot,\Omega)$ (restricted to $H^s_{g_\varepsilon}(\Omega)$) converge as $\varepsilon \to 0$ both in the variational and pointwise sense to

$$\mathcal{E}_0(v,\Omega) := \begin{cases} \mathcal{E}(v,\Omega) & \text{if } v \in H^s_g(\Omega;\{\pm 1\})\,, \\ +\infty & \text{otherwise}\,. \end{cases}$$

Now it is worth noting that

$$\mathcal{E}(v,\Omega) = 2\gamma_{n,s}P_{2s}\big(\{v=1\},\Omega\big) \quad \forall v \in H^s_g(\Omega;\{\pm 1\})\,, \tag{1.14}$$

where $P_{2s}(E,\Omega)$ is the exactly *fractional s-perimeter* in Ω of a set $E \subseteq \mathbb{R}^n$ as defined in (1.3).

2. Fractional harmonic maps

The main aim of this chapter is to introduce the concept of the fractional Harmonic maps. We start by recalling the concept of classical Harmonic maps.

2.1. The Classical Harmonic Maps

Harmonic maps can be seen as higher dimensional generalizations of geodesics and are defined as critical points of the Dirichlet energy with respect to perturbations in the target. If the target is a sphere, the Euler-Lagrange equation for harmonic maps reads (in the sense of distributions)

$$-\Delta v_* \perp T_{v_*}\mathbb{S}^{d-1} \quad \Longleftrightarrow \quad -\Delta v_* = |\nabla v_*|^2 v_* \,. \tag{2.1}$$

A main feature of this system is its critical structure concerning regularity. Indeed, the right hand side in equation (2.1) has *a priori* no better integrability than L^1. This is precisely the borderline case where linear elliptic regularity does not apply. In dimension $n = 2$, harmonic maps are smooth by the famous result of HÉLEIN [29]. However, equation (2.1) admits highly discontinuous solutions in dimensions $n \geqslant 3$ [46], and specific assumptions are required to obtain at least partial regularity (*i.e.*, smoothness outside some "small set"). In addition to this regularity issue, equation (2.1) turns out to be invariant under composition with conformal maps in dimension $n = 2$. This invariance implies that sequences of harmonic maps are usually non-compact in the energy space [34].

As far as the regularity of the Harmonic maps in 2D (i.e. $n = 2$) is concerned, a different proof than that of HÉLEIN can be given by using the relation between div-curl quantities and the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ famously discovered by Coifman, Lions, Meyer, and Semmes [13]. To state their result, let us recall that $L^p\left(\bigwedge^1 \mathbb{R}^n\right)$ denotes the L^p space of 1-forms on $\mathbb{R}^n$, or equivalently the space of vector fields $L^p(\mathbb{R}^n, \mathbb{R}^n)$.

THEOREM 1 (Coifman-Lions-Meyer-Semmes (CLMS)). *Let $F \in L^p\left(\bigwedge^1 \mathbb{R}^n\right)$ and $g \in \dot{W}^{1,p'}(\mathbb{R}^n)$ where $1 < p < \infty$ and $p' = \frac{p}{p-1}$. Then, if $\operatorname{div} F = 0$ we have $F \cdot dg \in \mathcal{H}^1(\mathbb{R}^n)$. Moreover the following estimate holds true*

$$||F \cdot dg||_{\mathcal{H}^1(\mathbb{R}^n)} \preceq ||F||_{L^p\left(\bigwedge^1 \mathbb{R}^n\right)} ||dg||_{L^{p'}\left(\bigwedge^1 \mathbb{R}^n\right)}$$

.

Continuity of 2D Harmonic Maps into Spheres using CLMS. Clearly, a Harmonic map $u \in \dot{W}^{1,2}\left(\mathbb{R}^2, \mathbb{R}^d\right)$ satisfies (2.1). If we use Einstein's summation convention then as a consequence of the fact $|u| = 1$ we can write

$$u^k du^k = \frac{1}{2} d\,|u|^2 = 0.$$

Therefore we can rewrite (2.1) as

$$-\Delta u^i = \Omega_{ik} \cdot du^k, \text{ where } \quad \Omega_{ik} := u^i du^k - u^k du^i. \tag{2.2}$$

As consequence of Shatah's [55] result, (2.1) is equivalent to the following conservation law

$$\operatorname{div} \Omega_{ik} = 0, \text{ for } i, k \in \{1, \ldots, d\}. \tag{2.3}$$

Therefore in view of Theorem 1, we have that

$$\Delta u^i \in \mathcal{H}^1(\mathbb{R}^2), \text{ for all } i = 1, \dots, d. \tag{2.4}$$

Now one can use Calderon-Zygmund theory to show that u is continuous. See [53] for more details. □

2.2. Boundary Harmonic Maps

To define Boundary Harmonic Maps we need to restrict ourselves to the class of admissible open sets. The definition of them is the following.

DEFINITION 4 (Admissible Open Sets). *For any bounded open set $\Omega \subseteq \mathbb{R}^{n+1}_+$, we shall say that Ω is an* **admissible open set** *whenever*

- *$\partial\Omega$ is Lipschitz regular;*
- *the open set $\partial^0\Omega \subseteq \mathbb{R}^n$ defined by*

$$\partial^0\Omega := \left\{ x \in \partial\Omega \cap \mathbb{R}^n : B_r^+(x) \subseteq \Omega \textit{ for some } r > 0 \right\},$$

  is non empty and has Lipschitz boundary;
- *$\partial\Omega = \partial^+\Omega \cup \overline{\partial^0\Omega}$ where $\partial^+\Omega = \partial\Omega \cap \mathbb{R}^{n+1}_+$.*

DEFINITION 5 (Weak Boundary Harmonic Maps). *Let $\Omega \subseteq \mathbb{R}^{n+1}_+$ be an admissible bounded open set, and let $u \in H^1(\Omega;\mathbb{R}^m)$ be such that $|u| = 1$ $\mathscr{H}^n$-a.e. on $\partial^0\Omega$. We say that u is weakly harmonic in Ω with respect to the (partially) free boundary condition $u(\partial^0\Omega) \subseteq \mathbb{S}^{m-1}$ if*

$$\int_\Omega \nabla u \cdot \nabla \Phi \, \mathrm{d}x = 0 \tag{2.5}$$

for all $\Phi \in H^1(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega)$ compactly supported $\Omega \cup \partial^0\Omega$ and satisfying

$$\Phi(x) \in T_{u(x)}\mathbb{S}^{m-1} \quad \mathscr{H}^n\text{-a.e. on } \partial^0\Omega. \tag{2.6}$$

In short, we shall say that u is a weak $(\mathbb{S}^{m-1}, \partial^0\Omega)$-boundary harmonic map in Ω.

Defintion 5 can be motivated by the fact that

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} \int_\Omega |\nabla u_t|^2 \, \mathrm{d}x \right) \right]_{t=0} = \int_\Omega \nabla u \cdot \nabla \Phi \, \mathrm{d}x$$

for variations u_t of the form

$$u_t := \frac{u + t\Phi}{\sqrt{1 + t^2|\Phi|^2}},$$

with Φ satisfying (2.6). Such variations u_t belong to $H^1(\Omega;\mathbb{R}^m)$, and they satisfy the constraint $|u_t| = 1$ $\mathscr{H}^n$-a.e. on $\partial^0\Omega$ by (2.6). Actually, if $u \in H^1(\Omega;\mathbb{R}^m)$ is a weak $(\mathbb{S}^{m-1},\partial^0\Omega)$-boundary harmonic map in Ω, a standard truncation argument shows that (2.5) holds for any $\Phi \in H^1(\Omega;\mathbb{R}^m)$ with compact support in $\Omega \cup \partial^0\Omega$ and satisfying (2.6). Moreover, choosing Φ with compact support in Ω in formula (2.5) shows that $\Delta u = 0$ in Ω. Integrating by parts in (2.5) then yields

$$\int_{\partial^0\Omega} \frac{\partial u}{\partial \nu} \cdot \Phi \, \mathrm{d}\mathscr{H}^n = 0$$

for all smooth functions Φ compactly supported in $\Omega \cup \partial^0\Omega$ and satisfying (2.6). Denoting $H^{1/2}_{00}(\omega)$ the space of functions in $H^{1/2}(\mathbb{R}^n)$ vanishing outside ω and by $H^{-1/2}(\omega)$ its topological dual, the integral above has to be understood in the $H^{-1/2} - H^{1/2}_{00}$ duality sense. Therefore, the weak $(\mathbb{S}^{m-1},\partial^0\Omega)$-boundary harmonic map system can be reformulated as

$$\begin{cases} \Delta u = 0 & \text{in } \Omega\,, \\ \dfrac{\partial u}{\partial \nu} \perp T_u\mathbb{S}^{m-1} & \text{in } H^{-1/2}(\partial^0\Omega)\,. \end{cases} \tag{2.7}$$

A larger class of problems

In view of the discussion above, let us mention that $(\mathbb{S}^{m-1},\partial^0\Omega)$-boundary harmonic maps belong to a larger class of harmonic maps known in the literature as *harmonic maps with partially free boundary*, see [4, 16, 17, 18, 26, 27, 28, 51] and references therein. In most studies, one considers a smooth compact Riemannian manifold $\mathcal{M}$ without boundary (that we can assume to be isometrically embedded in some Euclidean space by the Nash embedding Theorem), and $\mathcal{N}$ a smooth closed submanifold of $\mathcal{M}$. The boundary portion $\partial^0\Omega$ is called *the partially free boundary*, and $\mathcal{N}$ is *the supporting manifold*. Then $\mathcal{M}$-valued (weak) harmonic maps in Ω with the partially free boundary condition $u(\partial^0\Omega) \subseteq \mathcal{N}$ are defined as critical points of the Dirichlet energy under the constraints $u(x) \in \mathcal{M}$ for a.e. $x \in \Omega$ and $u(x) \in \mathcal{N}$ for $\mathscr{H}^n$-a.e. $x \in \partial^0\Omega$. For $(\mathbb{S}^{m-1},\partial^0\Omega)$-boundary harmonic maps, we may consider the submanifold $\mathcal{N} = \mathbb{S}^{m-1}$ of the target $\mathcal{M} = \mathbb{R}^m$. However, to apply known results on harmonic maps, the compactness of $\mathcal{M}$ is usually required. A way to avoid this problem is to consider *bounded* $(\mathbb{S}^{m-1},\partial^0\Omega)$-boundary harmonic maps, noticing that $\mathbb{S}^{m-1}$ can be viewed as a submanifold of a flat torus $\mathcal{M} = \mathbb{R}^m/r\mathbb{Z}^m$ with factor $r > 2$. Indeed, if u is a $(\mathbb{S}^{m-1},\partial^0\Omega)$-boundary harmonic map in Ω satisfying $\|u\|_{L^\infty(\Omega)} < r/2$, then it can be considered as a $\mathcal{M}$-valued harmonic map. We finally point out that such L^∞-bound is not too restrictive in many applications. For instance, if we consider an additional Dirichlet boundary condition $u = g$ on $\partial^+\Omega$ with $g : \overline{\partial^+\Omega} \to \mathbb{S}^{m-1}$ smooth, then we usually obtain $\|u\|_{L^\infty(\Omega)} \leqslant 1$ through some maximum principle.

DEFINITION 6 (Stationary Boundary Harmonic Maps). *Let $\Omega \subseteq \mathbb{R}^{n+1}_+$ be an admissible bounded open set, and let $u \in H^1(\Omega;\mathbb{R}^m)$ be a weak $(\mathbb{S}^{m-1},\partial^0\Omega)$-boundary*

harmonic map in Ω. *We say that* u *is stationary if*

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{\Omega}\left|\nabla(u\circ\Phi_t)\right|^2\mathrm{d}x\right)\right]_{t=0}=0 \tag{2.8}$$

for any differentiable 1-parameter group of smooth diffeomorphisms $\Phi_t:\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$ *satisfying*

- $\Phi_0=\mathrm{id}_{\mathbb{R}^{n+1}}$;
- $\Phi_t(\mathbb{R}^n)\subseteq\mathbb{R}^n$;
- $\Phi_t-\mathrm{id}_{\mathbb{R}^{n+1}}$ *is compactly supported, and* $\mathrm{supp}(\Phi_t-\mathrm{id}_{\mathbb{R}^{n+1}})\cap\overline{\mathbb{R}}^{n+1}_{+}\subseteq\Omega\cup\partial^0\Omega$.

2.3. Regularity of Fractional Harmonic Maps

In this section our main objective is to present the proofs the regularity theorems regarding fractional harmonic maps obtained in [40] and [42]. We start by recalling different notions of fractional Harmonic maps. Fractional harmonic maps into a sphere were originally introduced in [14] for $s=1/2$ and $n=1$.

DEFINITION 7 (Weak $s-$harmonic map). *A mapping* $v:\mathbb{R}^n\to\mathbb{S}^{d-1}$ *(of finite fractional Dirichlet energy) is called a weakly s-harmonic map in* ω *if*

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}\left(\frac{v+t\varphi}{|v+t\varphi|},\omega\right)\right]_{t=0}=0\qquad\forall\varphi\in\mathscr{D}(\omega;\mathbb{R}^d).$$

Here the energy $\mathcal{E}$ *is same as defined in* (1.12).

As shown in [40] for $s=1/2$, this condition leads (in the weak sense) to the Euler-Lagrange equation

$$(-\Delta)^s v(x)=\left(\frac{\gamma_{n,s}}{2}\int_{\mathbb{R}^n}\frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}y\right)v(x)\quad\text{in }\Omega. \tag{2.9}$$

For *any* set $E\subseteq\mathbb{R}^n$ of finite $2s$-perimeter in Ω, the function $v=\chi_E-\chi_{\mathbb{R}^n\setminus E}$ turns out to satisfy equation (2.9). In other words, if we identify $\{\pm1\}$ with $\{\pm1\}\times\{0\}\subseteq\mathbb{R}\times\mathbb{R}^{d-1}$, the function $\chi_E-\chi_{\mathbb{R}^n\setminus E}$ is a weakly s-harmonic map into $\mathbb{S}^{d-1}$ in the open set ω. As a consequence, no regularity can be expected for weakly s-harmonic maps for $s<1/2$. This is of course in analogy with the non-regularity result of [46] for usual weakly harmonic maps into a manifold (for $n\geqslant3$). Stationary s-harmonic maps into $\mathbb{S}^{d-1}$ are defined as weakly s-harmonic maps satisfying the additional stationarity condition $\delta\mathcal{E}(v,\omega)=0$ (where $\delta\mathcal{E}(\cdot,\omega)$ denotes the first inner variation of $\mathcal{E}(\cdot,\omega)$). One may expect that, for such s-harmonic maps, some partial regularity holds (see [14, 40] in the case $s=1/2$). In view of (1.14), if a set $E_*\subseteq\mathbb{R}^n$ satisfies (1.5) (i.e., whose boundary is a stationary nonlocal $2s$-minimal surface in ω), then the function $\chi_{E_*}-\chi_{\mathbb{R}^n\setminus E_*}$ is a stationary s-harmonic map in ω. It shows that, for general stationary

s-harmonic maps into a sphere, the singular set (or discontinuity set) can have a positive $\mathcal{H}^{n-1}$-measure if $s < 1/2$ (compare to the vanishing $\mathcal{H}^{n-1}$-measure of the singular set for stationary $1/2$-harmonic maps, see [40]).

Next, we want to define minimizing maps. To do that, we need to define some energy spaces.

H^s-spaces for $s \in (0,1/2)$

We are interested in the class of functions

$$\widehat{H}^s(\Omega) := \left\{ v \in L^2_{\mathrm{loc}}(\mathbb{R}^n) : \mathcal{E}(v,\Omega) < \infty \right\}.$$

The following properties hold for any bounded open subsets Ω and Ω' of $\mathbb{R}^n$:

- $\widehat{H}^s(\Omega)$ is a linear space;
- $\widehat{H}^s(\Omega) \subseteq \widehat{H}^s(\Omega')$ whenever $\Omega' \subseteq \Omega$, and $\mathcal{E}(v,\Omega') \leqslant \mathcal{E}(v,\Omega)$;
- $\widehat{H}^s(\Omega) \cap H^s_{\mathrm{loc}}(\mathbb{R}^n) \subseteq \widehat{H}^s(\Omega')$;
- $H^s_{\mathrm{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \subseteq \widehat{H}^s(\Omega)$.

It is straightforward to show that $\widehat{H}^s(\Omega)$ is actually a Hilbert space for the scalar product induced by the norm $v \mapsto \left(\|v\|^2_{L^2(\Omega)} + \mathcal{E}(v,\Omega)\right)^{1/2}$ (see e.g. [40]). If the functions of the energy spaces are vector valued then we simply denote them by $\widehat{H}^s(\Omega;\mathbb{R}^d)$. Given a compact smooth submanifold $\mathcal{N} \subseteq \mathbb{R}^d$ without boundary the space the space $\widehat{H}^s(\omega;\mathcal{N})$ is defined by

$$\widehat{H}^s(\omega;\mathcal{N}) := \left\{ u \in \widehat{H}^s(\omega;\mathbb{R}^d) : u(x) \in \mathcal{N} \text{ a.e. on } \mathbb{R} \right\}.$$

An elementary property concerning $\widehat{H}^s(\omega;\mathbb{R}^m)$ is given in Lemma 1 below whose proof is simple and can be found in [40, Appendix A] for $s = \frac{1}{2}$.

LEMMA 1. *Let $x_0 \in \omega$ and $\rho > 0$ be such that the open disc in $\mathbb{R}^n$ centered at $x_0 \in \mathbb{R}^n$ and radius ρ, $D_\rho(x_0) \subseteq \omega$. There exists a constant $C_\rho > 0$, independent of x_0, such that*

$$\int_{\mathbb{R}^n} \frac{|v(x)|^2}{(|x-x_0|+1)^{n+2s}}\,\mathrm{d}x \leqslant C_\rho\left(\mathcal{E}\big(v,D_\rho(x_0)\big) + \|v\|^2_{L^2(D_\rho(x_0))}\right)$$

for all $v \in \widehat{H}^s(\omega;\mathbb{R}^m)$.

DEFINITION 8 (Minimizing s-Harmonic maps). *We say that $u \in \widehat{H}^s(\omega;\mathcal{N})$ is a minimizing s-harmonic map in ω if*

$$\mathcal{E}(u,\omega) \leqslant \mathcal{E}(\widetilde{u},\omega)$$

for every $\widetilde{u} \in \widehat{H}^s(\omega;\mathcal{N})$ such that $\mathrm{spt}(\widetilde{u}-u) \subseteq \omega$.

Exactly as in the case $s = 1/2$ (see [40, Remark 4.24]), a minimizing s-harmonic map satisfies the Euler-Lagrange equation

$$(-\Delta)^s u \perp \mathrm{Tan}(u, \mathcal{N}) \quad \text{in } \mathscr{D}'(\omega)\,.$$

In terms of scaling, this equation turns out to be supercritical (since $s \in (0, 1/2)$), and one may expect that minimizing s-harmonic maps are singular, exactly as it happens for (classical) minimizing harmonic maps in dimensions greater than three [52].

DEFINITION 9 (Stationary Fractional Harmonic Maps). *Let $v \in \widehat{H}^{1/2}(\omega; \mathbb{R}^m)$ be a weak 1/2-harmonic map into $\mathbb{S}^{m-1}$ in ω. We say that v is stationary in ω if*

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{E}\big(v \circ \phi_t, \omega\big)\right]_{t=0} = 0$$

for any differentiable 1-parameter group of smooth diffeomorphisms $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ satisfying

- $\phi_0 = \mathrm{id}_{\mathbb{R}^n}$;
- $\phi_t - \mathrm{id}_{\mathbb{R}^n}$ *is compactly supported, and* $\mathrm{supp}(\phi_t - \mathrm{id}_{\mathbb{R}^n}) \subseteq \omega$.

To prove some regularity of the fractional harmonic maps it necessary to derive some monotonicity formula. So far the only available technique to derive such formula is to use some extension technique in the upper-half space. Indeed, the problem can be rephrased as a degenerate regularity problem for harmonic maps with free boundary, once we use the so-called Caffarelli-Silvestre extension [12]. It would a major achievement to find a monotone quantity at the level of the nonlocal equation itself.

Minimizing weighted harmonic maps

For a bounded admissible open subset $\Omega \subseteq \mathbb{R}^{n+1}_+$, we consider the weighted spaces

$$L^2(\Omega; \mathbb{R}^d, y^a d\mathbf{x}) := \left\{ v \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^d) : y^{\frac{a}{2}} |v| \in L^2(\Omega) \right\} \quad \text{with } a := 1 - 2s > 0\,,$$

and

$$H^1(\Omega; \mathbb{R}^d, y^a d\mathbf{x}) := \left\{ v \in L^2(\Omega, y^a d\mathbf{x}) : \nabla v \in L^2(\Omega, y^a d\mathbf{x}) \right\}.$$

We refer to [42, Section 2] for the main properties of these spaces that we shall use. We simply recall that a map $v \in H^1(\Omega; \mathbb{R}^d, y^a d\mathbf{x})$ has a well defined trace on $\partial^0\Omega$, and the trace operator from $H^1(\Omega; \mathbb{R}^d, y^a d\mathbf{x})$ into $L^2(\partial^0\Omega; \mathbb{R}^d)$ is a compact linear operator.

On $H^1(\Omega, y^a dxdy)$, we define the weighted Dirichlet energy

$$\mathbf{E}_s(v, \Omega) := \frac{1}{2} \int_\Omega y^a |\nabla v|^2 \, d\mathbf{x}\,.$$

DEFINITION 10 (Minimizing Weighted Harmonic Map). *Let $\Omega \subseteq \mathbb{R}^{n+1}_+$ be a bounded admissible open set, and consider a map $v \in H^1(\Omega;\mathbb{R}^d, y^a d\mathbf{x})$ such that $v(\mathbf{x}) \in \mathcal{N}$ a.e. on $\partial^0\Omega$. We say that v is a minimizing weighted harmonic map in Ω with respect to the (partially) free boundary $v(\partial^0\Omega) \subseteq \mathcal{N}$ if*

$$\mathbf{E}_s(v,\Omega) \leqslant \mathbf{E}_s(w,\Omega)$$

for every competitor $w \in H^1(\Omega, y^a d\mathbf{x})$ satisfying $w(\mathbf{x}) \in \mathcal{N}$ a.e. on $\partial^0\Omega$, and such that $\mathrm{spt}(w-v) \subseteq \Omega \cup \partial^0\Omega$. *In short, we shall say u is a minimizing weighted harmonic map with free boundary in Ω.*

REMARK 1. Using variations supported in Ω, one obtains that a minimizing weighted harmonic map v with free boundary in Ω satisfies

$$\mathrm{div}(y^a \nabla v) = 0 \quad \text{in } \Omega\,. \tag{2.10}$$

In particular, $v \in C^\infty(\Omega)$ by standard elliptic theory. The regularity issue is then at (and only at) the free boundary $\partial^0\Omega$. Arguing exactly as [17, Section 2], one obtains

$$y^a \partial_y v \perp \mathrm{Tan}(v,\mathcal{N}) \quad \text{on } \partial^0\Omega$$

in the duality sense. In other words, the (full) Euler-Lagrange equation derived from minimality is

$$\int_\Omega y^a \nabla v \cdot \nabla \zeta \, d\mathbf{x} = 0 \tag{2.11}$$

for every $\zeta \in H^1(\Omega;\mathbb{R}^d, y^a d\mathbf{x})$ satisfying $\zeta(\mathbf{x}) \in \mathrm{Tan}(v(\mathbf{x}),\mathcal{N})$ for a.e. $\mathbf{x} \in \partial^0\Omega$, and such that $\mathrm{spt}(\zeta) \subseteq \Omega \cup \partial^0\Omega$.

Extending fractional harmonic maps

We now move on the extension procedure of [12]. Given a bounded open interval $\omega \subseteq \mathbb{R}$, we define the extension $u^{\mathrm{e}} : \mathbb{R}^{n+1}_+ \to \mathbb{R}^d$ of a map $u \in \widehat{H}^s(\omega;\mathbb{R}^d)$ by

$$u^{\mathrm{e}}(x,y) := \sigma_{n,s} \int_{\mathbb{R}} \frac{t^{2s} u(t)}{(|x-y|^2+t^2)^{\frac{n+2s}{2}}}\, dt \quad \text{with } \sigma_{n,s} := \pi^{-\frac{n}{2}} \frac{\Gamma(\frac{n+2s}{2})}{\Gamma(s)}\,. \tag{2.12}$$

This extension can be referred to as *fractional harmonic extension* of u as it solves

$$\begin{cases} \mathrm{div}(y^a \nabla u^{\mathrm{e}}) = 0 & \text{in } \mathbb{R}^{n+1}_+\,, \\ u^{\mathrm{e}} = u & \text{on } \mathbb{R}^n \simeq \partial\mathbb{R}^{n+1}_+\,. \end{cases} \tag{2.13}$$

It turns out that $u^{\mathrm{e}} \in H^1(\Omega;\mathbb{R}^d, y^a d\mathbf{x})$ for every bounded admissible open set $\Omega \subseteq \mathbb{R}^{n+1}_+$ such that $\overline{\partial^0\Omega} \subseteq \omega$. In addition, $u^{\mathrm{e}} \in L^\infty(\mathbb{R}^{n+1}_+)$ whenever $u \in L^\infty(\mathbb{R}^n)$, and $\|u^{\mathrm{e}}\|_{L^\infty(\mathbb{R}^{n+1}_+)} \leqslant \|u\|_{L^\infty}(\mathbb{R})$. We refer to [42, Section 2] for further details.

The following propositions draw links between s-harmonic maps and harmonic maps with free boundary. Their proofs follow as in [40, Proposition 4.6, 4.9, 4.15].

PROPOSITION 1 (**Minimality transfer**). *Let $\omega \subseteq \mathbb{R}^n$ be a bounded open interval, and $u \in \widehat{H}^s(\omega;\mathcal{N})$ a minimizing s-harmonic map in ω. Then u^{e} is a minimizing weighted harmonic map in Ω with free boundary in every bounded admissible open set $\Omega \subseteq \mathbb{R}^{n+1}_+$ satisfying $\overline{\partial^0\Omega} \subseteq \omega$.*

PROPOSITION 2 (**Criticality transfer**). *Let $v \in \widehat{H}^{1/2}(\omega;\mathbb{R}^m)$ be a weak 1/2-harmonic map into $\mathbb{S}^{m-1}$ in ω, and let v^{e} be its harmonic extension to $\mathbb{R}^{n+1}_+$ given by (4.5). Then v^{e} is a weak $(\mathbb{S}^{m-1},\partial^0\Omega)$-boundary harmonic map in Ω for every admissible bounded open set $\Omega \subseteq \mathbb{R}^{n+1}_+$ such that $\overline{\partial^0\Omega} \subseteq \omega$.*

PROPOSITION 3 (**Stationarity transfer**). *Let $v \in \widehat{H}^{1/2}(\omega;\mathbb{R}^m)$ be a stationary weak 1/2-harmonic map into $\mathbb{S}^{m-1}$ in ω. Let v^{e} be the harmonic extension of v in $\mathbb{R}^{n+1}_+$. Then v^{e} is a stationary weak $(\mathbb{S}^{m-1},\partial^0\Omega)$-boundary harmonic map in Ω for every admissible bounded open set $\Omega \subseteq \mathbb{R}^{n+1}_+$ such that $\overline{\partial^0\Omega} \subseteq \omega$.*

The Monotonicity Formula

We present the fundamental monotonicity formula involving the following density function: for a point $\mathbf{x}_0 = (x_0,0) \in \partial^0\Omega$ and $r>0$ such that $B^+_r(\mathbf{x_0}) \subseteq \Omega$, we set

$$\Theta_v(\mathbf{x}_0,r) := \frac{1}{r^{1-2s}}\mathbf{E}_s\big(v,B^+_r(\mathbf{x}_0)\big)\,.$$

LEMMA 2. *For every $\mathbf{x}_0 \in \partial^0\Omega$, the limit*

$$\Theta_v(\mathbf{x}_0) := \lim_{r\downarrow 0}\Theta_v(\mathbf{x}_0,r)$$

exists, and the function $\Theta_v : \partial^0\Omega \to [0,\infty)$ is upper semicontinuous. In addition,

$$\Theta_v(\mathbf{x}_0,r) - \Theta_v(\mathbf{x}_0) = \int_{B^+_r(\mathbf{x}_0)} y^a\,\frac{|(\mathbf{x}-\mathbf{x}_0)\cdot\nabla v|^2}{|\mathbf{x}-\mathbf{x}_0|^{3-2s}}\,d\mathbf{x}\,. \tag{2.14}$$

Small energy regularity

To prove Hölder continuity it is standard to show some energy improvement under a small oscillation condition. This leads to an improved energy decay, which in turn implies Hölder continuity as in the classical Morrey's lemma.

THEOREM 2 (**Energy improvement**). *There exist constants $r_0 = r_0(s,\mathcal{N}) \in (0,1/2)$ and $\varepsilon_2 = \varepsilon_2(s,\mathcal{N}) > 0$ such that the following holds. If $v \in H^1(B^+_1;\mathbb{R}^d, y^a d\mathbf{x})$ is a minimizing weighted harmonic map in B^+_1 satisfying $\mathbf{E}_s(v,B^+_1) \leqslant \varepsilon_2^2$, then*

$$\frac{1}{r_0^{1-2s}}\mathbf{E}_s(v,B^+_{r_0}) < \frac{1}{2}\mathbf{E}_s(v,B^+_1)\,.$$

COROLLARY 1 (**Energy decay**). *If $v \in H^1(B^+_{2R};\mathbb{R}^d, y^a d\mathbf{x})$ is a minimizing weighted harmonic map in B^+_{2R} satisfying $\mathbf{E}_s(v, B^+_{2R}) \leqslant \varepsilon_2^2 R^{1-2s}$, then*

$$\frac{1}{r^{1-2s}}\mathbf{E}_s\big(v, B^+_r(\mathbf{x})\big) \leqslant CR^{-\beta}r^{\beta} \quad \textit{for all } \mathbf{x} \in \partial^0 B^+_R \textit{ and } 0 < r \leqslant R\,,$$

for some exponent $\beta \in (0,1)$ depending only on s and $\mathcal{N}$.

In turn, this last corollary implies Hölder continuity at the boundary as in Morrey's lemma.

COROLLARY 2. *In addition to Corollary 1, v is Hölder continuous on $\partial^0 B^+_R$ with Hölder exponent $\beta/2$.*

Compactness of Minimizing s-harmonic Maps

THEOREM 3. *Let $\omega \subseteq \mathbb{R}$ be a bounded open interval, and $\{u_n\}_{n\in\mathbb{N}} \subseteq \widehat{H}^s(\omega;\mathcal{N})$ a sequence of minimizing s-harmonic maps in ω. Assume that $\sup_n \mathcal{E}(u_n,\omega) < \infty$, and $u_n \to u$ in $L^2_{\mathrm{loc}}(\mathbb{R})$. Then $u \in \widehat{H}^s(\omega;\mathcal{N})$ is a minimizing s-harmonic map in ω, $u_n \to u$ strongly in $H^s_{\mathrm{loc}}(\omega)$, and $\mathcal{E}(u_n,\omega') \to \mathcal{E}(u,\omega')$ for every open interval such that $\overline{\omega}' \subseteq \omega$.*

As a consequence of this theorem we have the following result.

COROLLARY 3. *In addition to Theorem 3, if $\{x_n\} \subseteq \omega$ is a sequence converging to $x \in \omega$, then*

$$\limsup_{n\to\infty} \Theta_{u_n^{\mathrm{e}}}(x_n) \leqslant \Theta_{u^{\mathrm{e}}}(x)\,.$$

PROPOSITION 4 (**Tangent maps**). *Let $\rho_n \to 0$ be an arbitrary sequence. There is a (not relabeled) subsequence such that $u_{x_0,\rho_n} \to u_0$ strongly in $H^s_{\mathrm{loc}}(\mathbb{R})$, where u_0 is a minimizing s-harmonic map in every bounded open interval of the form*

$$u_0(x) := \begin{cases} a & \textit{if } x > 0\,, \\ b & \textit{if } x < 0\,, \end{cases} \tag{2.15}$$

for some $a, b \in \mathcal{N}$. In addition, $\Theta_{u_0^{\mathrm{e}}}(0,r) = \Theta_{u_0^{\mathrm{e}}}(0) = \Theta_{u^{\mathrm{e}}}(x_0)$ for every $r > 0$.

REMARK 2. If u is continuous at x_0, the limit u_0 obtained in Proposition 4 is obviously the constant map equal to $u(x_0)$. As a consequence, if u is continuous at x_0, then $\Theta_{u^{\mathrm{e}}}\big((x_0,0)\big) = 0$.

Regularity of $s-$Harmonic Maps

THEOREM 4. *Let $\omega \subseteq \mathbb{R}$ be a bounded open interval. For $s \in (0,1/2)$, let $u \in \widehat{H}^s(\omega;\mathcal{N})$ be a minimizing s-harmonic map in ω. Then u is locally Hölder continuous in ω away from a locally finite set of points.*

Proof. Let us consider the set

$$S := \left\{x \in \omega : \Theta_{u^{\mathrm{e}}}\big((x,0)\big) \geqslant 2^{2s-1}\varepsilon_2^2\right\}, \tag{2.16}$$

where $\varepsilon_2 > 0$ is the constant given by Theorem 2. Since $\Theta_{u^{\mathrm{e}}}$ is upper semicontinuous, S is a relatively closed subset of ω. Moreover, Corollaries 1 & 2 together with Lemma 2 implies that u is locally Hölder continuous in $\omega \setminus S$. To prove Theorem 4, it then remains to show that S has no accumulation point in ω. We argue by contradiction assuming that there is a sequence $\{x_n\} \subseteq S$ such that $x_n \to x \in \omega$. Without loss of generality, we may assume that $x_n > x$. Setting $\rho_n := x_n - x$, we consider the sequence $u_n := u_{x_0,\rho_n}$, and then apply Proposition 4 to find a (not relabeled) subsequence and a minimizing s-harmonic map u_0 of the form (2.15) such that $u_n \to u_0$. In view of Corollary 3 we have

$$\Theta_{u_0^{\mathrm{e}}}\big((1,0)\big) \geqslant \limsup_{n\to\infty} \Theta_{u_n^{\mathrm{e}}}\big((1,0)\big) = \limsup_{n\to\infty} \Theta_{u^{\mathrm{e}}}\big((x_n,0)\big) \geqslant \varepsilon_2 .$$

On the other hand, by the explicit form (2.15), the map u_0 is continuous at 1. Hence, $\Theta_{u_0^{\mathrm{e}}}\big((1,0)\big) = 0$ by Remark 2, contradiction. □

THEOREM 5. *For $s \in (0,1/2)$ and $d \geq 2$, let $u \in \widehat{H}^s(\omega;\mathbb{S}^{d-1})$ be a minimizing s-harmonic map in ω. Then u is locally Hölder continuous in ω.*

Proof. Recall that we assume now that $\mathcal{N} = \mathbb{S}^{d-1}$. In view of the proof of Theorem 4, it is enough to show that the set S defined in (2.16) is empty. Assume by contradiction that $S \neq \emptyset$. We may then assume without loss of generality that $0 \in S$. Let u_0 be a s-minimizing harmonic map produced by Proposition 4, i.e., u_0 is the limit of the rescaled map u_{0,ρ_n} for some sequence $\rho_n \to 0$. Then $\Theta_{u_0^{\mathrm{e}}}(0) \geqslant \varepsilon_2 > 0$, so that u_0 is not constant. In other words, in the form (2.15) the two vectors $a, b \in \mathbb{S}^{d-1}$ are distinct. Upon working in the plane passing through a, b, and the origin, there is no loss of generality assuming that $d = 2$, that is $\mathcal{N} = \mathbb{S}^1$. Moreover, rotating coordinates in the image if necessary, we can assume that

$$a = (\alpha,\beta) \text{ and } b = (-\alpha,\beta),$$

with $0 < \alpha \leqslant 1$ and $0 \leqslant \beta < 1$ satisfying $\alpha^2 + \beta^2 = 1$. Then set

$$a^* := (-\beta,\alpha) \text{ and } b^* := (\beta,\alpha).$$

Note that $a^* \perp a$ and $b^* \perp b$. We define for $t \in \mathbb{R}$,

$$u_t(x) := \begin{cases} \dfrac{a + ta^*}{\sqrt{1+t^2}} & \text{if } 0 < x < 1, \\ \dfrac{b + tb^*}{\sqrt{1+t^2}} & \text{if } -1 < x < 0, \\ u_0(x) & \text{otherwise}. \end{cases}$$

One can easily check that $u_t \in \widehat{H}^s\big((-2,2);\mathbb{S}^1\big)$, and since $\operatorname{spt}(u_t - u) \subseteq (-2,2)$, the map u_t is an admissible competitor for the minimality of u_0 in $(-2,2)$. In other words,

$\mathcal{E}\big(u_0,(-2,2)\big) \leqslant \mathcal{E}\big(u_t,(-2,2)\big)$, which in turn yields $\mathcal{E}\big(u_0,(-1,1)\big) \leqslant \mathcal{E}\big(u_t,(-1,1)\big)$ since $u_t = u_0$ outside $(-1,1)$. Therefore,

$$\left[\frac{d}{dt}\mathcal{E}\big(u_t,(-1,1)\big)\right]_{t=0} = 0 \quad \text{and} \quad \left[\frac{d^2}{dt^2}\mathcal{E}\big(u_t,(-1,1)\big)\right]_{t=0} \geqslant 0\,.$$

Now we expand $\mathcal{E}\big(u_t,(-1,1)\big)$ as

$$\begin{aligned}
\mathcal{E}\big(u_t,(-1,1)\big) = &\left(\gamma_s \int_0^1 \int_{-1}^0 \frac{dxdy}{|x-y|^{1+2s}}\right) \frac{\big|(a-b)+t(a^*-b^*)\big|^2}{1+t^2} \\
&+ \left(\gamma_s \int_0^1 \int_1^{+\infty} \frac{dxdy}{|x-y|^{1+2s}}\right) \frac{\big|(1-\sqrt{1+t^2})a+ta^*\big|^2}{1+t^2} \\
&+ \left(\gamma_s \int_0^1 \int_{-\infty}^{-1} \frac{dxdy}{|x-y|^{1+2s}}\right) \frac{\big|a+ta^*-b\sqrt{1+t^2}\big|^2}{1+t^2} \\
&+ \left(\gamma_s \int_{-1}^0 \int_{-\infty}^{-1} \frac{dxdy}{|x-y|^{1+2s}}\right) \frac{\big|(1-\sqrt{1+t^2})b+tb^*\big|^2}{1+t^2} \\
&+ \left(\gamma_s \int_{-1}^0 \int_1^{+\infty} \frac{dxdy}{|x-y|^{1+2s}}\right) \frac{\big|b+tb^*-a\sqrt{1+t^2}\big|^2}{1+t^2}\,.
\end{aligned}$$

It then follows that

$$\left[\frac{d}{dt}\mathcal{E}\big(u_t,(-1,1)\big)\right]_{t=0} = -C\alpha\beta$$

for some constant $C = C(s) > 0$. The first order condition implies $\alpha\beta = 0$, and thus

$$a = (1,0)\,,\ b = (-1,0)\,,\ \text{and } a^* = b^* = (0,1)\,.$$

As a consequence, using the symmetry in the integrals above,

$$\begin{aligned}
\mathcal{E}\big(u_t,(-1,1)\big) = &\left(4\gamma_s \int_0^1 \int_{-1}^0 \frac{dxdy}{|x-y|^{1+2s}}\right) \frac{1}{1+t^2} \\
&+ \left(2\gamma_s \int_0^1 \int_1^{+\infty} \frac{dxdy}{|x-y|^{1+2s}}\right) \frac{(1-\sqrt{1+t^2})^2+t^2}{1+t^2} \\
&+ \left(2\gamma_s \int_0^1 \int_{-\infty}^{-1} \frac{dxdy}{|x-y|^{1+2s}}\right) \frac{(1+\sqrt{1+t^2})^2+t^2}{1+t^2}\,.
\end{aligned}$$

An elementary computation now yields

$$\begin{aligned}\left[\frac{d^2}{dt^2}\mathcal{E}\big(u_t,(-1,1)\big)\right]_{t=0} &= -8\gamma_s\int_0^1\int_{-1}^0\frac{dxdy}{|x-y|^{1+2s}}+4\gamma_s\int_0^1\int_1^{+\infty}\frac{dxdy}{|x-y|^{1+2s}}\\ &\qquad -4\gamma_s\int_0^1\int_{-\infty}^{-1}\frac{dxdy}{|x-y|^{1+2s}}\\ &= -4\gamma_s\int_0^1\int_{-1}^0\frac{dxdy}{|x-y|^{1+2s}}+4\gamma_s\int_0^1\int_1^{+\infty}\frac{dxdy}{|x-y|^{1+2s}}\\ &\qquad -4\gamma_s\int_0^1\int_{-\infty}^{0}\frac{dxdy}{|x-y|^{1+2s}}\\ &= -4\gamma_s\int_0^1\int_{-1}^0\frac{dxdy}{|x-y|^{1+2s}}\\ &< 0\,,\end{aligned}$$

contradicting the second order condition for minimality. □

Regularity of $1/2$ Harmonic Maps

THEOREM 6. *Let $\omega\subseteq\mathbb{R}^n$ be a smooth bounded open set, and let $v_*\in\widehat{H}^{1/2}(\omega;\mathbb{R}^m)\cap L^\infty(\mathbb{R}^n)$ be a weak 1/2-harmonic map into $\mathbb{S}^{m-1}$ in ω. Then $v_*\in C^\infty\big(\omega\setminus\mathrm{sing}(v_*)\big)$, where $\mathrm{sing}(v_*)$ denotes the complement of the largest open set on which v_* is continuous. Moreover,*

(i) *if $n=1$, then $\mathrm{sing}(v_*)\cap\omega=\emptyset$;*

(ii) *if $n\geqslant 2$ and v_* is stationary, then $\mathscr{H}^{n-1}\big(\mathrm{sing}(v_*)\cap\omega\big)=0$;*

(iii) *if v_* is minimizing, then $\dim_{\mathscr{H}}\big(\mathrm{sing}(v_*)\cap\omega\big)\leqslant n-2$ for $n\geqslant 3$, and $\mathrm{sing}(v_*)\cap\omega$ is discrete for $n=2$.*

The proof of this theorem is a consequence of the regularity of harmonic maps with partially free boundary. We state the result in the form of the following theorem.

THEOREM 7. *Let $\Omega\subseteq\mathbb{R}^{n+1}_+$ be an admissible bounded open set. If $u\in H^1(\Omega;\mathbb{R}^m)\cap L^\infty(\Omega)$ is a weak $(\mathbb{S}^{m-1},\partial^0\Omega)$-boundary harmonic map in Ω, then $u\in C^\infty\big((\Omega\cup\partial^0\Omega)\setminus\mathrm{sing}(u)\big)$. Moreover,*

(i) *if $n=1$, then $\mathrm{sing}(u)=\emptyset$;*

(ii) *if $n\geqslant 2$ and u is stationary, then $\mathscr{H}^{n-1}(\mathrm{sing}(u))=0$;*

(iii) *if u is minimizing, then $\dim_{\mathscr{H}}(\mathrm{sing}(u))\leqslant n-2$ for $n\geqslant 3$, and $\mathrm{sing}(u)$ is discrete for $n=2$.*

Although the the proof of Theorem 6 follows from the regularity theory of boundary harmonic maps, it is worth mentioning a recent development by Schikorra and Mazowiecka [33], where they have established fractional versions of div-curl lemma and Wente's lemma [61]. We will finish the discussion by stating their results.

Fractional Gradient and Divergence: We will denote by $\mathcal{M}(\mathbb{R}^n)$ the set of all functions measurable with respect to the Lebesgue measure dx. Furthermore, the space of measurable (off diagonal) vector fields $\mathcal{M}(\bigwedge^1_{od}\mathbb{R}^n)$ is the space of functions $F:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ measurable with respect to the measure $\frac{dxdy}{|x-y|^n}$. Here "od" stands for "off diagonal". Observe that we do not require antisymmetry $F(x,y)=-F(y,x)$. On the space of measurable vector fields we define the scalar product. For two vector fields $F,G\in\mathcal{M}(\bigwedge^1_{od}\mathbb{R}^n)$ set

$$\langle F,G\rangle(x):=F\cdot G(x):=\int_{\mathbb{R}^n}F(x,y)G(x,y)\frac{dy}{|x-y|^n}.$$

In particular we denote $||F||_2(x):=\sqrt{\langle F,F\rangle(x)}$ and more generally,

$$||F||_p(x):=\left(\int_{\mathbb{R}^n}|F(x,y)|^p\frac{dy}{|x-y|^n}\right)^{\frac{1}{p}}.$$

The $s-gradient\ d_s:\mathcal{M}(\mathbb{R}^n)\to\mathcal{M}(\bigwedge^1_{od}\mathbb{R}^n)$ acting on functions $g:\mathbb{R}^n\to\mathbb{R}$ takes the form $d_sg(x,y):=\frac{g(x)-g(y)}{|x-y|^s}\in\mathcal{M}(\bigwedge^1_{od}\mathbb{R}^n)$. In particular,

$$F\cdot d_sg(x)=\int_{\mathbb{R}^n}F(x,y)\frac{g(x)-g(y)}{|x-y|^s}\frac{dy}{|x-y|^n}.$$

We will call the dual operation to the $s-gradient\ d_s$, the $s-divergence$ div_s. It maps a vector field $F\in\mathcal{M}(\bigwedge^1_{od}\mathbb{R}^n)$ into a function $\mathrm{div}_sF\in\mathcal{M}(\mathbb{R}^n)$. Its distributional definition is

$$\mathrm{div}(F)[\phi]=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}F(x,y)d_s\phi(x,y)\frac{dxdy}{|x-y|^n},\phi\in C^\infty_c(\mathbb{R}^n). \tag{2.17}$$

In particular we say a vector field is *divergence free* if its *s-divergence* is 0 in the sense of distribution given above. Moreover, the canonical relation to the fractional Laplacian holds true: $\mathrm{div}_sd_s=-(-\Delta)^s$, in the following sense

$$\int_{\mathbb{R}^n}d_sf\cdot d_sg(x)dx=\int_{\mathbb{R}^n}(-\Delta)^sf(x)g(x)dx.$$

For $p\in(1,\infty)$ the natural L^p-space on vector fields F, is induced by the norm $||F||_{L^p(\bigwedge^1_{od}\mathbb{R}^n)}:=\left|\left|||F||_p(.)\right|\right|_{L^p(\mathbb{R}^n)}$. Observe that, $[g]_{W^{s,p}(\mathbb{R}^n)}=||d_sg||_{L^p(\bigwedge^1_{od}\mathbb{R}^n)}$. Lastly, for $\Omega\subseteq\mathbb{R}^n$ we denote

$$||F||_{L^p(\bigwedge^1_{od}\Omega)}:=\left(\iint_{(\Omega\times\mathbb{R}^n)\cup(\mathbb{R}^n\times\Omega)}|F(x,y)|^p\frac{dxdy}{|x-y|^n}\right)^{\frac{1}{p}}.$$

Now we are in situation to state the counterpart of Theorem 1 derived in [33].

THEOREM 8 (div-curl quantity and Hardy space). *Let $s \in (0,1)$, $p \in (0,\infty)$. For $F \in L^p(\bigwedge^1_{od}\mathbb{R}^n)$ and $g \in W^{s,p'}(\mathbb{R}^n)$ assume that* $\mathrm{div}_s F = 0$. *Then $F \cdot d_s g \in \mathcal{H}^1(\mathbb{R}^n)$ and we have the estimate*

$$||F \cdot d_s g||_{\mathcal{H}^1(\mathbb{R}^n)} \leqslant C\,||F||_{L^p\left(\bigwedge^1_{od}\mathbb{R}^n\right)}\ ||d_s g||_{L^{p'}\left(\bigwedge^1_{od}\mathbb{R}^n\right)},$$

or equivalently, for any $\phi \in C^\infty_c(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \phi F \cdot d_s g dx \leqslant C[\phi]_{BMO(\mathbb{R}^n)}\,||d_s g||_{L^{p'}\left(\bigwedge^1_{od}\mathbb{R}^n\right)}.$$

Here, $C > 0$ is a uniform constant depending on s, p and the dimension n.

As a first immediate application let us state the following fractional version of Wente's lemma [33].

COROLLARY 4 (Fractional Wente Lemma). *Let $s \in (0,\infty)$, $p \in (1,\infty)$. For $F \in L^p(\bigwedge^1_{od}\mathbb{R}^n)$ and $g \in W^{s,p'}(\mathbb{R}^n)$ assume that* $\mathrm{div}_s F = 0$. *Moreover, let $T[\phi]$ be a linear operator such that for some $\Lambda > 0$*

$$T[\phi] \leqslant \Lambda \left|\left|(-\Delta)^{\frac{n}{4}}\phi\right|\right|_{L^{(2,\infty)}(\mathbb{R}^n)},\ \forall \phi \in C^\infty_c(\mathbb{R}^n),$$

where $L^{(2,\infty)}$ denotes the weak L^2 space. Then any distributional solution $u \in \dot{W}^{\frac{n}{2},2}(\mathbb{R}^n)$ to

$$(-\Delta)^{\frac{n}{2}} u = F \cdot d_s g + T \text{ in } \mathbb{R}^n,$$

is continuous and if $\lim_{|x|\to\infty} u(x) = 0$, *then*

$$||u||_{L^\infty(\mathbb{R}^n)} \leqslant C\left(||F||_{L^p\left(\bigwedge^1_{od}\mathbb{R}^n\right)}||d_s g||_{L^{p'}\left(\bigwedge^1_{od}\mathbb{R}^n\right)} + \Lambda\right)$$

for a uniform constant $C > 0$ depending only on s, p and n.

Continuity of $1/2$-Harmonic maps in $\mathbb{R}$: Let $v \in \widehat{H}^{\frac{1}{2}}(\mathbb{R},\mathbb{S}^{m-1})$ be any weak harmonic map i.e. satisfies (2.9). Then it is easy to see

$$(-\Delta)^{\frac{1}{2}} v^i(x) = \langle \Omega_{i,k}, d_{\frac{1}{2}} v^k\rangle(x) + T(x),$$

where

$$\Omega_{ik}(x,y) := v^i(x) d_{\frac{1}{2}} v^k(x,y) - v^k(x) d_{\frac{1}{2}} v^i(x,y) \text{ and } T(x) := \langle d_{\frac{1}{2}} v^i, v^k(x) d_{\frac{1}{2}} v^k\rangle(x).$$

As proved in [33, Lemma 3.1], $\mathrm{div}_{\frac{1}{2}}\Omega_{ik} = 0$ for all $i,k \in \{1,\dots,m\}$. On the other hand T satisfies

$$\left|\int_{\mathbb{R}} T(x)\phi(x)dx\right| \preceq \left|\left|(-\Delta)^{\frac{1}{4}}\phi\right|\right|_{L^{(2,\infty)}(\mathbb{R})},\ \ \forall \phi \in C^\infty_c(\mathbb{R}).$$

Therefore straightforward application of Fractional Wente's Lemma (i.e. Corollary 4) proves the continuity of $1/2$-Harmonic maps in dimension 1. □

The link with Minimal Surfaces with Free boundary

We now put the previously defined half-harmonic maps into general targets in perspective with the so-called minimal surfaces with free boundary. The definition of 1/2-harmonic maps extends to a more general target submanifold $\mathcal{N} \subseteq \mathbb{R}^m$ in the following way. Assuming that $\mathcal{N}$ is smooth, compact, and without boundary, the nearest point retraction $\pi_{\mathcal{N}}$ on $\mathcal{N}$ is smooth in a small tubular neighborhood of $\mathcal{N}$. We then say that a map $v \in \widehat{H}^{1/2}(\omega;\mathbb{R}^m)$ satisfying $v(x) \in \mathcal{N}$ for a.e. $x \in \omega$, is weakly 1/2-harmonic into $\mathcal{N}$ in ω if

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{E}\left(\pi_{\mathcal{N}}(v+t\varphi),\omega\right)\right]_{t=0} = 0$$

for all $\varphi \in H^{1/2}_{00}(\omega;\mathbb{R}^m) \cap L^\infty(\omega)$ compactly supported in ω. The associated Euler-Lagrange equation reads

$$(-\Delta)^{\frac{1}{2}} v \perp T_v\mathcal{N} \quad \text{in } H^{-1/2}_{00}(\omega)\,,$$

i.e., $\left\langle (-\Delta)^{\frac{1}{2}} v, \varphi \right\rangle_\omega = 0$ for all $\varphi \in H^{1/2}_{00}(\omega;\mathbb{R}^m)$ satisfying $\varphi(x) \in T_{v(x)}\mathcal{N}$ for a.e. $x \in \omega$.

Concerning the definition of $(\mathcal{N},\partial^0\Omega)$-valued boundary harmonic maps, we simply reproduce Definition 5 replacing $\mathbb{S}^{m-1}$ by the manifold $\mathcal{N}$. The notions of minimality and stationarity for both boundary harmonic and 1/2-harmonic maps remain unchanged. As a result, we have the same partial regularity theory for $(\mathcal{N},\partial^0\Omega)$-boundary harmonic maps as stated in Theorem 7 by the general regularity results in [28, 17, 18, 51]. As a consequence, we have the same partial regularity theory for 1/2-harmonic maps into $\mathcal{N}$ as stated in Theorem 6. We now have the following lemma.

LEMMA 3. *Let $v \in \dot{H}^{1/2}(\mathbb{R};\mathbb{R}^m)$ be a nontrivial entire 1/2-harmonic map into $\mathcal{N} \hookrightarrow \mathbb{R}^m$, and let v^{e} be its harmonic extension to $\mathbb{R}^2_+$. Then v^{e} is either a conformal or an anti-conformal transformation of $\overline{\mathbb{R}}^2_+$ into the closed unit disc $\overline{\mathbb{D}}$.*

If $m = 3$ and $\mathcal{N}$ is a surface, it shows that the image of $\mathbb{R}^2_+$ by v^{e} is a minimal surface in $\mathbb{R}^3$ whose boundary lies in $\mathcal{N}$, and meets $\mathcal{N}$ orthogonaly, see [59]. If $\mathcal{N}$ is a closed boundary curve, then the image of $\mathbb{R}^2_+$ by v^{e} is a minimal surface spanned by $\mathcal{N}$.

3. Singular perturbation of half-Ginzburg-Landau

3.1. Ginzburg-Landau Energy and Harmonic Maps

The Ginzburg-Landau equation is precisely a vectorial version of the Allen-Cahn equation (1.2) and is given by the following equation.

$$\begin{aligned} -\Delta v &= \frac{1}{\varepsilon^2}\left(1-|v|^2\right)v, \quad \text{in } \omega, \\ v &= g, \quad \text{on } \partial\omega. \end{aligned} \tag{3.1}$$

Here ω is a smooth bounded domain in $\mathbb{R}^n$ ($n \geqslant 1$), $v : \mathbb{R}^n \to \mathbb{R}^m$ ($m \geqslant 2$) and $g : \mathbb{R}^n \to \mathbb{R}^m$ is a smooth function satisfying $|g| = 1$. This local Ginzburg-Landau equation has a geometrical flavour like in the scalar case, and the limiting system in the asymptotic $\varepsilon \to 0$ is given by the harmonic map equation into the sphere $\mathbb{S}^{m-1}$ (or into more general manifolds according to the potential well). The weak convergence of critical point of the local Ginzburg-Landau energy towards (weakly) harmonic maps has been proved by LIN &WANG [35, 36, 37] (see also [12]). In the spirit of the blow-up analysis for harmonic maps developed in [34], the authors have determined in [35, 36, 37] the geometrical nature of the possible defect measure arising in the weak convergence process, as well as the rectifiability of the energy concentration set.

3.2. The fractional Ginzburg-Landau equation

We consider for the entire section a bounded open set $\omega \subseteq \mathbb{R}^n$ with Lipschitz boundary.

Energy spaces and weak formulation

We recall that the fractional Laplacian can be defined via the following weak formulation. For a given map $v \in \widehat{H}^{1/2}(\omega;\mathbb{R}^m)$, we define the distribution $(-\Delta)^{\frac{1}{2}} v$ through its action on $\widehat{H}^{1/2}(\omega;\mathbb{R}^m)$ by setting

$$\left\langle (-\Delta)^{\frac{1}{2}} v, \varphi \right\rangle_\omega := \frac{\gamma_n}{2} \iint_{\omega\times\omega} \frac{(v(x)-v(y))\cdot(\varphi(x)-\varphi(y))}{|x-y|^{n+1}}\,\mathrm{d}x\mathrm{d}y + \gamma_n \iint_{\omega\times\omega^c} \frac{(v(x)-v(y))\cdot(\varphi(x)-\varphi(y))}{|x-y|^{n+1}}\,\mathrm{d}x\mathrm{d}y\,. \tag{3.2}$$

Clearly, $\mathcal{E}(v,\omega) = \frac{1}{2}\left\langle (-\Delta)^{\frac{1}{2}} v, v\right\rangle_\omega$. We define $H^{1/2}_{00}(\omega;\mathbb{R}^m)$ as the completion of $\mathcal{D}(\omega,\mathbb{R}^m)$ under the norm $(\mathcal{E}(v,\omega))^{\frac{1}{2}}$ and denote the topological dual of $H^{1/2}_{00}(\omega;\mathbb{R}^m)$ by $H^{-1/2}(\omega;\mathbb{R}^m)$. Then the restriction of $(-\Delta)^{\frac{1}{2}} v$ to $H^{1/2}_{00}(\omega;\mathbb{R}^m)$ belongs to $H^{-1/2}(\omega;\mathbb{R}^m)$ with the estimate

$$\|(-\Delta)^{\frac{1}{2}} v\|^2_{H^{-1/2}(\omega)} \leqslant \mathcal{E}(v,\omega)\,, \tag{3.3}$$

which follows Cauchy-Schwarz Inequality. Noticing that $\mathcal{E}(v,\omega)$ is quadratic in v, we observe that formula (3.2) is precisely the *symmetric bilinear form* associated to $\mathcal{E}(\cdot,\omega)$. In particular, the restriction of $(-\Delta)^{\frac{1}{2}} v$ to $H^{-1/2}(\omega;\mathbb{R}^m)$ corresponds to the first variation of the energy $\mathcal{E}(v,\omega)$ with respect to pertubations supported in ω, *i.e.*,

$$\left\langle (-\Delta)^{\frac{1}{2}} v, \varphi \right\rangle_\omega = \left[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(v+t\varphi,\omega)\right]_{t=0} \tag{3.4}$$

for all $\varphi \in H^{1/2}_{00}(\omega;\mathbb{R}^m)$.

We are interested in weak solutions $v_\varepsilon \in \widehat{H}^{1/2}(\omega;\mathbb{R}^m) \cap L^4(\omega)$ of the fractional Ginzburg-Landau equation

$$(-\Delta)^{\frac{1}{2}} v_\varepsilon = \frac{1}{\varepsilon}(1-|v_\varepsilon|^2)v_\varepsilon \quad \text{in } \omega. \tag{3.5}$$

Here the notion of weak solution is understood in the duality sense according to the formulation (3.2) of the fractional Laplacian, *i.e.*,

$$\left\langle (-\Delta)^{\frac{1}{2}} v_\varepsilon, \varphi \right\rangle_\omega = \frac{1}{\varepsilon}\int_\omega (1-|v_\varepsilon|^2)v_\varepsilon \cdot \varphi \, \mathrm{d}x \quad \text{for all } \varphi \in H^{1/2}_{00}(\omega;\mathbb{R}^m) \cap L^4(\omega)$$

(or equivalently for all $\varphi \in \mathscr{D}(\omega;\mathbb{R}^m)$). By (3.4), such solutions correspond to critical points in ω of the *Ginzburg-Landau* $1/2$*-energy*

$$\mathcal{E}_\varepsilon(v,\omega) := \mathcal{E}(v,\omega) + \frac{1}{4\varepsilon}\int_\omega (1-|v|^2)^2 \, \mathrm{d}x. \tag{3.6}$$

In other words, we are interested in maps $v_\varepsilon \in \widehat{H}^{1/2}(\omega;\mathbb{R}^m) \cap L^4(\omega)$ satisfying

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\varepsilon(v_\varepsilon + t\varphi,\omega)\right]_{t=0} = 0 \quad \text{for all } \varphi \in H^{1/2}_{00}(\omega;\mathbb{R}^m) \cap L^4(\omega).$$

Among all kinds of critical points are the *minimizers*. We say that $v_\varepsilon \in \widehat{H}^{1/2}(\omega;\mathbb{R}^m) \cap L^4(\omega)$ is a minimizer of $\mathcal{E}_\varepsilon$ in ω if

$$\mathcal{E}_\varepsilon(v_\varepsilon,\omega) \leqslant \mathcal{E}_\varepsilon(v_\varepsilon + \varphi,\omega) \quad \text{for all } \varphi \in H^{1/2}_{00}(\omega;\mathbb{R}^m) \cap L^4(\omega)$$

(or equivalently for all $\varphi \in \mathscr{D}(\omega;\mathbb{R}^m)$). The most standard way to construct minimizing solutions (and in particular solutions of (3.5)) is to minimize $\mathcal{E}_\varepsilon(\cdot,\omega)$ under an exterior Dirichlet condition. More precisely, given a map $g \in \widehat{H}^{1/2}(\omega;\mathbb{R}^m) \cap L^4(\omega)$, one consider the minimization problem

$$\min\left\{\mathcal{E}_\varepsilon(v,\omega) : v \in H^{1/2}_g(\omega;\mathbb{R}^m) \cap L^4(\omega)\right\}, \tag{3.7}$$

whose resolution follows directly from the Direct Method of Calculus of Variations. Here, the natural energy space $H^{1/2}_g(\omega;\mathbb{R}^m)$ with exterior Dirichlet data g can be characterized by

$$H^{1/2}_g(\omega;\mathbb{R}^m) = g + H^{1/2}_{00}(\omega;\mathbb{R}^m). \tag{3.8}$$

Extension

To investigate the qualitative properties of solutions of (3.5), we rely on the harmonic extension to the half space $\mathbb{R}^{n+1}_+$ introduced in the previous section (See (2.12)). To make the connection more transparent we invoke a lemma from [40].

LEMMA 4. *Let $\omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. For every $v \in \widehat{H}^{1/2}(\omega;\mathbb{R}^m)$ we have*

$$\left\langle (-\Delta)^{\frac{1}{2}} v, \varphi \right\rangle_\omega = \left\langle \frac{\partial v^{\mathrm{e}}}{\partial \nu}, \varphi \right\rangle \quad \textit{for all } \varphi \in H^{1/2}_{00}(\omega;\mathbb{R}^m),$$

where v^{e} is the harmonic extension of v to $\mathbb{R}^{n+1}_+$ given by (4.5).

Using Lemma 4 we conclude that, if $v_\varepsilon \in \widehat{H}^{1/2}(\omega;\mathbb{R}^m) \cap L^4(\omega)$ is a weak solution of (3.5), then its harmonic extension $v^{\mathrm{e}}_\varepsilon$ given by (2.12) weakly solves

$$\begin{cases} \Delta v^{\mathrm{e}}_\varepsilon = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \dfrac{\partial v^{\mathrm{e}}_\varepsilon}{\partial \nu} = \dfrac{1}{\varepsilon}\left(1 - |v^{\mathrm{e}}_\varepsilon|^2\right) v^{\mathrm{e}}_\varepsilon & \text{on } \omega. \end{cases} \tag{3.9}$$

To give a weak formulation of (3.9), we state the following lemma from [40] (See Lemma 2.6).

LEMMA 5. *Let $\omega \subseteq \mathbb{R}^n$ be a bounded open set. For every $v \in \widehat{H}^{1/2}(\omega;\mathbb{R}^m)$, the harmonic extension v^{e} given by (4.5) belongs to $H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+ \cup \omega;\mathbb{R}^m) \cap L^2_{\mathrm{loc}}(\overline{\mathbb{R}}^{n+1}_+)$. In addition, for every $x_0 \in \omega$, $R > 0$ and $\rho > 0$ such that $D_{3\rho}(x_0) \subseteq \omega$, there exist constant $C_{R,\rho} > 0$ and $C_R > 0$, independent of v and x_0, such that*

$$\left\|v^{\mathrm{e}}\right\|^2_{L^2(B^+_R(x_0))} \leqslant C_{R,\rho}\left(\mathcal{E}\big(v, D_{2\rho}(x_0)\big) + \|v\|^2_{L^2(D_{2\rho}(x_0))}\right),$$

and

$$\left\|\nabla v^{\mathrm{e}}\right\|^2_{L^2(B^+_\rho(x_0))} \leqslant C_\rho\left(\mathcal{E}\big(v, D_{2\rho}(x_0)\big) + \|v\|^2_{L^2(D_{2\rho}(x_0))}\right).$$

REMARK 3. By the previous lemma, for any $v \in \widehat{H}^{1/2}(\omega;\mathbb{R}^m) \cap H^{1/2}_{\mathrm{loc}}(\mathbb{R}^n)$, the harmonic extension v^{e} belongs to $H^1_{\mathrm{loc}}(\overline{\mathbb{R}}^{n+1}_+;\mathbb{R}^m)$, and for any $R > 0$,

$$\left\|v^{\mathrm{e}}\right\|^2_{H^1(B^+_R)} \leqslant C_R\left(\mathcal{E}\big(v, D_{2R}\big) + \|v\|^2_{L^2(D_{2R})}\right).$$

If $v \in \widehat{H}^{1/2}(\omega;\mathbb{R}^m)$ for some bounded open set $\omega \subseteq \mathbb{R}^n$ with Lipschitz boundary, we now observe that v^{e} admits a distributional (exterior) normal derivative $\partial_\nu v^{\mathrm{e}}$ on ω by its harmonicity in $\mathbb{R}^{n+1}_+$. More precisely, we define $\partial_\nu v^{\mathrm{e}}$ through its action on $\varphi \in \mathscr{D}(\omega;\mathbb{R}^m)$ by

$$\left\langle \frac{\partial v^{\mathrm{e}}}{\partial \nu}, \varphi \right\rangle := \int_{\mathbb{R}^{n+1}_+} \nabla v^{\mathrm{e}} \cdot \nabla \Phi \, \mathrm{d}x, \tag{3.10}$$

where Φ is any smooth extension of φ compactly supported in $\mathbb{R}^{n+1}_+ \cup \omega$. Note that the right hand side of (3.10) is well defined by Lemma 5. Using the harmonicity of v^{e} and the divergence theorem, it is routine to check that the integral in (3.10) does

not depend on the choice of the extension Φ, and that $\partial_\nu v^{\mathrm{e}}$ coincides with the classical exterior normal derivative of v^{e} on $\omega \subseteq \partial\mathbb{R}^{n+1}_+$ whenever v is smooth. It turns out that $\partial_\nu v^{\mathrm{e}}$ coincides with the distribution $(-\Delta)^{\frac{1}{2}} v$.

In view of Lemma 5 and (3.10), the weak sense for this system corresponds to

$$\int_{\mathbb{R}^{n+1}_+} \nabla v^{\mathrm{e}}_\varepsilon \cdot \nabla \Phi \,\mathrm{d}x = \frac{1}{\varepsilon} \int_\omega \left(1 - |v^{\mathrm{e}}_\varepsilon|^2\right) v^{\mathrm{e}}_\varepsilon \cdot \Phi \,\mathrm{d}\mathscr{H}^n \tag{3.11}$$

for all $\Phi \in H^1(\mathbb{R}^{n+1}_+;\mathbb{R}^m) \cap L^4(\omega)$ compactly supported in $\mathbb{R}^{n+1}_+ \cup \omega$. System (3.9) also has a variational structure. Indeed, considering an admissible bounded open set $\Omega \subseteq \mathbb{R}^{n+1}_+$ such that $\overline{\partial^0\Omega} \subseteq \omega$, and setting for $u \in H^1(\Omega;\mathbb{R}^m) \cap L^4(\partial^0\Omega)$,

$$E_\varepsilon(u,\Omega) := \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \frac{1}{4\varepsilon}\int_{\partial^0\Omega} (1-|u|^2)^2 \,\mathrm{d}\mathscr{H}^n, \tag{3.12}$$

the weak formulation (3.11) can be rephrased as

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} E_\varepsilon\left(v^{\mathrm{e}}_\varepsilon + t\Phi, \Omega\right)\right]_{t=0} = 0$$

for all $\Phi \in H^1(\Omega;\mathbb{R}^m) \cap L^4(\partial^0\Omega)$ compactly supported in $\Omega \cup \partial^0\Omega$. Hence v^{e} is a critical point of E_ε in Ω for all admissible bounded open set $\Omega \subseteq \mathbb{R}^{n+1}_+$ such that $\overline{\partial^0\Omega} \subseteq \omega$. The energy E_ε is what we may refer to as *Ginzburg-Landau boundary energy*.

If v_ε turns out to be minimizing, we can transfer the minimality of v_ε to $v^{\mathrm{e}}_\varepsilon$ with the help of the next Proposition which is taken from [40] (See Proposition 3.4). In the following , we say that $u \in H^1(\Omega;\mathbb{R}^m) \cap L^4(\partial^0\Omega)$ is minimizer of E_ε in Ω if

$$E_\varepsilon(u,\Omega) \leqslant E_\varepsilon(u+\Phi,\Omega)$$

for all $\Phi \in H^1(\Omega;\mathbb{R}^m) \cap L^4(\partial^0\Omega)$ compactly supported in $\Omega \cup \partial^0\Omega$.

PROPOSITION 5 (**Minimality transfer**). *Let $v_\varepsilon \in \widehat{H}^{1/2}(\omega;\mathbb{R}^m) \cap L^4(\omega)$ be a minimizer of $\mathcal{E}_\varepsilon$ in ω, and let $v^{\mathrm{e}}_\varepsilon$ be its harmonic extension to $\mathbb{R}^{n+1}_+$. Then $v^{\mathrm{e}}_\varepsilon$ is a minimizer of E_ε in every admissible bounded open set $\Omega \subseteq \mathbb{R}^{n+1}_+$ such that $\overline{\partial^0\Omega} \subseteq \omega$.*

Regularity

Concerning regularity for system (3.5), it suffices to investigate the boundary regularity of the harmonic extension which satisfies (3.9). For this issue we can rely on the well known regularity theory for linear Neumann boundary value problems, see *e.g.* [24]. The following theorem deals with regularity in the interior of the free boundary, and a simple proof can be found in [9, Lemma 2.2] (which actually deals with scalar valued equations, but a quick inpection of the proof shows that the result still holds in the vectorial case).

THEOREM 9. *Let $R>0$, and let $u_\varepsilon \in H^1(B^+_{2R};\mathbb{R}^m)\cap L^\infty(B^+_{2R})$ be a weak solution of*

$$\begin{cases} \Delta u_\varepsilon = 0 & \text{in } B^+_{2R}\,, \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = \dfrac{1}{\varepsilon}(1-|u_\varepsilon|^2)u_\varepsilon & \text{on } D_{2R}\,. \end{cases}$$

Then $u \in C^\infty\big(B^+_R \cup D_R\big)$.

Applying Theorem 9 to system (3.9) yields interior regularity for *bounded* variational solutions of the fractional Ginzburg-Landau system (3.5).

COROLLARY 5. *Let $v_\varepsilon \in \widehat{H}^{1/2}(\omega;\mathbb{R}^m)\cap L^\infty(\mathbb{R}^n)$ be a weak solution of (3.5). Then $v_\varepsilon \in C^\infty(\omega)$.*

Dealing with arbitrary critical points of $\mathcal{E}_\varepsilon(\cdot,\omega)$ in $H^{1/2}_g(\omega;\mathbb{R}^m)$ for a smooth domain ω and a smooth (exterior) Dirichlet data g, one may wonder if regularity holds accross the boundary $\partial\omega$. If v_ε is such a critical point and $v^{\mathrm{e}}_\varepsilon$ is its harmonic extension, we end up with the mixed boundary value problem

$$\begin{cases} \Delta v^{\mathrm{e}}_\varepsilon = 0 & \text{in } \mathbb{R}^{n+1}_+\,, \\ \dfrac{\partial v^{\mathrm{e}}_\varepsilon}{\partial \nu} = \dfrac{1}{\varepsilon}\big(1-|v^{\mathrm{e}}_\varepsilon|^2\big)v^{\mathrm{e}}_\varepsilon & \text{on } \omega\,, \\ v^{\mathrm{e}}_\varepsilon = g & \text{on } \mathbb{R}^n\setminus\omega\,. \end{cases} \tag{3.13}$$

Even if we obviously expect regularity in the interior of ω, it might not hold across the edge $\partial\omega$, since solutions of linear mixed boundary value problems are usually not better than Hölder continous (and 1/2 is in general the best possible Hölder exponant, see *e.g.* [47] and the references therein). Hölder continuity for variational solutions of (non homogeneous) linear mixed boundary value problems follows from the general results of STAMPACCHIA [56, 57], and an estimate on the order of regularity is given by the classical result of SHAMIR [54]. In our case, it is enough to prove an *a priori* global L^∞-bound on v_ε to derive from [54] the regularity stated in the following theorem.

THEOREM 10. *Assume that $\partial\omega$ is smooth. Let $g \in C^1(\mathbb{R}^n;\mathbb{R}^m)\cap L^\infty(\mathbb{R}^n)$, and let $v_\varepsilon \in H^{1/2}_g(\omega;\mathbb{R}^m)\cap L^4(\omega)$ be a weak solution of (3.5). Then $v_\varepsilon \in C^\infty(\omega)$ and v_ε is α-Hölder continuous near $\partial\omega$ for every $\alpha \in (0,1/2)$.*

As mentioned in the few lines above, the proof of Theorem 10 rely on the boundedness of v_ε. Indeed, once the L^∞-bound on v_ε is obtained, the right hand side in the Neumann equation of (3.13) remains bounded. This is enough to apply the regularity result of [54], and then deduce the Hölder continuity of $v^{\mathrm{e}}_\varepsilon$ (and thus of v_ε). The higher order regularity away from $\partial\omega$ follows from Theorem 9.

Small energy estimate for Ginzburg-Landau boundary reactions

THEOREM 11. *Let $\varepsilon > 0$ and $R > 0$ such that $\varepsilon \leqslant R$. There exist constants $\eta_0 > 0$ and $C_0 > 0$ independent of ε and R such that for each map $u_\varepsilon \in C^2(B_R^+ \cup D_R; \mathbb{R}^m)$ satisfying $|u_\varepsilon| \leqslant 1$ and solving*

$$\begin{cases} \Delta u_\varepsilon = 0 & \text{in } B_R^+ , \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = \dfrac{1}{\varepsilon}(1-|u_\varepsilon|^2)u_\varepsilon & \text{on } D_R , \end{cases} \tag{3.14}$$

the condition $E_\varepsilon(u_\varepsilon, B_R^+) \leqslant \eta_0 R^{n-1}$ implies

$$\sup_{B_{R/4}^+} |\nabla u_\varepsilon|^2 + \sup_{D_{R/4}} \frac{(1-|u_\varepsilon|^2)^2}{\varepsilon^2} \leqslant \frac{C_0}{R^2}\eta_0 . \tag{3.15}$$

As usual in Ginzburg-Landau or harmonic map problems, the first main ingredient to derive regularity estimates is a useful monotonicity formula. This is the purpose of the following lemma.

LEMMA 6 (**Monotonicity formula**). *Let $R > 0$ and $u_\varepsilon \in C^2(B_R^+ \cup D_R; \mathbb{R}^m)$ satisfying (3.14). Then, for every $x_0 \in D_R$ and every $0 < \rho < r < \operatorname{dist}(x_0, \partial D_R)$,*

$$\begin{aligned} \frac{1}{r^{n-1}} E_\varepsilon\big(u_\varepsilon, B_r^+(x_0)\big) - \frac{1}{\rho^{n-1}} E_\varepsilon\big(u_\varepsilon, B_\rho^+(x_0)\big) \\ = \int_\rho^r \frac{1}{t^{n-1}} \left(\int_{\partial^+ B_t(x_0)} \left| \frac{\partial u_\varepsilon}{\partial \nu} \right|^2 \mathrm{d}\mathscr{H}^n \right) \mathrm{d}t \\ + \frac{1}{4\varepsilon} \int_\rho^r \frac{1}{t^n} \left(\int_{D_t(x_0)} (1-|u_\varepsilon|^2)^2 \, \mathrm{d}\mathscr{H}^n \right) \mathrm{d}t . \end{aligned}$$

The second key ingredient in Ginzburg-Landau problems is the so-called *clearing-out property* which essentially asserts that a solution must take values in a small neighborhood of the potential-well whenever the energy is small enough. This is precisely the object of the following lemma. The proof uses some ingredients from [51, Lemma 3.1] suitably modified to fit the Ginzburg-Landau setting.

LEMMA 7 (**Clearing-out**). *Let $\varepsilon \in (0,1]$. There exists a constant $\eta_1 > 0$ independent of ε such that for each map $u_\varepsilon \in C^2(B_1^+ \cup D_1; \mathbb{R}^m)$ satisfying $|u_\varepsilon| \leqslant 1$ and (3.14) with $R = 1$, the condition $E_\varepsilon(u_\varepsilon, B_1^+) \leqslant \eta_1$ implies*

$$|u_\varepsilon| \geqslant \frac{1}{2} \quad \text{in } \overline{B}_{1/2}^+ . \tag{3.16}$$

3.3. Asymptotics for the fractional Ginzburg-Landau equation

Convergence to boundary harmonic maps and defect measures

The following theorem is the main result of this section. The proof is rather standards, once the previous lemma are known.

THEOREM 12. *Let $\Omega \subseteq \mathbb{R}^{n+1}_+$ be an admissible bounded open set. Let $\varepsilon_k \downarrow 0$ be an arbitrary sequence, and let $\{u_k\}_{k\in\mathbb{N}} \subseteq H^1(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega)$ be such that for every $k \in \mathbb{N}$, $|u_k| \leqslant 1$, and u_k weakly solves*

$$\begin{cases} \Delta u_k = 0 & \text{in } \Omega\,, \\ \dfrac{\partial u_k}{\partial \nu} = \dfrac{1}{\varepsilon_k}(1-|u_k|^2)u_k & \text{on } \partial^0\Omega\,. \end{cases}$$

If $\sup_k E_{\varepsilon_k}(u_k,\Omega) < \infty$, then there exist a (not relabeled) subsequence and a bounded weak $(\mathbb{S}^{m-1},\partial^0\Omega)$-boundary harmonic map u_* in Ω such that $u_k \rightharpoonup u_*$ weakly in $H^1(\Omega)$ as $k \to \infty$. In addition, there exist a finite nonnegative Radon measure $\mu_{\rm sing}$ on $\partial^0\Omega$ and a relatively closed set $\Sigma \subseteq \partial^0\Omega$ of finite $(n-1)$-dimensional Hausdorff measure such that

(i) $|\nabla u_k|^2 \mathscr{L}^{n+1}\mathbb{L}\Omega \rightharpoonup |\nabla u_*|^2 \mathscr{L}^{n+1}\mathbb{L}\Omega + \mu_{\rm sing}$ *weakly* as Radon measures on $\Omega \cup \partial^0\Omega$;*

(ii) $\dfrac{(1-|u_k|^2)^2}{\varepsilon_k} \to 0$ *in $L^1_{\rm loc}(\partial^0\Omega)$;*

(iii) $\Sigma = \mathrm{supp}(\mu_{\rm sing}) \cup \mathrm{sing}(u_*)$*;*

(iv) $u_k \to u_*$ *in $C^{2,\alpha}_{\rm loc}\big((\Omega\cup\partial^0\Omega)\setminus\Sigma\big)$ for every $\alpha \in (0,1)$;*

(v) *if $n = 1$ the set Σ is finite and $u_* \in C^\infty(\Omega\cup\partial^0\Omega)$.*

Proof. Step 1. First notice that Theorem 9 yields $u_k \in C^\infty(\Omega\cup\partial^0\Omega)$. By assumption on u_k, we can find a (not relabeled) subsequence such that $u_k \rightharpoonup u_*$ weakly in $H^1(\Omega)$ for some map $u_* \in H^1(\Omega;\mathbb{R}^m)$. Since $|u_k| \leqslant 1$ and u_k harmonic in Ω, we deduce that $u_k \to u_*$ in $C^l_{\rm loc}(\Omega)$ for every $l \in \mathbb{N}$, u_* is harmonic in Ω, and $|u_*| \leqslant 1$ in Ω. On the other hand, $|u_k| \to 1$ in $L^2(\partial^0\Omega)$, and we infer from the compact imbedding $H^1(\Omega) \hookrightarrow L^2(\partial^0\Omega)$ that $|u_*| = 1$ $\mathscr{H}^n$-a.e. on $\partial^0\Omega$.

It then remains to analyse the asymptotic behavior of u_k near $\partial^0\Omega$. Setting

$$\mu_k := |\nabla u_k|^2 \mathscr{L}^{n+1}\mathbb{L}\Omega + \frac{1}{2\varepsilon_k}(1-|u_k|^2)^2 \mathscr{H}^n\mathbb{L}\partial^0\Omega\,,$$

we have $\sup_k \mu_k(\Omega\cup\partial^0\Omega) < \infty$. Hence we can find a further subsequence such that

$$\mu_k \rightharpoonup \mu := |\nabla u_*|^2 \mathscr{L}^{n+1}\mathbb{L}\Omega + \mu_{\rm sing}\,, \tag{3.17}$$

weakly* as Radon measures on $\Omega\cup\partial^0\Omega$ for some nonnegative $\mu_{\rm sing}$ on $\Omega\cup\partial^0\Omega$. Notice that the local smooth convergence of u_k to u_* in Ω implies that

$$\operatorname{supp}(\mu_{\rm sing})\subseteq\partial^0\Omega \tag{3.18}$$

(here $\operatorname{supp}(\mu_{\rm sing})$ denotes the relative support of $\mu_{\rm sing}$ in $\Omega\cup\partial^0\Omega$). By Lemma 6, we have

$$\rho^{1-n}\mu_k(B_\rho(x))\leqslant r^{1-n}\mu_k(B_r(x)) \tag{3.19}$$

for every $x\in\partial^0\Omega$ and every $0<\rho<r<\operatorname{dist}(x,\partial^+\Omega)$. Therefore,

$$\rho^{1-n}\mu(B_\rho(x))\leqslant r^{1-n}\mu(B_r(x)) \tag{3.20}$$

for every $x\in\partial^0\Omega$ and every $0<\rho<r<\operatorname{dist}(x,\partial^+\Omega)$. As a consequence, the $(n-1)$-dimensional density

$$\Theta^{n-1}(\mu,x):=\lim_{r\downarrow 0}\frac{\mu(B_r(x))}{\omega_{n-1}r^{n-1}} \tag{3.21}$$

exists and is finite at every point $x\in\partial^0\Omega$. Here ω_{n-1} denotes the volume of the unit ball in $\mathbb{R}^{n-1}$ if $n\geqslant 2$, and $\omega_0=1$. Note that (3.17) and (3.19) yield

$$\Theta^{n-1}(\mu,x)\leqslant\frac{C}{\big(\operatorname{dist}(x,\partial^+\Omega)\big)^{n-1}}\sup_{k\in\mathbb{N}}E_{\varepsilon_k}(u_k,\Omega)<\infty\quad\text{for all } x\in\partial^0\Omega\,. \tag{3.22}$$

On the other hand, by the smooth convergence of u_k toward u_* in Ω,

$$\Theta^{n-1}(\mu,x)=0\quad\text{for all } x\in\Omega\,.$$

In addition, we observe that $x\in\partial^0\Omega\mapsto\Theta^{n-1}(\mu,x)$ is upper semicontinuous.

Next we define the concentration set

$$\Sigma:=\left\{x\in\partial^0\Omega:\inf_r\left\{\liminf_{k\to\infty}r^{1-n}\mu_k(B_r(x)):0<r<\operatorname{dist}(x,\partial^+\Omega)\right\}\geqslant\eta_0\right\},$$

where $\eta_0>0$ is the constant given by Theorem 11. From (3.19) and (3.20) we infer that

$$\Sigma=\left\{x\in\partial^0\Omega:\lim_{r\downarrow 0}\liminf_{k\to\infty}r^{1-n}\mu_k(B_r(x))\geqslant\eta_0\right\}=\left\{x\in\partial^0\Omega:\lim_{r\downarrow 0}r^{1-n}\mu(B_r(x))\geqslant\eta_0\right\},$$

and consequently,

$$\Sigma=\left\{x\in\partial^0\Omega:\Theta^{n-1}(\mu,x)\geqslant\frac{\eta_0}{\omega_{n-1}}\right\}. \tag{3.23}$$

In particular, Σ is a relatively closed subset of $\partial^0\Omega$ (since $\Theta(\mu,\cdot)$ is upper semicontinuous). Moreover, by a well known property of upper densities (see *e.g.* [2, Theorem 2.56]), we have

$$\frac{\eta_0}{\omega_{n-1}}\mathscr{H}^{n-1}(\Sigma)\leqslant\mu(\Sigma)<\infty\,. \tag{3.24}$$

If $n=1$, it obviously implies that Σ is finite.

Step 2. Let us now show that $u_k \to u_*$ in $C^{2,\alpha}_{\mathrm{loc}}\big((\Omega\cup\partial^0\Omega)\setminus\Sigma\big)$ for every $\alpha\in(0,1)$. In view of the local smooth convergence of u_k in Ω, it suffices to prove the claim near $\partial^0\Omega$. To this purpose, let us fix $x_0\in\partial^0\Omega\setminus\Sigma$ and $0<r<\mathrm{dist}(x_0,\partial^+\Omega\cup\Sigma)$ such that $r^{1-n}\mu(B_r(x_0))<\eta_0$. By the monotony in (3.20), we may assume without loss of generality that $\mu(\partial B_r(x_0))=0$. Then $\lim_k\mu_k(B_r(x_0))=\mu(B_r(x_0))$, which in turn implies that $r^{1-n}\mu_k(B_r(x_0))\leqslant\eta_0$ for k sufficiently large. Taking k even larger we have $\varepsilon_k\leqslant r$, and we can then apply Theorem 11 and Lemma 7 to deduce that $|\nabla u_k|\leqslant C_r$ and $1/2\leqslant|u_k|\leqslant 1$ in $B^+_{r/4}(x_0)$. This is now enough to reproduce the convergence proof of Proposition 5.5, Lemma 5.6, and Remark 5.7 derived in [40]. It shows that $u_k\to u_*$ in $C^{2,\alpha}_{\mathrm{loc}}\big(B^+_{r/4}(x_0)\cup D_{r/4}(x_0)\big)$. We finally notice that Theorem 11 also provides the estimate $(1-|u_k|^2)^2\leqslant C_r\varepsilon_k^2$ in $D_{r/4}(x_0)$. As a consequence,

$$\lim_{k\to\infty}\frac{1}{\varepsilon_k}\int_{D_{r/4}(x_0)}(1-|u_k|^2)^2\,\mathrm{d}\mathscr{H}^n=0\,, \tag{3.25}$$

a fact that we shall use later on.

Step 3. Let us now prove that u_* a weak $\mathbb{S}^{m-1}$-boundary harmonic map in $\Omega\cup\partial^0\Omega$. We distinguish the two cases $n=1$ and $n\geqslant 2$.

Case 1, $n=1$. Let $\Phi\in H^1(\Omega;\mathbb{R}^m)\cap L^\infty(\Omega)$ with compact support in $\Omega\cup\partial^0\Omega$ such that $\Phi(x)\in T_{u_*(x)}\mathbb{S}^{m-1}$ for $\mathscr{H}^n$-a.e. $x\in\partial^0\Omega$. By Step 1 the set $\mathrm{supp}\,\Phi\cap\Sigma$ contains finitely many points $b_1,\dots,b_L$. Then, fix an arbitray cut-off function $\zeta\in C^\infty(\mathbb{R}^{n+1};[0,1])$ such that $\zeta=0$ in a small neighborhood of each b_l. We set $\widetilde{\Phi}:=\zeta\Phi$, so that $\widetilde{\Phi}$ has compact support in $(\Omega\cup\partial^0\Omega)\setminus\Sigma$. From the convergence of u_k established in Step 2, we have $|(1-|u_k|^2)||\widetilde{\Phi}|\leqslant C\varepsilon_k$ on $\partial^0\Omega$, and thus

$$\lim_{k\to\infty}\frac{1}{\varepsilon_k}\int_{\partial^0\Omega}(1-|u_k|^2)u_k\cdot\widetilde{\Phi}\,\mathrm{d}\mathscr{H}^n=\lim_{k\to\infty}\frac{1}{\varepsilon_k}\int_{\partial^0\Omega}(1-|u_k|^2)(u_k-u_*)\cdot\widetilde{\Phi}\,\mathrm{d}\mathscr{H}^n=0\,,$$

by dominated convergence. On the other hand,

$$\lim_{k\to\infty}\int_\Omega\nabla u_k\cdot\nabla\widetilde{\Phi}\,\mathrm{d}x=\int_\Omega\nabla u_*\cdot\nabla\widetilde{\Phi}\,\mathrm{d}x\,,$$

and we deduce that

$$\int_\Omega\nabla u_*\cdot\nabla\widetilde{\Phi}\,\mathrm{d}x=0\,.$$

Given an arbitrary $\delta>0$, we now choose the cut-off function ζ of the form $\zeta(x)=\chi_\delta(x)\widehat{\zeta}(x)$ where $\widehat{\zeta}\in C^\infty(\mathbb{R}^{n+1};[0,1])$ satisfies $\widehat{\zeta}=0$ in a small neighborhood of b_l only for $l\geqslant 2$, and $\chi_\delta\in C^\infty_c(\mathbb{R}^{n+1};[0,1])$ satisfies $\chi_\delta=0$ in $B_\delta(b_1)$, $\chi_\delta=1$ outside $B_{2\delta}(b_1)$, and $|\nabla\chi_\delta|\leqslant C/\delta$. Setting $\widehat{\Phi}=\widehat{\zeta}\Phi$, we have

$$\int_\Omega\chi_\delta\nabla u_*\cdot\nabla\widehat{\Phi}\,dx+\int_{\Omega\cap B^+_{2\delta}(b_1)}\widehat{\Phi}\cdot(\nabla\chi_\delta\cdot\nabla u_*)\,\mathrm{d}x=0\,. \tag{3.26}$$

Using Cauchy-Schwarz inequality we estimate

$$\left|\int_{\Omega\cap B^+_{2\delta}(b_1)} \widehat{\Phi}\cdot(\nabla\chi_\delta\cdot\nabla u_*)\,\mathrm{d}x\right| \leqslant C\|\Phi\|_{L^\infty(\Omega)}\left(\int_{\Omega\cap B^+_{2\delta}(b_1)}|\nabla u_*|^2\,\mathrm{d}x\right)^{1/2} \underset{\delta\downarrow 0}{\longrightarrow} 0\,.$$

Therefore, letting $\delta\downarrow 0$ in (3.26) leads to

$$\int_\Omega \nabla u_*\cdot\nabla\widehat{\Phi}\,\mathrm{d}x = 0\,.$$

We then repeat the argument for each point b_l to reach the conclusion

$$\int_\Omega \nabla u_*\cdot\nabla\Phi\,\mathrm{d}x = 0\,. \tag{3.27}$$

Hence u_* is a weak $\mathbb{S}^{m-1}$-boundary harmonic map in $\Omega\cup\partial^0\Omega$, and $u_*\in C^\infty(\Omega\cup\partial^0\Omega)$ by Theorem 7.

Case 2, $n\geqslant 2$. Consider an arbitrary Φ as in Case 1, and write $K:=\operatorname{supp}\Phi$. From Step 1 we know that $\mathscr{H}^{n-1}(\Sigma\cap K)<\infty$. By [19, Theorem 3, p.154] it implies that $\operatorname{cap}_2(\Sigma\cap K)=0$ where cap_2 denotes the Newtonian capacity. Moreover, the proof of [19, Theorem 3, p.154] provides a sequence of functions $\{\chi_l\}_{l\in\mathbb{N}}$ such that $\chi_l\in\dot{H}^1(\mathbb{R}^{n+1})$, $0\leqslant\chi_l\leqslant 1$, $\chi_l=0$ in a neighborhood of $\Sigma\cap K$, $\chi_l\to 1$ a.e. as $l\to\infty$, and

$$\lim_{l\to\infty}\int_{\mathbb{R}^{n+1}}|\nabla\chi_l|^2\,\mathrm{d}x = 0\,.$$

Arguing as in Case 1, we obtain that

$$0=\int_\Omega \nabla u_*\cdot\nabla(\chi_l\Phi)\,\mathrm{d}x = \int_\Omega \chi_l\nabla u_*\cdot\nabla\Phi\,dx + \int_\Omega \Phi\cdot(\nabla\chi_l\cdot\nabla u_*)\,\mathrm{d}x\,, \tag{3.28}$$

and we estimate

$$\left|\int_\Omega \Phi\cdot(\nabla\chi_l\cdot\nabla u_*)\,\mathrm{d}x\right| \leqslant C\|\Phi\|_{L^\infty(\Omega)}\|\nabla u_*\|_{L^2(\Omega)}\left(\int_{\mathbb{R}^{n+1}}|\nabla\chi_l|^2\,\mathrm{d}x\right)^{1/2} \underset{l\to\infty}{\longrightarrow} 0\,.$$

Letting $l\to\infty$ in (3.28) then shows that (3.27) holds, whence u_* is a weak $\mathbb{S}^{m-1}$-boundary harmonic map in $\Omega\cup\partial^0\Omega$.

Step 4. We conclude the proof by showing that $\Sigma=\operatorname{supp}(\mu_{\mathrm{sing}})\cup\operatorname{sing}(u_*)$. If x_0 does not belong to $\operatorname{supp}(\mu_{\mathrm{sing}})\cup\operatorname{sing}(u_*)$, we can find $r_0>0$ such that $\mu_{\mathrm{sing}}(B_r(x_0))=0$ for all $r<r_0$. Hence,

$$r^{1-n}\mu(B_r(x_0)) = \lim_{n\to\infty} r^{1-n}\mu_k(B_r(x_0)) = \frac{r^{1-n}}{2}\int_{B_r(x_0)\cap\Omega}|\nabla u_*|^2\,\mathrm{d}x$$

for all $r<r_0$. Since u_* is smooth in a neighborhood of x_0 by Theorem 7, we deduce that $\Theta^{n-1}(\mu,x_0)=0$, and thus $x_0\notin\Sigma$ by (3.23).

Let us now assume that $x_0 \notin \Sigma$. If $x_0 \in \Omega$ then $x_0 \notin \mathrm{supp}(\mu_{\mathrm{sing}}) \cup \mathrm{sing}(u_*)$ by (3.18) and the smoothness of u_* in Ω. If $x_0 \in \partial^0\Omega$ we deduce from (3.25) and the convergence established in Step 2 that $x_0 \notin \mathrm{sing}(u_*)$, and

$$\mu(B_r(x_0)) = \lim_{k\to\infty} \mu_k(B_r(x_0)) = \frac{1}{2}\int_{B_r(x_0)\cap\Omega} |\nabla u_*|^2 \,\mathrm{d}x$$

for a radius $r > 0$ sufficiently small. Therefore, $\mu_{\mathrm{sing}}(B_r(x_0)) = 0$, and thus x_0 does not belong to $\mathrm{supp}(\mu_{\mathrm{sing}})$.

Step 5. In view of (3.17), it only remains to prove (ii). Let K be a compact subset of $\partial^0\Omega$, and set $\delta_0 := \frac{1}{2}\mathrm{dist}(K, \partial^+\Omega)$. For $\delta \in (0,\delta_0)$, we define $\Sigma_\delta := \{x \in \partial^0\Omega : \mathrm{dist}(x,\Sigma) \leqslant \delta\}$. Then, $K \cap \Sigma_\delta$ is a compact set. From Step 4 and the local smooth convergence of u_k toward u_* outside Σ, we deduce that $\varepsilon_k^{-1}\int_{K\setminus\Sigma_\delta}(1-|u_k|^2)^2\,\mathrm{d}\mathscr{H}^n \to 0$. On the other hand, for any $x_0 \in K \cap \Sigma_\delta$ we have $B_\delta^+(x_0) \subseteq \Omega$, and we infer from the monotonicity formula in Lemma 6 that

$$\frac{1}{4\varepsilon_k}\int_{D_\delta(x_0)}(1-|u_k|^2)^2\,\mathrm{d}\mathscr{H}^n \leqslant \frac{\delta^{n-1}}{\delta_0^{n-1}} E_{\varepsilon_k}(u_k, B^+_{\delta_0}(x_0))\,.$$

Hence,

$$\limsup_{k\to\infty} \frac{1}{4\varepsilon_k}\int_{D_\delta(x_0)}(1-|u_k|^2)^2\,\mathrm{d}\mathscr{H}^n \leqslant \frac{\delta^{n-1}}{\delta_0^{k-1}} \sup_{n\in\mathbb{N}} E_{\varepsilon_k}(u_k,\Omega) \underset{\delta\downarrow 0}{\longrightarrow} 0\,.$$

Then, by a standard covering argument we deduce that

$$\lim_{k\to\infty} \frac{1}{\varepsilon_k}\int_{\Sigma_\delta}(1-|u_k|^2)^2\,\mathrm{d}\mathscr{H}^n = 0\,,$$

and thus $\varepsilon_k^{-1}\int_K(1-|u_k|^2)^2\,\mathrm{d}\mathscr{H}^n \to 0$ as $k \to \infty$. □

To complete this subsection, we now prove the $(n-1)$-rectifiability of the defect measure μ_{sing} through the celebrated PREISS criteria [45].

PROPOSITION 6. *Assume that $n \geqslant 2$. In Theorem 12 the set Σ is countably $\mathscr{H}^{n-1}$-rectifiable, and the defect measure μ_{sing} satisfies*

$$\mu_{\mathrm{sing}} = \theta\mathscr{H}^{n-1}\mathbb{L}\Sigma \tag{3.29}$$

for some positive Borel function $\theta : \Sigma \to (0,\infty)$.

Proof. By a well known property of Sobolev functions (see *e.g.* [62, (3.3.28)]), we have

$$\lim_{r\downarrow 0} \frac{1}{r^{n-1}}\int_{\Omega\cap B_r^+(x)} |\nabla u_*|^2\,\mathrm{d}x = 0 \quad \text{for } \mathscr{H}^{n-1}\text{-a.e. } x \in \Sigma\,. \tag{3.30}$$

Therefore (3.21) yields

$$\Theta^{n-1}(\mu_{\rm sing},x) := \lim_{r\downarrow 0}\frac{\mu_{\rm sing}(B_r(x))}{\omega_{n-1}r^{n-1}} = \Theta^{n-1}(\mu,x) \quad \text{for } \mathscr{H}^{n-1}\text{-a.e. } x\in\Sigma. \tag{3.31}$$

On the other hand, we derive from (3.22) that

$$\Theta^{*,n-1}(\mu_{\rm sing},x) := \limsup_{r\downarrow 0}\frac{\mu_{\rm sing}(B_r(x))}{\omega_{n-1}r^{n-1}} \leqslant \frac{C}{\big(\mathrm{dist}(x,\partial^+\Omega)\big)^{n-1}}\sup_{k\in\mathbb{N}} E_{\varepsilon_k}(u_k,\Omega) < \infty$$

for all $x\in\Sigma$. Since $\mathrm{supp}(\mu_{\rm sing})\subseteq\Sigma$, we infer from [2, Theorem 2.56] that $\mu_{\rm sing}$ is absolutely continuous with respect to $\mathscr{H}^{n-1}\llcorner\Sigma$. Then we deduce from (3.31) and (3.21)-(3.22)-(3.23) that

$$\Theta^{n-1}(\mu_{\rm sing},\cdot) = \Theta^{n-1}(\mu,\cdot)\in(0,\infty) \quad \mu_{\rm sing}\text{-a.e.},$$

and according to PREISS rectifiability criteria [45], it implies that $\mu_{\rm sing}$ is a $(n-1)$-rectifiable measure (see *e.g.* [2, Definition 2.59]). Next we infer from [2, Theorem 2.83], (3.31), and (3.22)-(3.23) that $\mu_{\rm sing}$ is of the form (3.29) with $\theta(x) := \Theta^{n-1}(\mu,x)\in(0,\infty)$. As a consequence, Σ is a countably $\mathscr{H}^{n-1}$-rectifiable set. □

Stationarity defect and generalized varifolds

In this subsection we assume that $n\geqslant 2$, and our discussion departs from Theorem 12. In this theorem, we point out that the limiting $(\mathbb{S}^{m-1},\partial^0\Omega)$-boundary harmonic map u_* is *a priori* only weakly harmonic, and might not be stationary. On the other hand, u_* arises as a weak limit of (smooth) critical points of the boundary Ginzburg-Landau energy E_ε, so that the lack of stationarity of u_* should be quantified. For the classical Ginzburg-Landau system, the analoguous question is treated by LIN & WANG in [37] where they show that the possible stationarity defect is related to the defect measure through an explicit formula. One can prove that a similar formula holds in the Ginzburg-Landau boundary context, and this is the object of the following theorem which is established in [40].

THEOREM 13. *Assume that $n\geqslant 2$. In Theorem 12, the limiting $(\mathbb{S}^{m-1},\partial^0\Omega)$-boundary harmonic map u_* and the defect measure $\mu_{\rm sing}$ represented in (3.29) satisfy*

$$\int_\Omega\left(|\nabla u_*|^2\mathrm{div}\mathbf{X} - 2\sum_{i,j=1}^{n+1}(\partial_i u_*\cdot\partial_j u_*)\partial_j\mathbf{X}_i\right)\mathrm{d}x + \int_\Sigma \theta\,\mathrm{div}_\Sigma\mathbf{X}\,\mathrm{d}\mathscr{H}^{n-1} = 0$$

for all vector fields $\mathbf{X}=(\mathbf{X}_1,\dots,\mathbf{X}_{n+1})\in C^1(\overline{\Omega};\mathbb{R}^{n+1})$ compactly supported in $\Omega\cup\partial^0\Omega$ and satisfying $\mathbf{X}_{n+1}=0$ on $\partial^0\Omega$.

In the spirit of [37], the proof of Theorem 13 relies on the notion of *generalized varifold* introduced by AMBROSIO-SONER [3]. See the proof of [40, Theorem 6.4] for details.

Asymptotics without exterior Dirichlet condition

We start in this subsection with a general case where no exterior condition is imposed. We obtain here the most important convergence results. The proofs are based on by now well-known arguments based on trace theory.

THEOREM 14. *Let $\omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let $\varepsilon_k \downarrow 0$ be an arbitrary sequence, and let $\{v_k\}_{k\in\mathbb{N}} \subseteq \widehat{H}^{1/2}(\omega;\mathbb{R}^m) \cap L^\infty(\mathbb{R}^n)$ be such that for every $k \in \mathbb{N}$, $|v_k| \leqslant 1$, and v_k weakly solves*

$$(-\Delta)^{\frac{1}{2}} v_k = \frac{1}{\varepsilon_k}(1-|v_k|^2)v_k \quad \text{in } \omega. \tag{3.32}$$

If $\sup_k \mathcal{E}_{\varepsilon_k}(v_k,\omega) < \infty$, then there exist a (not relabeled) subsequence and $v_* \in \widehat{H}^{1/2}(\omega;\mathbb{R}^m)$ a bounded weak 1/2-harmonic map into $\mathbb{S}^{m-1}$ in ω such that $v_k \rightharpoonup v_*$ weakly in $H^{1/2}(\omega) \cap L^2_{\mathrm{loc}}(\mathbb{R}^n)$ as $k \to \infty$. In addition, there exist a nonnegative Radon measure μ_{sing} on ω, a countably $\mathscr{H}^{n-1}$-rectifiable relatively closed set $\Sigma \subseteq \omega$ of locally finite $(n-1)$-dimensional Hausdorff measure in ω, and a Borel function $\theta : \Sigma \to (0,\infty)$ such that

(i) $|\nabla v_k^{\mathrm{e}}|^2 \mathscr{L}^{n+1} L_{\mathbb{R}^{n+1}_+} \overset{*}{\rightharpoonup} |\nabla v_*^{\mathrm{e}}|^2 \mathscr{L}^{n+1} L_{\mathbb{R}^{n+1}_+} + \mu_{\mathrm{sing}}$ *locally weakly* as Radon measures on $\mathbb{R}^{n+1}_+ \cup \omega$;*

(ii) $\dfrac{(1-|v_k|^2)^2}{\varepsilon_k} \to 0$ *in $L^1_{\mathrm{loc}}(\omega)$;*

(iii) $\mu_{\mathrm{sing}} = \theta \mathscr{H}^{n-1} \mathbb{L}\Sigma$;

(iv) $v_* \in C^\infty(\omega \setminus \Sigma)$ *and $v_k \to v_*$ in $C^{2,\alpha}_{\mathrm{loc}}(\omega \setminus \Sigma)$ for every $0 < \alpha < 1$;*

(v) *if $n \geqslant 2$, the limiting 1/2-harmonic map v_* and the defect measure μ_{sing} satisfy*

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}\big(v_* \circ \phi_t, \omega\big)\right]_{t=0} = \frac{1}{2}\int_\Sigma \mathrm{div}_\Sigma X \, \mathrm{d}\mu_{\mathrm{sing}}$$

for all vector fields $X \in C^1(\mathbb{R}^n;\mathbb{R}^n)$ compactly supported in ω, where $\{\phi_t\}_{t\in\mathbb{R}}$ denotes the flow on $\mathbb{R}^n$ generated by X;

(vi) *if $n = 1$, the set Σ is locally finite in ω and $v_* \in C^\infty(\omega)$.*

Proof. Step 1. From the assumptions $|v_k| \leqslant 1$ and $\sup_k \mathcal{E}_{\varepsilon_k}(v_k,\omega) < \infty$, we first deduce from Lemma 1 that the sequence $\{v_k\}$ is bounded in $L^2(\mathbb{R}^n,\mathfrak{m})$, where the measure $\mathfrak{m}$ is defined by

$$\mathfrak{m} := (1+|z|)^{-(n+1)}\,\mathrm{d}z. \tag{3.33}$$

Therefore, we can find a subsequence and $v_* \in L^2(\mathbb{R}^n,\mathfrak{m})$ such that $v_k \rightharpoonup v_*$ weakly in $L^2(\mathbb{R}^n,\mathfrak{m})$. In particular, $v_k \rightharpoonup v_*$ weakly in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$. On the other hand,

the uniform energy bound also shows that $|v_k|^2 \to 1$ in $L^2(\omega)$, and $\{v_k\}$ is bounded in $H^{1/2}(\omega)$. Hence $v_k \rightharpoonup v_*$ weakly in $H^{1/2}(\omega)$, and from the compact embedding $H^{1/2}(\omega) \hookrightarrow L^2(\omega)$, it implies that $v_k \to v_*$ strongly in $L^2(\omega)$. In particular, $|v_*| = 1$ a.e. in ω.

We now claim that $v_* \in \widehat{H}^{1/2}(\omega;\mathbb{R}^m)$, and more precisely that

$$\mathcal{E}(v_*,\omega) \leqslant \liminf_{k\to\infty} \mathcal{E}(v_k,\omega) < \infty .$$

In view of the weak convergence of $\{v_k\}$ in $H^{1/2}(\omega)$, it remains to show that

$$\iint_{\omega\times\omega^c} \frac{|v_*(x)-v_*(y)|^2}{|x-y|^{n+1}}\,\mathrm{d}x\mathrm{d}y \leqslant \liminf_{k\to\infty} \iint_{\omega\times\omega^c} \frac{|v_k(x)-v_k(y)|^2}{|x-y|^{n+1}}\,\mathrm{d}x\mathrm{d}y . \tag{3.34}$$

We fix $R>0$ and $0<\delta<R/2$ such that $\omega \subseteq D_{R/2}$. Set $\omega_\delta := \{x\in\mathbb{R}^n : \mathrm{dist}(x,\omega)<\delta\}$, and write

$$\begin{aligned}\iint_{\omega\times(\omega_\delta)^c\cap D_R} \frac{|v_k(x)-v_k(y)|^2}{|x-y|^{n+1}}\,\mathrm{d}x\mathrm{d}y &= \int_\omega |v_k(x)|^2 w_1(x)\,\mathrm{d}x \\ &+ \int_{(\omega_\delta)^c\cap D_R} |v_k(x)|^2 w_2(x)\,\mathrm{d}x - 2\int_{(\omega_\delta)^c\cap D_R} v_k(x)\cdot f_k(x)\,\mathrm{d}x =: I_k + II_k + III_k ,\end{aligned} \tag{3.35}$$

where

$$w_1(x) := \int_{(\omega_\delta)^c\cap D_R} \frac{\mathrm{d}y}{|x-y|^{n+1}}, \quad w_2(x) := \int_\omega \frac{\mathrm{d}y}{|x-y|^{n+1}}, \quad f_k(x) := \int_\omega \frac{v_k(y)\,\mathrm{d}y}{|x-y|^{n+1}} .$$

By the strong $L^2(\omega)$-convergence of v_k toward v_*, f_k converges strongly in $L^2((\omega_\delta)^c \cap D_R)$ to the function $f_*(x) := \int_\omega \frac{v_*(y)\,\mathrm{d}y}{|x-y|^{n+1}}$. From the weak $L^2_{\mathrm{loc}}(\mathbb{R}^n)$-convergence of v_k, we deduce that

$$\int_\omega |v_*(x)|^2 w_1(x)\,\mathrm{d}x \leqslant \liminf_{k\to\infty} I_k , \qquad \int_{(\omega_\delta)^c\cap D_R} |v_*(x)|^2 w_2(x)\,\mathrm{d}x \leqslant \liminf_{k\to\infty} II_k ,$$

and

$$\lim_{k\to\infty} III_k = -2\int_{(\omega_\delta)^c\cap D_R} v_*(x)\cdot f_*(x)\,\mathrm{d}x .$$

Therefore,

$$\iint_{\omega\times(\omega_\delta)^c\cap D_R} \frac{|v_*(x)-v_*(y)|^2}{|x-y|^{n+1}}\,\mathrm{d}x\mathrm{d}y \leqslant \liminf_{k\to\infty} \iint_{\omega\times\omega^c} \frac{|v_k(x)-v_k(y)|^2}{|x-y|^{n+1}}\,\mathrm{d}x\mathrm{d}y ,$$

and (3.34) follows letting $R\to\infty$ and $\delta\to 0$.

We end this first step showing that $v_n^{\mathrm{e}} \rightharpoonup v_*^{\mathrm{e}}$ weakly in $H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+\cup\omega)$. Indeed, we start deducing from continuity of the trace of harmonic extension (See [40, Lemma 2.4]) that $v_n^{\mathrm{e}} \rightharpoonup v_*^{\mathrm{e}}$ weakly in $L^2_{\mathrm{loc}}(\overline{\mathbb{R}}^{n+1}_+)$. On the other hand, the uniform energy bound together with Lemma 5 and standard estimates on harmonic functions, shows that $\{v_k^{\mathrm{e}}\}$ is bounded in $H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+\cup\omega)$, whence the announced weak convergence.

Step 2. Let us now consider an increasing sequence $\{\Omega_l\}_{l\in\mathbb{N}}$ of bounded admissible open sets such that $\overline{\partial^0\Omega_l}\subseteq\omega$ for every $l\in\mathbb{N}$, $\cup_l\Omega_l=\mathbb{R}^{n+1}_+$, and $\cup_l\partial^0\Omega_l=\omega$. Using Lemma 1. Moreover, if $v\in L^\infty(\mathbb{R}^n)$, then $v^{\mathrm{e}}\in L^\infty(\mathbb{R}^{n+1}_+)$ and

$$\|v^{\mathrm{e}}\|_{L^\infty(\mathbb{R}^{n+1}_+)}\leqslant\|v\|_{L^\infty(\mathbb{R}^n)}\,. \tag{3.36}$$

By (3.36), Step 1, and the discussions in Section 3.2, $v_k^{\mathrm{e}}\in H^1(\Omega_l;\mathbb{R}^m)\cap L^\infty(\Omega_l)$ satisfies $|v_k^{\mathrm{e}}|\leqslant 1$ and solves

$$\begin{cases}\Delta v_k^{\mathrm{e}}=0 & \text{in } \Omega_l\,,\\ \dfrac{\partial v_k^{\mathrm{e}}}{\partial\nu}=\dfrac{1}{\varepsilon_k}(1-|v_k^{\mathrm{e}}|^2)v_k^{\mathrm{e}} & \text{on } \partial^0\Omega_l\,,\end{cases}$$

for every $l\in\mathbb{N}$. In addition, we have proved in Step 1 that $\sup_k E_{\varepsilon_k}(v_k^{\mathrm{e}},\Omega_l)<\infty$ for every $l\in\mathbb{N}$. Therefore, we can find a further subsequence such that the conclusions of Theorem 12 & 13 hold in every Ω_l, and v_*^{e} is the limiting $(\mathbb{S}^{m-1},\partial^0\Omega_l)$-boundary harmonic map in each Ω_l by Step 1. This yields the announced conclusions on the defect measure μ_{sing} and on the concentration set Σ stated in (i), (ii), (iii), and (iv). The stationarity relation stated in (v) between v_* and μ_{sing} is in turn a direct consequence of Theorem 13 and a lemma derived in [40, Lemma 4.16]. Then it only remains to prove that v_* is a weak 1/2-harmonic map into $\mathbb{S}^{m-1}$ in ω. Note that, it is enough to check that $\langle(-\Delta)^{\frac12}v_*,\varphi\rangle_\omega=0$ for every $\varphi\in H^{1/2}_{00}(\omega;\mathbb{R}^m)\cap L^\infty(\omega)$ compactly supported in ω satisfying $v_*\cdot\varphi=0$ a.e. in ω (See [40, Proposition 4.2] for more details). Given such a test function φ, we consider an arbitrary extension $\Phi\in H^1(\mathbb{R}^{n+1}_+;\mathbb{R}^m)\cap L^\infty(\mathbb{R}^{n+1}_+)$ of φ which is compactly supported in $\mathbb{R}^{n+1}_+\cup\omega$. Then $\operatorname{supp}\Phi\subseteq\Omega_l\cup\partial^0\Omega_l$ for l large enough. Since v_*^{e} is a weak $(\mathbb{S}^{m-1},\partial^0\Omega_l)$-boundary harmonic map in Ω_l, we infer from Lemma 4 that

$$\langle(-\Delta)^{\frac12}v_*,\varphi\rangle_\omega=\int_{\Omega_l}\nabla v_*^{\mathrm{e}}\cdot\nabla\Phi\,\mathrm{d}x=0\,,$$

and the proof is complete. □

REMARK 4. We notice that Theorem 12 actually implies that $v_k^{\mathrm{e}}\to v_*^{\mathrm{e}}$ in $C^{2,\alpha}_{\mathrm{loc}}\big(\mathbb{R}^{n+1}_+\cup(\omega\setminus\Sigma)\big)$ for every $\alpha\in(0,1)$. In view of Lemma 4, it shows that $(-\Delta)^{\frac12}v_k\to(-\Delta)^{\frac12}v_*$ in $C^{1,\alpha}_{\mathrm{loc}}(\omega\setminus\Sigma)$ for every $\alpha\in(0,1)$.

Asymptotics with Dirichlet exterior condition

This section is devoted to proof of following Theorem derived in [40]. To this aim, we consider for the rest of this subsection a smooth bounded open set $\omega\subseteq\mathbb{R}^n$, and a smooth exterior Dirichlet condition $g:\mathbb{R}^n\to\mathbb{R}^m$ satisfying $|g|=1$ in $\mathbb{R}^n\setminus\omega$. We set $\mathrm{e}(v,\omega)$ to be the *nonlocal energy density in* ω given by

$$\mathrm{e}\big(v(x),\omega\big):=\frac{\gamma_n}{2}\int_\omega\frac{|v(x)-v(y)|^2}{|x-y|^{n+1}}\,\mathrm{d}y+\gamma_n\int_{\mathbb{R}^n\setminus\omega}\frac{|v(x)-v(y)|^2}{|x-y|^{n+1}}\,\mathrm{d}y\,. \tag{3.37}$$

THEOREM 15. *Let $\omega \subseteq \mathbb{R}^n$ be a smooth bounded open set, and let $g : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map satisfying $|g| = 1$ in $\mathbb{R}^n \setminus \omega$. Let $\varepsilon_k \downarrow 0$ be an arbitrary sequence, and let $\{v_k\}_{k\in\mathbb{N}} \subseteq H^{1/2}_g(\omega;\mathbb{R}^m) \cap L^4(\omega)$ be such that for each $k \in \mathbb{N}$, v_k weakly solves*

$$\begin{cases} (-\Delta)^{\frac{1}{2}} v_k = \dfrac{1}{\varepsilon_k}(1-|v_k|^2)v_k & \text{in } \omega, \\ v_k = g & \text{in } \mathbb{R}^n \setminus \omega. \end{cases}$$

If $\sup_k \mathcal{E}_{\varepsilon_k}(v_k,\omega) < \infty$, then there exist a (not relabeled) subsequence and $v_* \in H^{1/2}_g(\omega;\mathbb{R}^m)$ a weak 1/2-harmonic map into $\mathbb{S}^{m-1}$ in ω such that $\{v_k - v_*\} \rightharpoonup 0$ weakly in $H^{1/2}_{00}(\omega)$. In addition, there exist a finite nonnegative Radon measure $\mu_{\rm sing}$ on ω, a countably $\mathscr{H}^{n-1}$-rectifiable relatively closed set $\Sigma \subseteq \omega$ of finite $(n-1)$-dimensional Hausdorff measure, and a Borel function $\theta : \Sigma \to (0,\infty)$ such that

(i) $\mathrm{e}(v_k,\omega)\mathscr{L}^n\llcorner\omega \overset{*}{\rightharpoonup} \mathrm{e}(v_*,\omega)\mathscr{L}^n\llcorner\omega + \mu_{\rm sing}$ *weakly* as Radon measures on ω;*

(ii) $\dfrac{(1-|v_k|^2)^2}{\varepsilon_k} \to 0$ *in $L^1_{\rm loc}(\omega)$;*

(iii) $\dfrac{1-|v_k(x)|^2}{\varepsilon_k} \rightharpoonup \dfrac{\gamma_n}{2}\displaystyle\int_{\mathbb{R}^n} \dfrac{|v_*(x)-v_*(y)|^2}{|x-y|^{n+1}}\,\mathrm{d}y + \mu_{\rm sing}$ *in $\mathscr{D}'(\omega)$;*

(iv) $\mu_{\rm sing} = \theta\mathscr{H}^{n-1}\llcorner\Sigma$;

(v) $v_* \in C^\infty(\omega\setminus\Sigma)$ *and $v_n \to v_*$ in $C^{2,\alpha}_{\rm loc}(\omega\setminus\Sigma)$ for every $0<\alpha<1$;*

(vi) *if $n \geqslant 2$, the limiting 1/2-harmonic map v_* and the defect measure $\mu_{\rm sing}$ satisfy the stationarity relation*

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}\big(v_*\circ\phi_t,\omega\big)\right]_{t=0} = \frac{1}{2}\int_\Sigma \mathrm{div}_\Sigma X\,\mathrm{d}\mu_{\rm sing} \tag{3.38}$$

for all vector fields $X \in C^1(\mathbb{R}^n;\mathbb{R}^n)$ compactly supported in ω, where $\{\phi_t\}_{t\in\mathbb{R}}$ denotes the flow on $\mathbb{R}^n$ generated by X;

(vii) *if $n=1$, the set Σ is finite and $v_* \in C^\infty(\omega)$.*

Proof. Given an arbitrary sequence $\varepsilon_k \downarrow 0$, we consider $\{v_k\}_{k\in\mathbb{N}} \subseteq H^{1/2}_g(\omega;\mathbb{R}^m)\cap L^4(\omega)$ such that for each $k\in\mathbb{N}$, v_k weakly solves (3.32), and $\sup_k \mathcal{E}_{\varepsilon_k}(v_k,\omega) < \infty$. We recall that under such assumptions, $v_k \in C^{0,\alpha}_{\rm loc}(\mathbb{R}^n)\cap C^\infty(\mathbb{R}^n\setminus\partial\omega)$ for every $0<\alpha<1/2$, and that $|v_k| \leqslant 1$ in $\mathbb{R}^n$. Therefore, we can apply Theorem 14 to find a (not relabeled) subsequence such that $v_k \rightharpoonup v_*$ weakly in $H^{1/2}(\omega)\cap L^2_{\rm loc}(\mathbb{R}^n)$ for some map $v_* \in \widehat{H}^{1/2}(\omega;\mathbb{R}^m)$ which is a bounded weak 1/2-harmonic map into $\mathbb{S}^{m-1}$ in ω, and such that all the conclusions of Theorem 14 hold. Since v_k is constantly equal to g outside ω, we also infer that $v_* \in H^{1/2}_g(\omega;\mathbb{R}^m)$.

It now remains to prove that

(a) $v_k - v_* \rightharpoonup 0$ weakly in $H^{1/2}_{00}(\omega)$;

(b) μ_{sing} is a finite measure on ω, $\mathscr{H}^{n-1}(\Sigma) < \infty$, and statements (i) and (iii) in Theorem 15 hold.

Proof of (a). We first notice that $v_k - v_* \in H^{1/2}_{00}(\omega;\mathbb{R}^m)$ for every $k \in \mathbb{N}$ by (3.8). Hence,

$$[v_k - v_*]^2_{H^{1/2}(\mathbb{R}^n)} = \mathcal{E}(v_k - v_*, \omega) \leqslant 2\big(\mathcal{E}(v_k,\omega) + \mathcal{E}(v_*,\omega)\big)\,,$$

so that $\{v_k - v_*\}$ in bounded in $H^{1/2}_{00}(\omega)$. Since $\{v_k - v_*\} \rightharpoonup 0$ weakly in $H^{1/2}(\omega)$, it remains to show that

$$\iint_{\omega\times\omega^c} \frac{(v_k(x) - v_*(x))\cdot\varphi(x)}{|x-y|^{n+1}}\,\mathrm{d}x\mathrm{d}y \to 0$$

for every $\varphi \in H^{1/2}_{00}(\omega;\mathbb{R}^m)$. First notice that $v_k \to v_*$ strongly in $L^2(\omega)$ by the compact embedding $H^{1/2}(\omega) \hookrightarrow L^2(\omega)$. Then, by density of smooth maps compactly supported in ω, it suffices to consider the case where $\varphi \in \mathscr{D}(\omega)$. For such a test function φ, the assertion is easily proved using the dominated convergence Theorem and the fact that $v_k \to v_*$ a.e. in ω (up to a further subsequence if necessary). □

Proof of (b). From the uniform energy bound, we may find a further subsequence such that

$$\mathrm{e}(v_k,\omega)\,\mathscr{L}^n\mathbb{L}\omega \overset{*}{\rightharpoonup} \mathrm{e}(v_*,\omega)\,\mathscr{L}^n\mathbb{L}\omega + \mu_{\text{def}}$$

weakly* as Radon measures on ω for some finite nonnegative measure μ_{def}. We thus have to prove that $\mu_{\text{def}} \equiv \mu_{\text{sing}}$. It will then show that μ_{sing} is finite, and that $\mathscr{H}^{n-1}(\Sigma) < \infty$ (since $\mathscr{H}^{n-1}(\Sigma)$ is controlled by the total variation of μ_{sing}, see (3.24)).

Let us now fix $\varphi \in \mathscr{D}(\omega)$ arbitrary. We notice that

$$\begin{aligned}\int_\omega \mathrm{e}(v_k,\omega)\varphi\,\mathrm{d}x &= \langle(-\Delta)^{\frac12}v_k, \varphi v_k\rangle_\omega - \frac{\gamma_n}{2}\iint_{\omega\times\omega}\frac{(v_k(x)-v_k(y))\cdot v_k(y)(\varphi(x)-\varphi(y))}{|x-y|^{n+1}}\,\mathrm{d}x\mathrm{d}y\\ &\quad - \gamma_n\iint_{\omega\times\omega^c}\frac{(v_k(x)-g(y))\cdot g(y)\varphi(x)}{|x-y|^{n+1}}\,\mathrm{d}x\mathrm{d}y\\ &=: I_k - II_k - III_k\,.\end{aligned}$$

We consider a function $\Phi \in C^\infty(\overline{\mathbb{R}}^{n+1}_+)$ compactly supported in $\mathbb{R}^{n+1}_+ \cup \omega$ such that $\Phi_{|\mathbb{R}^n} = \varphi$, and set $K := \operatorname{supp}(\Phi)$. We observe that Φv^{e}_k is a smooth function compactly supported by K, so that Lemma 4 yields

$$\langle(-\Delta)^{\frac12}v_k, \varphi v_k\rangle_\omega = \int_{\mathbb{R}^{k+1}_+}\nabla v^{\mathrm{e}}_k\cdot\nabla(\Phi v^{\mathrm{e}}_k)\,\mathrm{d}x = \int_{\mathbb{R}^{n+1}_+}|\nabla v^{\mathrm{e}}_k|^2\Phi\,\mathrm{d}x + \int_K \nabla v^{\mathrm{e}}_k\cdot(v^{\mathrm{e}}_k\nabla\Phi)\,\mathrm{d}x\,.$$

From *(a)* we deduce that $v_k^{\mathrm{e}} \rightharpoonup v_*^{\mathrm{e}}$ weakly in $H^1_{\mathrm{loc}}(\overline{\mathbb{R}}^{n+1}_+)$, so that $\nabla v_n^{\mathrm{e}} \rightharpoonup \nabla v_*^{\mathrm{e}}$ weakly in $L^2(K)$ and $v_k^{\mathrm{e}} \to v_*^{\mathrm{e}}$ strongly in $L^2(K)$. Together with item (i) in Theorem 14, it yields

$$\begin{aligned}\langle(-\Delta)^{\frac{1}{2}} v_k, \varphi v_k\rangle_\omega \underset{k\to\infty}{\longrightarrow} \int_{\mathbb{R}^{n+1}_+} |\nabla v_*^{\mathrm{e}}|^2 \Phi \,\mathrm{d}x + \int_\omega \varphi \,\mathrm{d}\mu_{\mathrm{sing}} + \int_K \nabla v_*^{\mathrm{e}} \cdot (v_*^{\mathrm{e}} \nabla \Phi)\,\mathrm{d}x \\ = \int_{\mathbb{R}^{n+1}_+} \nabla v_*^{\mathrm{e}} \cdot \nabla(\Phi v_*^{\mathrm{e}})\,\mathrm{d}x + \int_\omega \varphi \,\mathrm{d}\mu_{\mathrm{sing}}\,.\end{aligned}$$

By Lemma 4 again, we have thus proved that

$$\langle(-\Delta)^{\frac{1}{2}} v_k, \varphi v_k\rangle_\omega \underset{k\to\infty}{\longrightarrow} \langle(-\Delta)^{\frac{1}{2}} v_*, \varphi v_*\rangle_\omega + \int_\omega \varphi \,\mathrm{d}\mu_{\mathrm{sing}}\,, \tag{3.39}$$

where we have used the (elementary) fact that $\varphi v_* \in H^{1/2}_{00}(\omega;\mathbb{R}^m)$.

On the other hand, we have that $v_k \to v_*$ a.e. on $\mathbb{R}^n$, eventually after the extraction of a further subsequence. Using that $\varphi \in \mathscr{D}(\omega)$, we deduce by dominated convergence that

$$II_k \to \frac{\gamma_n}{2} \iint_{\omega\times\omega} \frac{(v_*(x)-v_*(y))\cdot v_*(y)(\varphi(x)-\varphi(y))}{|x-y|^{n+1}}\,\mathrm{d}x\mathrm{d}y \tag{3.40}$$

and

$$III_k \to \gamma_n \iint_{\omega\times\omega^c} \frac{(v_*(x)-g(y))\cdot g(y)\varphi(x)}{|x-y|^{n+1}}\,\mathrm{d}x\mathrm{d}y \tag{3.41}$$

as $k\to\infty$. Gathering (3.39), (3.40), and (3.41) leads to

$$\int_\omega \mathrm{e}(v_k,\omega)\varphi\,\mathrm{d}x \underset{k\to\infty}{\longrightarrow} \int_\omega \mathrm{e}(v_*,\omega)\varphi\,\mathrm{d}x + \int_\omega \varphi\,\mathrm{d}\mu_{\mathrm{sing}}\,,$$

and thus $\mu_{\mathrm{def}} = \mu_{\mathrm{sing}}$ by the arbitrariness of φ.

To derive item (iii), we just notice that

$$\begin{aligned}\int_\omega \frac{(1-|v_k|^2)}{\varepsilon_k}\varphi\,\mathrm{d}x &= \int_\omega \frac{(1-|v_k|^2)^2}{\varepsilon_k}\varphi\,\mathrm{d}x + \frac{1}{\varepsilon_k}\int_\omega (1-|v_k|^2)v_k\cdot(\varphi v_k)\,\mathrm{d}x \\ &= \int_\omega \frac{(1-|v_k|^2)^2}{\varepsilon_k}\varphi\,\mathrm{d}x + \langle(-\Delta)^{\frac{1}{2}} v_k, \varphi v_k\rangle_\omega\,,\end{aligned}$$

so that the announced convergence follows from item (ii) and (3.39). □

□

Asymptotics for Dirichlet minimizers

We finally consider solutions of the minimization problem (3.7), and we show that, in this case, no concentration occurs by minimality.

THEOREM 16. *Let $\omega \subseteq \mathbb{R}^n$ be a smooth bounded open set, and let $g : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map satisfying $|g| = 1$ in $\mathbb{R}^n \setminus \omega$, and such that*

$$H_g^{1/2}(\omega;\mathbb{S}^{m-1}) := \left\{ v \in H_g^{1/2}(\omega;\mathbb{R}^m) : |v| = 1 \text{ a.e. in } \omega \right\} \neq \emptyset . \tag{3.42}$$

Let $\varepsilon_k \downarrow 0$ be an arbitrary sequence, and let $\{v_k\}_{k\in\mathbb{N}} \subseteq H_g^{1/2}(\omega;\mathbb{R}^m) \cap L^4(\omega)$ be such that for each $k \in \mathbb{N}$,

$$v_k \in \operatorname{argmin} \left\{ \mathcal{E}_{\varepsilon_k}(v,\omega) : v \in H_g^{1/2}(\omega;\mathbb{R}^m) \cap L^4(\omega) \right\} . \tag{3.43}$$

Then there exist a (not relabeled) subsequence and $v_* \in H_g^{1/2}(\omega;\mathbb{R}^m)$ a minimizing 1/2-harmonic map into $\mathbb{S}^{m-1}$ in ω such that $v_k - v_* \to 0$ strongly in $H_{00}^{1/2}(\omega)$. In addition,

(i) $\mathcal{E}_{\varepsilon_k}(v_k,\omega) \to \mathcal{E}(v_*,\omega)$;

(ii) $\dfrac{1-|v_k(x)|^2}{\varepsilon_k} \rightharpoonup \dfrac{\gamma_n}{2} \displaystyle\int_{\mathbb{R}^n} \frac{|v_*(x)-v_*(y)|^2}{|x-y|^{n+1}} \, dy$ *in* $\mathscr{D}'(\omega)$;

(iii) $v_n \to v_*$ *in* $C^{2,\alpha}_{\mathrm{loc}}(\omega \setminus \operatorname{sing}(v_*))$ *for every* $0 < \alpha < 1$;

(iv) *if* $n = 1$, *then* $\omega \cap \operatorname{sing}(v_*) = \emptyset$, *while* $\dim_{\mathscr{H}}\big(\operatorname{sing}(v_*) \cap \omega\big) \leqslant n-2$ *for* $n \geqslant 3$, *and* $\operatorname{sing}(v_*) \cap \omega$ *is discrete for* $n = 2$.

Proof. By assumption (3.42) there exists $\widetilde{g} \in H_g^{1/2}(\omega;\mathbb{R}^m)$ such that $|\widetilde{g}| = 1$ a.e. in ω. Then we infer from (3.43) that for all $k \in \mathbb{N}$,

$$\mathcal{E}_{\varepsilon_k}(v_k,\omega) \leqslant \mathcal{E}_{\varepsilon_k}(\widetilde{g},\omega) = \mathcal{E}(\widetilde{g},\omega) .$$

Therefore $\sup_k \mathcal{E}_{\varepsilon_k}(v_k,\omega) < \infty$, and we can extract a (not relabeled) subsequence such that the conclusions of Theorem 15 do hold for a map $v_* \in \widehat{H}_g^{1/2}(\omega;\mathbb{R}^m)$ which is a bounded weak 1/2-harmonic map into $\mathbb{S}^{m-1}$ in ω, and a finite measure μ_{sing} on ω. By the compact embedding $H^{1/2}(\omega) \hookrightarrow L^2(\omega)$, we may assume that $v_k \to v_*$ strongly in $L^2(\omega)$. Since $v_k = v_* = g$ outside ω, we deduce that $v_k \to v_*$ a.e. in $\mathbb{R}^n$, eventually up to a further subsequence. By (3.42) and the lower semicontinuity of the 1/2-Dirichlet energy with respect to a.e. pointwise convergence, we have

$$\mathcal{E}(v,\omega) \geqslant \liminf_{k\to\infty} \mathcal{E}_{\varepsilon_k}(v_k,\omega) \geqslant \liminf_{k\to\infty} \mathcal{E}(v_k,\omega) \geqslant \mathcal{E}(v_*,\omega)$$

for all $v \in H_g^{1/2}(\omega;\mathbb{S}^{m-1})$. Hence v_* is a minimizing 1/2-harmonic map into $\mathbb{S}^{m-1}$ in ω, and item (iv) follows from Theorem 6. On the other hand, the minimality of v_k yields

$$\mathcal{E}_{\varepsilon_k}(v_k,\omega) \leqslant \mathcal{E}_{\varepsilon_k}(v_*,\omega) = \mathcal{E}(v_*,\omega) .$$

Combining the two previous inequalities leads to item (i). In turn, it implies that $\mu_{\mathrm{sing}} = 0$, and then yields item (ii). The conclusion in item (iii) is a straightforward

consequence of Theorem 12 - item (iii) - together with the fact that $\mu_{\text{sing}} = 0$. It now only remains to prove the strong convergence of $v_k - v_*$ to 0 in $H^{1/2}_{00}(\omega)$. To this purpose, we have to prove that

$$\lim_{k\to\infty} \left\langle (-\Delta)^{\frac{1}{2}} v_k, v_* \right\rangle_\omega = \left\langle (-\Delta)^{\frac{1}{2}} v_*, v_* \right\rangle_\omega = 2\mathcal{E}(v_*, \omega)\,. \tag{3.44}$$

Indeed, if (3.44) holds, then item (i) yields

$$[v_k - v_*]^2_{H^{1/2}(\mathbb{R}^n)} = \mathcal{E}(v_k - v_*, \omega) = \mathcal{E}(v_k, \omega) + \mathcal{E}(v_*, \omega) - \left\langle (-\Delta)^{\frac{1}{2}} v_k, v_* \right\rangle_\omega \underset{k\to\infty}{\longrightarrow} 0\,.$$

To show (3.44) we first write

$$\begin{aligned}\left\langle (-\Delta)^{\frac{1}{2}} v_k, v_* \right\rangle_\omega = \frac{\gamma_n}{2} \iint_{\omega\times\omega} \frac{(v_k(x) - v_k(y)) \cdot (v_*(x) - v_*(y))}{|x-y|^{n+1}}\, \mathrm{d}x\mathrm{d}y \\ + \gamma_n \iint_{\omega\times\omega^c} \frac{(v_k(x) - v_*(y)) \cdot (v_*(x) - v_*(y))}{|x-y|^{n+1}}\, \mathrm{d}x\mathrm{d}y =: I_k + II_k\,.\end{aligned}$$

Since $v_k \rightharpoonup v_*$ weakly in $H^{1/2}(\omega)$, we have

$$I_k \underset{k\to\infty}{\longrightarrow} \frac{\gamma_n}{2} \iint_{\omega\times\omega} \frac{|v_*(x) - v_*(y)|^2}{|x-y|^{n+1}}\, \mathrm{d}x\mathrm{d}y\,.$$

Next we set $V_k(x,y) := (v_k(x) - v_*(y)) = (v_k(x) - v_k(y))$. Then $\{V_k\}$ is bounded in $L^2(\omega\times\omega^c, \Lambda)$ for the weighted measure $\Lambda := |x-y|^{-n-1}\mathscr{L}^n_x \otimes \mathscr{L}^n_y$. Extracting a further subsequence if necessary, V_k is thus weakly converging in $L^2(\omega\times\omega^c, \Lambda)$ to some function V. Using the strong convergence in $L^2(\omega)$ of v_k, it is easy to see that $V(x,y) = (v_*(x) - v_*(y))$ (See the proof of [40, Lemma 8.2]). As a consequence,

$$\lim_{k\to\infty} \gamma_n \iint_{\omega\times\omega^c} \frac{(v_k(x) - v_*(y)) \cdot (v_*(x) - v_*(y))}{|x-y|^{n+1}}\, \mathrm{d}x\mathrm{d}y = \gamma_n \iint_{\omega\times\omega^c} \frac{|v_*(x) - v_*(y)|^2}{|x-y|^{n+1}}\, \mathrm{d}x\mathrm{d}y\,,$$

and (3.44) is proved. □

4. Singular perturbation of fractional Allen-Cahn in the fully nonlocal regime

In Section 1.3, we have introduced the concept of non local minimal surface and in Section 1.4 we have described the connection between the fractional Allen-Cahn energy minimizer and non local minimal surfaces. The goal of this section is to further explore the connection. A typical result that we are interested to prove is given by following Theorem collected from [41].

THEOREM 17. *Assume that* $s \in (0, 1/2)$ *and that* (H1)-(H2)-(H3) *hold. Let* $\Omega \subseteq \mathbb{R}^n$ *be a smooth and bounded open set. For a given sequence* $\varepsilon_k \downarrow 0$*, let* $\{g_k\}_{k\in\mathbb{N}} \subseteq C^{0,1}_{\text{loc}}(\mathbb{R}^n)$ *be such that* $\sup_k \|g_k\|_{L^\infty(\mathbb{R}^n\setminus\Omega)} < \infty$ *and* $g_k \to g$ *in* $L^1_{\text{loc}}(\mathbb{R}^n \setminus \Omega)$ *for a function*

g satisfying $|g| = 1$ a.e. in $\mathbb{R}^n \setminus \Omega$. For each $k \in \mathbb{N}$, let $v_k \in H^s_{g_k}(\Omega) \cap L^p(\Omega)$ be a weak solution of

$$\begin{cases} (-\Delta)^s v_k + \dfrac{1}{\varepsilon_k^{2s}} W'(v_k) = 0 & \text{in } \Omega\,, \\ v_k = g_k & \text{in } \mathbb{R}^n \setminus \Omega\,. \end{cases} \tag{4.1}$$

If $\sup_k \mathcal{E}_{\varepsilon_k}(v_k, \Omega) < \infty$, then there exist a (not relabeled) subsequence and a set $E_* \subseteq \mathbb{R}^n$ of finite $2s$-perimeter in Ω such that

(i) *$v_k \to v_* := \chi_{E_*} - \chi_{\mathbb{R}^n \setminus E_*}$ strongly in $H^{s'}_{\mathrm{loc}}(\Omega) \cap L^2_{\mathrm{loc}}(\mathbb{R}^n)$ for every $s' < \min(2s, 1/2)$;*

(ii) *the set $E_* \cap \Omega$ is open;*

(iii) *the boundary $\partial E_* \cap \Omega$ is a stationary nonlocal $2s$-minimal surface in Ω (i.e., relation (1.5) holds).*

In addition, for every smooth open set $\Omega' \subseteq \Omega$ such that $\overline{\Omega'} \subseteq \Omega$,

(iv) *$\mathcal{E}(v_k, \Omega') \to 2\gamma_{n,s} P_{2s}(E_*, \Omega')$;*

(v) *$\int_{\Omega'} W(v_k)\,\mathrm{d}x = O(\varepsilon_k^{\min(4s,\alpha)})$ for every $\alpha \in (0,1)$;*

(vi) *$\dfrac{-1}{\varepsilon_k^{2s}} W'(v_k) \to \left(\dfrac{\gamma_{n,s}}{2} \displaystyle\int_{\mathbb{R}^n} \dfrac{|v_*(x) - v_*(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y \right) v_*(x)$ strongly in $H^{-s}(\Omega')$ and weakly in $L^{\bar{p}}(\Omega')$ for every $\bar{p} < 1/2s$;*

(vii) *$v_k \to v_*$ in $C^{1,\alpha}_{\mathrm{loc}}(\Omega \setminus \partial E_*)$ for some $\alpha = \alpha(n,s) \in (0,1)$;*

(viii) *for each $t \in (-1,1)$, the level set $L^t_k := \{v_k = t\}$ converges locally uniformly in Ω to $\partial E_* \cap \Omega$, i.e., for every compact set $K \subseteq \Omega$ and every $r > 0$,*

$$L^t_k \cap K \subseteq \mathcal{T}_r(\partial E_* \cap \Omega) \quad \text{and} \quad \partial E_* \cap K \subseteq \mathcal{T}_r(L^t_k \cap \Omega)$$

whenever k is large enough. Here, $\mathcal{T}_r(A)$ represents the open tubular neighborhood of radius r of a set A.

Comparing this result to what is known on the classical Allen-Cahn equation (1.2), we can now say that the main difference lies in the strong compactness of solutions , and the resulting continuity of the energy. In some sense, such compactness is not really surprising as one may guess that $H^{s'}$-regularity with $s' \in (0,1/2)$ is not strong enough to capture folding of interfaces. The key argument in proving compactness in the energy space rests on the fractional scaling of the equation and the *Marstrand's Theorem* (see e.g. [38]), a purely measure theoretic result. In the same flavour, strong convergence of solutions to the p-Ginzburg-Landau system (involving the p-Laplacian) towards stationary p-harmonic maps has been proved in [60] for non-integer values of the exponent p. Compactness at the $H^{s'}$-level with $s' < \min(2s, 1/2)$ is in turn a much

more delicate issue. We establish such compactness combining fine elliptic estimates in the region $|v_k| \simeq 1$ together with quantitative estimates on the volume of the sublevel sets $\{|v_k| \lesssim 1/2\}$. To derive these volume estimates, we apply the *quantitative stratification principle* of singular sets introduced in [11] (in the context of harmonic maps and minimal currents) and generalized to an abstract framework in [23]. We point out that this stratification principle does not apply verbatim to our setting since solutions of (1.8) are smooth, and non trivial adjustments have to be made. To the best of our knowledge, this is the first time that the quantitative stratification principle is applied to an Allen-Cahn (or Ginzburg-Landau) type equation.

REMARK 5. We emphasize that Theorem 17 applies to minimizing solutions of (4.1) since the function $\chi_\Omega - g_k\chi_{\mathbb{R}^n\setminus\Omega}$ is an admissible competitor of uniformly bounded energy. In particular, this theorem extends the result of [50] for $s \in (0,1/2)$ to arbitrary solutions (with uniformly bounded energy) together with a full set of new estimates. However, if we assume that each v_k is minimizing, i.e., $\mathcal{E}_{\varepsilon_k}(v_k,\Omega) \leqslant \mathcal{E}_{\varepsilon_k}(w,\Omega)$ for every $w \in H^s_{g_k}(\Omega)$, then [50] shows that the limiting set E_* is a minimizing nonlocal minimal surface in Ω in the sense of [10], i.e., E_* satisfies (1.4).

REMARK 6. Non trivial examples of (entire) stationary nonlocal minimal surfaces have been constructed in [15]. These examples are nonlocal analogues of classical minimal surfaces such as catenoids, or Lawson cones (see also [6, 7] for Delaunay type surfaces with constant nonlocal mean curvature). It would be very interesting to construct solutions of the fractional Allen-Cahn equation concentrating as $\varepsilon \to 0$ on such surfaces.

4.1. Surface With Prescribed Mean Curvature

In proving Theorem 17, we actually investigate the more general case where (1.8) is replaced by

$$(-\Delta)^s v_\varepsilon + \frac{1}{\varepsilon^{2s}} W'(v_\varepsilon) = f_\varepsilon \quad \text{in } \Omega\,, \tag{4.2}$$

with a smooth right hand side f_ε controlled (with respect to ε) in a suitable Sobolev space. Considering such inhomogeneous equation is a way to analyse the asymptotic behavior of an arbitrary sequence of (smooth) functions $v_\varepsilon \in H^s_{g_\varepsilon}(\Omega)$ satisfying $\mathcal{E}_\varepsilon(v_\varepsilon,\Omega) = O(1)$ and

$$\left\|(-\Delta)^s v_\varepsilon + \varepsilon^{-2s} W'(v_\varepsilon)\right\|_{W^{1,q}(\Omega)} = O(1) \quad \text{as } \varepsilon \to 0\,,$$

for some suitable exponent q.

In this setting, we have proved that the main conclusions in Theorem 17 hold (see Theorem 4 for precise statements) with a limiting set E_* satisfying

$$\delta P_{2s}(E_*,\Omega)[X] = \frac{1}{\gamma_{n,s}} \int_{E_*\cap\Omega} \operatorname{div}(fX)\,\mathrm{d}x \qquad \forall X \in C^1_c(\Omega;\mathbb{R}^n)\,, \tag{4.3}$$

where f is the weak limit of f_ε in $W^{1,q}(\Omega)$ as $\varepsilon \to 0$. In view of (1.6), the boundary of E_* satisfies in the weak sense

$$\mathrm{H}^{(2s)}_{\partial E_*} = \frac{1}{\gamma_{n,s}} f \quad \text{on } \partial E_* \cap \Omega\,. \tag{4.4}$$

We shall refer to this equation as the *prescribed nonlocal (2s-)mean curvature equation* in Ω, and to weak solutions as *surfaces of prescribed nonlocal (2s-)mean curvature.*

4.2. The Extended Problem

As it is already explained, the best way to study fractional problem is to consider the extended problem. For that we consider the following extension

$$v^{\mathrm{e}}(x,z) := \sigma_{n,s} \int_{\mathbb{R}^n} \frac{z^{2s} v(y)}{\left(|x-y|^2 + z^2\right)^{\frac{n+2s}{2}}}\, \mathrm{d}y\,. \tag{4.5}$$

If $v \in \widehat{H}^s(\Omega)$ for a bounded open set $\Omega \subseteq \mathbb{R}^n$, we have the following estimates on v^{e}. The proof follows closely the arguments in [40, Lemma 2.7].

LEMMA 8. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. For every $v \in \widehat{H}^s(\Omega)$, the extension v^{e} given by (4.5) belongs to $H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+ \cup \Omega, |z|^a \mathrm{d}\mathbf{x}) \cap L^2_{\mathrm{loc}}\big(\overline{\mathbb{R}^{n+1}_+}, |z|^a \mathrm{d}\mathbf{x}\big)$. In addition, for every $x_0 \in \Omega$, $R > 0$, and $\rho > 0$ such that $D_{3\rho}(x_0) \subseteq \Omega$, there exist constants $C_{s,R,\rho} > 0$ and $C_{s,\rho} > 0$, independent of v and x_0, such that*

$$\left\|v^{\mathrm{e}}\right\|^2_{L^2(B^+_R(x_0), |z|^a \mathrm{d}\mathbf{x})} \leqslant C_{s,R,\rho} \left(\mathcal{E}\big(v, D_{2\rho}(x_0)\big) + \|v\|^2_{L^2(D_{2\rho}(x_0))} \right),$$

and

$$\left\|\nabla v^{\mathrm{e}}\right\|^2_{L^2(B^+_\rho(x_0), |z|^a \mathrm{d}\mathbf{x})} \leqslant C_{s,\rho} \left(\mathcal{E}\big(v, D_{2\rho}(x_0)\big) + \|v\|^2_{L^2(D_{2\rho}(x_0))} \right).$$

The extension v^{e} solves the equation

$$\begin{cases} \operatorname{div}(z^a \nabla v^{\mathrm{e}}) = 0 & \text{in } \mathbb{R}^{n+1}_+\,, \\ v^{\mathrm{e}} = v & \text{on } \partial\mathbb{R}^{n+1}_+\,. \end{cases} \tag{4.6}$$

$$(-\Delta)^s v(x) = -\lim_{z \downarrow 0} z^a \partial_z v^{\mathrm{e}}(x,z) = 2s \lim_{z \downarrow 0} \frac{v^{\mathrm{e}}(x,0) - v^{\mathrm{e}}(x,z)}{z^{2s}}$$

We defined the energy function via

$$\mathbf{E}(u,G) := \frac{d_s}{2} \int_G z^a |\nabla u|^2\, \mathrm{d}\mathbf{x}\,, \tag{4.7}$$

In future we will be interested in the following degenerated problem.

$$\begin{cases} \operatorname{div}(z^a \nabla u_\varepsilon) = 0 & \text{in } G, \\ d_s \partial_z^{(2s)} u_\varepsilon = \dfrac{1}{\varepsilon^{2s}} W'(u_\varepsilon) - f & \text{on } \partial^0 G, \end{cases} \tag{4.8}$$

The following representation formula will play a crucial part in future development. The proof can be found in [41, Corollary 2.14]

LEMMA 9. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and $G \subseteq \mathbb{R}^{n+1}_+$ be an admissible bounded open set such that $\overline{\partial^0 G} \subseteq \Omega$. For each $v \in \widehat{H}^s(\Omega)$, and each $X \in C^1(\mathbb{R}^n;\mathbb{R}^n)$ compactly supported in $\partial^0 G$, we have*

$$\delta\mathcal{E}(v,\Omega)[X] = \frac{d_s}{2}\int_G z^a \Big(|\nabla v^{\mathrm{e}}|^2 \operatorname{div}\mathbf{X} - 2(\nabla v^{\mathrm{e}} \otimes \nabla v^{\mathrm{e}}) : \nabla \mathbf{X}\Big)\,\mathrm{d}\mathbf{x} + \frac{d_s a}{2}\int_G z^{a-1}|\nabla v^{\mathrm{e}}|^2 \mathbf{X}_{n+1}\,\mathrm{d}\mathbf{x},$$

where $\mathbf{X} = (\mathbf{X}_1,\dots,\mathbf{X}_{n+1}) \in C^1(\overline{G};\mathbb{R}^{n+1})$ is any vector field compactly supported in $G \cup \partial^0 G$ satisfying $\mathbf{X} = (X,0)$ on $\partial^0 G$.

REMARK 7. For an admissible bounded open set $G \subseteq \mathbb{R}^{n+1}_+$ and $u \in H^1(G,|z|^a \mathrm{d}\mathbf{x})$, we can define *the first inner variation up to the boundary $\partial^0 G$ of $\mathbf{E}(\cdot,G)$ at u* as

$$\delta\mathbf{E}\big(u, G\cup\partial^0 G\big)[\mathbf{X}] := \left[\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}(u\circ\Phi_{-t},G)\right]_{t=0},$$

where $\{\Phi_t\}_{t\in\mathbb{R}}$ denotes the flow on $\mathbb{R}^{n+1}$ generated by a given vector field $\mathbf{X} = (X,\mathbf{X}_{n+1}) \in C^1(\overline{G};\mathbb{R}^{n+1})$ compactly supported in $G\cup\partial^0 G$ and satisfying $\mathbf{X}_{n+1} = 0$ on $\partial^0 G$. Then, one obtains

$$\delta\mathbf{E}\big(u, G\cup\partial^0 G\big)[\mathbf{X}] = \frac{d_s}{2}\int_G z^a \Big(|\nabla u|^2 \operatorname{div}\mathbf{X} - 2(\nabla u \otimes \nabla u) : \nabla \mathbf{X}\Big)\,\mathrm{d}\mathbf{x} + \frac{d_s a}{2}\int_G z^{a-1}|\nabla u|^2 \mathbf{X}_{n+1}\,\mathrm{d}\mathbf{x}. \tag{4.9}$$

Hence, we can rephrased the conclusion of Lemma 9 as $\delta\mathcal{E}(v,\Omega) = \delta\mathbf{E}\big(v^{\mathrm{e}}, G\cup\partial^0 G\big)$.

4.3. Estimates on the boundary reactions

Our strategy now consists in deriving a priori estimates for weak solutions of (4.8). Concerning regularity, the starting point is the following linear estimate given in [8, proof of Lemma 4.5].

LEMMA 10. *Let* $\mathbf{f} \in L^\infty(D_2)$ *and* $u \in H^1(B_2^+, |z|^a \mathrm{d}\mathbf{x}) \cap L^\infty(B_2^+)$ *be a weak solution of*

$$\begin{cases} \operatorname{div}(z^a \nabla u) = 0 & \text{in } B_2^+, \\ \partial_z^{(2s)} u = \mathbf{f} & \text{on } D_2. \end{cases} \tag{4.10}$$

There exist $\beta_* = \beta_*(n,s) \in (0,1)$*, and a positive constant* $\mathbf{c}_{n,s}$ *depending only on n and s such that*

$$\|u\|_{C^{0,\beta_*}(\overline{B_1}^+)} \leqslant \mathbf{c}_{n,s}\big(\|\mathbf{f}\|_{L^\infty(D_2)} + \|u\|_{L^\infty(B_2^+)}\big). \tag{4.11}$$

In addition, if $\mathbf{f} \in C^{0,\sigma}(D_2)$ *with* $\sigma \in (0,1)$*, then* $z^a \partial_z u \in C^{0,\gamma}(\overline{B}_1^+)$ *for some* $\gamma \in (0,1)$.

For $f \in C^{0,1}(D_2)$, bootstrapping estimate (4.11) yields the following interior regularity for bounded weak solutions of (4.8).

THEOREM 18. *Let* $f \in C^{0,1}(D_2)$ *and* $u_\varepsilon \in H^1(B_2^+, |z|^a d\mathbf{x}) \cap L^\infty(B_2^+)$ *be a weak solution of*

$$\begin{cases} \operatorname{div}(z^a \nabla u_\varepsilon) = 0 & \text{in } B_2^+, \\ d_s \partial_z^{(2s)} u_\varepsilon = \dfrac{1}{\varepsilon^{2s}} W'(u_\varepsilon) - f & \text{on } D_2. \end{cases} \tag{4.12}$$

Then $u_\varepsilon \in C^\infty(B_2^+)$*,* $u_\varepsilon \in C^{0,\beta_*}\big(\overline{B}_1^+\big)$*,* $\nabla_x u_\varepsilon \in C^{0,\beta_*}(\overline{B}_1^+)$*, and* $z^a \partial_z u_\varepsilon \in C^{0,\gamma}(\overline{B}_1^+)$ *for some* $\gamma \in (0,1)$.

4.4. Monotonicity Formula and Clearing Out Lemma

In this subsection, we present two of the main ingredients, an energy monotonicity, and a clearing-out property reminiscent of Ginzburg-Landau theories. We start with the fundamental *monotonicity formula*.

LEMMA 11. *Let* $q \in (\frac{n}{1+2s}, n)$*,* $R > 0$*, and* $\varepsilon > 0$*. Given* $f \in C^{0,1}(D_R)$*, let* $u_\varepsilon \in H^1(B_R^+, |z|^a \mathrm{d}\mathbf{x}) \cap L^\infty(B_R^+)$ *be a weak solution of*

$$\begin{cases} \operatorname{div}(z^a \nabla u_\varepsilon) = 0 & \text{in } B_R^+, \\ d_s \partial_z^{(2s)} u_\varepsilon = \dfrac{1}{\varepsilon^{2s}} W'(u_\varepsilon) - f & \text{on } D_R. \end{cases} \tag{4.13}$$

There exists a constant $\mathbf{c}_{n,q} > 0$ *(depending only on n and q) such that for every point* $\mathbf{x}_0 = (x_0, 0) \in D_R \times \{0\}$*, the function* $r \in (0, R - |\mathbf{x}_0|] \mapsto \Theta^\varepsilon_{u_\varepsilon}(f, x_0, r)$ *defined by*

$$\Theta^\varepsilon_{u_\varepsilon}(f, x_0, r) := \frac{1}{r^{n-2s}} \mathbf{E}_\varepsilon\big(u_\varepsilon, B_r^+(\mathbf{x}_0)\big)$$
$$+ \mathbf{c}_{n,q} \|u_\varepsilon\|_{L^\infty(D_R)} \int_0^r t^{\theta_q - 1} \|f\|_{\dot{W}^{1,q}(D_t(x_0))} \, \mathrm{d}t$$

with $\theta_q := 1 + 2s - n/q$*, is non-decreasing.*

LEMMA 12. *Let $q \in (\frac{n}{1+2s}, n)$. Given $b \geqslant 1$ and $T \geqslant 0$, there exists a non-decreasing function $\eta_{b,T} : (0,1) \to (0,\infty)$ depending only n, s, b, T, and W, such that the following holds. Let $R \in (0,1]$, $\varepsilon \in (0,R)$, and $f \in C^{0,1}(D_R)$ such that $\varepsilon^{2s}\|f\|_{L^\infty(D_R)} \leqslant T$. If $u_\varepsilon \in H^1(B_R^+, |z|^a \mathrm{d}\mathbf{x}) \cap L^\infty(B_R^+)$ is a weak solution of (4.13) satisfying $\|u_\varepsilon\|_{L^\infty(B_R^+)} \leqslant b$, and for some $\delta \in (0,1)$,*

$$\Theta^{\varepsilon}_{u_\varepsilon}(f,0,R) \leqslant \eta_{b,T}(\delta)\,, \tag{4.14}$$

then $\big||u_\varepsilon| - 1\big| \leqslant \delta$ on $D_{R/2}$.

4.5. Density Estimate

LEMMA 13. *If $u_* \in H^1(B_1^+, |z|^a \mathrm{d}\mathbf{x}) \cap L^\infty(B_1^+)$ satisfies*

$$\begin{cases} \operatorname{div}\big(z^a \nabla u_*\big) = 0 & \text{in } B_1^+\,, \\ u_* = 1 & \text{on } D_1\,, \end{cases}$$

then $u_* \in C^{0,\alpha}_{\mathrm{loc}}(B_1^+ \cup D_1)$, $\nabla_x u_* \in C^{0,\alpha}_{\mathrm{loc}}(B_1^+ \cup D_1)$, and $z^a \partial_z u_* \in C^{0,\alpha}_{\mathrm{loc}}(B_1^+ \cup D_1)$ for some $\alpha = \alpha(n,s) \in (0,1)$. Moreover, for every $r \in (0,1)$, $\|u_*\|_{C^{0,\alpha}(B_r^+)}$, $\|\nabla_x u_*\|_{C^{0,\alpha}(B_r^+)}$, and $\|z^a \partial_z u_*\|_{C^{0,\alpha}(B_r^+)}$ only depends n, s, r, and $\|u_*\|_{L^\infty(B_1^+)}$. In particular,

$$\lim_{r\to 0} \frac{1}{r^{n-2s}} \mathbf{E}\big(u_*, B_r^+(\mathbf{x}_0)\big) = 0 \tag{4.15}$$

locally uniformly with respect to $\mathbf{x}_0 \in D_1 \times \{0\}$.

4.6. Asymptotics for singular Perturbation of Fractional Allen-Cahn

The following theorem is the main result of this section. The proof is rather standard, except at the very last step where we use Marstrand's theorem.

THEOREM 19. *Let $G \subseteq \mathbb{R}^{n+1}_+$ be an admissible bounded open set, and $\varepsilon_k \downarrow 0$ a given sequence. Let $\{f_k\}_{k\in\mathbb{N}} \subseteq C^{0,1}(\partial^0 G)$ satisfying*

$$\sup_k \Big(\varepsilon_k^{2s}\|f_k\|_{L^\infty(\partial^0 G)} + \|f_k\|_{W^{1,q}(\partial^0 G)}\Big) < \infty \quad \text{for some } q \in (\frac{n}{1+2s}, n)\,.$$

Let $\{u_k\}_{k\in\mathbb{N}} \subseteq H^1(G, |z|^a \mathrm{d}\mathbf{x}) \cap L^\infty(G)$ satisfying $\sup_k \|u_k\|_{L^\infty(G)} < \infty$, and such that each u_k weakly solves

$$\begin{cases} \operatorname{div}\big(z^a \nabla u_k\big) = 0 & \text{in } G\,, \\ d_s \partial_z^{(2s)} u_k = \dfrac{1}{\varepsilon_k^{2s}} W'(u_k) - f_k & \text{on } \partial^0 G\,. \end{cases} \tag{4.16}$$

If $\sup_k \mathbf{F}_{\varepsilon_k}(u_k, G) < \infty$, then there exist a (not relabeled) subsequence, $u_* \in H^1(G, |z|^a \mathrm{d}\mathbf{x})$ and an open subset $E_* \subseteq \partial^0 G$ such that $u_* = \chi_{E_*} - \chi_{\partial^0 G \setminus E_*}$ on $\partial^0 G$,

$u_k \rightharpoonup u_*$ weakly in $H^1(G,|z|^a\mathrm{d}\mathbf{x})$, and $u_k \to u_*$ strongly in $H^1_{\mathrm{loc}}(G\cup\partial^0 G,|z|^a\mathrm{d}\mathbf{x})$ as $k\to\infty$.

In addition,

(i) $\varepsilon_k^{-2s}W(u_k)\to 0$ *in* $L^1_{\mathrm{loc}}(\partial^0 G)$*;*

(ii) $u_k\to u_*$ *in* $C^0_{\mathrm{loc}}(\partial^0 G\setminus\partial E_*)$*;*

(iii) *if* $\sup_k\|f_k\|_{L^\infty(\partial^0 G)}<\infty$*, then* $u_k\to u_*$ *in* $C^{0,\alpha}_{\mathrm{loc}}(\partial^0 G\setminus\partial E_*)$ *for every* $\alpha\in(0,\beta_*)$ *with* β_* *given by Lemma 10;*

(iv) *if* $\sup_k\|f_k\|_{C^{0,1}(\partial^0 G)}<\infty$*, then* $u_k\to u_*$ *in* $C^{1,\alpha}_{\mathrm{loc}}(\partial^0 G\setminus\partial E_*)$ *for every* $\alpha\in(0,\beta_*)$*;*

(v) *for each* $t\in(-1,1)$*, the level set* $L_k^t:=\{u_k=t\}$ *converges locally uniformly in* $\partial^0 G$ *to* $\partial E_*\cap\partial^0 G$*, i.e., for every compact set* $K\subseteq\partial^0 G$ *and every* $r>0$*,*

$$L_k^t\cap K\subseteq\mathcal{T}_r(\partial E_*\cap\partial^0 G)\quad\text{and}\quad\partial E_*\cap K\subseteq\mathcal{T}_r(L_k^t\cap\partial^0 G)\,,$$

whenever k *is large enough;*

(vi) *if* $f_k\rightharpoonup f_*$ *weakly in* $W^{1,q}(\partial^0 G)$*, then the function* u_* *satisfies*

$$\delta\mathbf{E}\big(u_*,G\cup\partial^0 G\big)[\mathbf{X}]=\int_{\partial^0 G}u_*\,\mathrm{div}(f_*X)\,\mathrm{d}x$$

for every vector field $\mathbf{X}=(X,\mathbf{X}_{n+1})\in C^1(\overline{G};\mathbb{R}^{n+1})$ *compactly supported in* $G\cup\partial^0 G$ *such that* $\mathbf{X}_{n+1}=0$ *on* $\partial^0 G$*.*

Proof. Step 1: Compactness. Let $b\geqslant 1$ such that $b\geqslant\sup_k\|u_k\|_{L^\infty(G)}$. By the assumptions on $\{u_k\}_{k\in\mathbb{N}}$, we have

$$\sup_k\mathbf{E}_{\varepsilon_k}(u_k,G)\leqslant\sup_k\big(\mathbf{F}_{\varepsilon_k}(u_k,G)+b\|f_k\|_{L^1(\partial^0 G)}\big)<\infty\,.$$

Hence there is a (not relabeled) subsequence such that $u_k\rightharpoonup u_*$ weakly in $H^1(G,|z|^a\mathrm{d}\mathbf{x})$. By the compact embedding $H^1(G,|z|^a\mathrm{d}\mathbf{x})\hookrightarrow L^1(G)$, we also have $u_k\to u_*$ strongly in $L^1(G)$. Since $|u_k|\leqslant b$, it implies that $|u_*|\leqslant b$ in G, and $u_k\to u_*$ strongly in $L^2(G,|z|^a\mathrm{d}\mathbf{x})$. Moreover, $u_k\to u_*$ in $C^\ell_{\mathrm{loc}}(G)$ for all $\ell\in\mathbb{N}$, so that $\mathrm{div}\big(z^a\nabla u_*\big)=0$ in G. On the other hand, the uniform energy bound implies $|u_k|\to 1$ in $L^1(\partial^0 G)$, and we infer from the continuity of the trace operator that $|u_*|=1$ on $\partial^0 G$.

We now wish to analyse the asymptotic behavior of u_k near $\partial^0 G$. For this we consider the measures

$$\mu_k:=\frac{d_s}{2}z^a|\nabla u_k|^2\mathcal{L}^{n+1}\mathbb{L}G+\frac{1}{\varepsilon_k^{2s}}W(u_k)\mathcal{H}^n\mathbb{L}\partial^0 G\,.$$

Since $\sup_k\mu_k(G\cup\partial^0 G)<\infty$, we can find a further subsequence such that

$$\mu_k\rightharpoonup\mu:=\frac{d_s}{2}z^a|\nabla u_*|^2\mathcal{L}^{n+1}\mathbb{L}G+\mu_{\mathrm{sing}}\,,\tag{4.17}$$

weakly* as Radon measures on $G \cup \partial^0 G$ for some finite nonnegative measure $\mu_{\rm sing}$. Notice that the local smooth convergence of u_k to u_* in G implies that

$$\mathrm{spt}(\mu_{\rm sing}) \subseteq \partial^0 G \tag{4.18}$$

(here $\mathrm{spt}(\mu_{\rm sing})$ denotes the relative support of $\mu_{\rm sing}$ in $G \cup \partial^0 G$).

Since $\partial^0 G$ is a Lipschitz domain of $\mathbb{R}^n$, there exits a constant C depending only on $\partial^0 G$ such that $\|f_k\|_{\dot W^{1,q}(\partial^0 G)} \leqslant C\|f_k\|_{W^{1,q}(\partial^0 G)}$. Then we set

$$T := \sup_k \left((2\varepsilon_k)^{2s} \|f_k\|_{L^\infty(\partial^0 G)} + \|f_k\|_{\dot W^{1,q}(\partial^0 G)} \right) < \infty .$$

Noticing that

$$\int_\rho^r t^{\theta_q - 1} \|f_k\|_{\dot W^{1,q}(D_t(x))} \, \mathrm{d}t \leqslant \frac{T}{\theta_q} (r^{\theta_q} - \rho^{\theta_q}) ,$$

we can apply the monotonicity to deduce that

$$\rho^{2s-n} \mu_k(B_\rho(\mathbf{x})) + \frac{b\,\mathbf{c}_{n,q}}{\theta_q} T \rho^{\theta_q} \leqslant r^{2s-n} \mu_k(B_r(\mathbf{x})) + \frac{b\,\mathbf{c}_{n,q}}{\theta_q} T r^{\theta_q} \tag{4.19}$$

for every $\mathbf{x} \in \partial^0 G$ and every $0 < \rho < r < \min\big(1, \mathrm{dist}(\mathbf{x}, \partial^+ G)\big)$. Therefore,

$$\rho^{2s-n} \mu(B_\rho(\mathbf{x})) + \frac{b\,\mathbf{c}_{n,q}}{\theta_q} T \rho^{\theta_q} \leqslant r^{2s-n} \mu(B_r(\mathbf{x})) + \frac{b\,\mathbf{c}_{n,q}}{\theta_q} T r^{\theta_q} \tag{4.20}$$

for every $\mathbf{x} \in \partial^0 G$ and every $0 < \rho < r < \min\big(1, \mathrm{dist}(\mathbf{x}, \partial^+ G)\big)$. As a consequence, the $(n-2s)$-dimensional density

$$\Theta^{n-2s}(\mu, \mathbf{x}) := \lim_{r \downarrow 0} \frac{\mu(B_r(\mathbf{x}))}{\omega_{n-2s} r^{n-2s}} \tag{4.21}$$

exists and is finite at every point $\mathbf{x} \in \partial^0 G$. Note that (4.17) and (4.19) yield

$$\Theta^{n-2s}(\mu, \mathbf{x}) \leqslant \frac{C}{\big(\mathrm{dist}(\mathbf{x}, \partial^+ G)\big)^{n-2s}} \sup_k \mathbf{E}_{\varepsilon_k}(u_k, G) + \frac{b\,\mathbf{c}_{n,q}}{\theta_q} T (\mathrm{diam}\, \partial^0 G)^{\theta_q} < \infty \tag{4.22}$$

for all $\mathbf{x} \in \partial^0 G$.

On the other hand, by the smooth convergence of u_k toward u_* in G,

$$\Theta^{n-2s}(\mu, \mathbf{x}) = 0 \quad \text{for all } x \in G .$$

In addition, we observe that $\mathbf{x} \in \partial^0 G \mapsto \Theta^{n-2s}(\mu, \mathbf{x})$ is upper semicontinuous Next we define the concentration set

$$\Sigma := \left\{ \mathbf{x} \in \partial^0 G : \inf_r \left\{ \liminf_{k \to \infty} r^{2s-n} \mu_k(B_r(\mathbf{x})) : \right.\right.$$
$$\left.\left. 0 < r < \min\big(1, \mathrm{dist}(\mathbf{x}, \partial^+ G)\big) \right\} \geqslant \theta_{b,T} \right\} , \tag{4.23}$$

where $\theta_{b,T} > 0$. From (3.19) and (3.20) we infer that

$$\Sigma = \left\{ \mathbf{x} \in \partial^0 G : \lim_{r\downarrow 0} \liminf_{k\to\infty} r^{2s-n} \mu_k(B_r(\mathbf{x})) \geqslant \theta_{b,T} \right\}$$
$$= \left\{ \mathbf{x} \in \partial^0 G : \lim_{r\downarrow 0} r^{2s-n} \mu(B_r(\mathbf{x})) \geqslant \theta_{b,T} \right\},$$

and consequently,

$$\Sigma = \left\{ \mathbf{x} \in \partial^0 G : \Theta^{n-2s}(\mu, \mathbf{x}) \geqslant \frac{\theta_{b,T}}{\omega_{n-2s}} \right\}. \tag{4.24}$$

In particular, Σ is a relatively closed subset of $\partial^0 G$ since $\Theta^{n-2s}(\mu, \cdot)$ is upper semicontinuous. Moreover, by a well known property of densities (see e.g. [2]), we have

$$\frac{\theta_{b,T}}{\omega_{n-2s}} \mathcal{H}^{n-2s}(\Sigma) \leqslant \mu(\Sigma) < \infty. \tag{4.25}$$

On the other hand, it follows from (4.22) and GMT that $\mu_{\text{sing}} \mathbb{L}\Sigma$ is absolutely continuous with respect to $\mathcal{H}^{n-2s}\mathbb{L}\Sigma$.

We now claim that $\text{spt}(\mu_{\text{sing}}) \subseteq \Sigma$. Indeed, for $\mathbf{x}_0 \in \partial^0 G \setminus \Sigma$, we can find a radius r such that $r^{2s-n}\mu(B_r(\mathbf{x}_0)) < \theta_{b,T}$ and $\mu(\partial B_r(\mathbf{x}_0)) = 0$. Then

$$\lim_{k\to\infty} \mathbf{E}_{\varepsilon_k}(u_k, B_r^+(\mathbf{x}_0)) = \mu(B_r(\mathbf{x}_0)) < \theta_{b,T} r^{n-2s},$$

and we deduce that $\mu_{\text{sing}}(B_{r/4}(\mathbf{x}_0)) = 0$. Hence

$$\mu_{\text{sing}}(\partial^0 G \setminus \Sigma) = 0,$$

and thus μ_{sing} is supported by Σ. In conclusion, we thus proved that μ_{sing} is absolutely continuous with respect to the Radon measure $\mathcal{H}^{n-2s}\mathbb{L}\Sigma$.

We are now ready to show that $\mu_{\text{sing}} \equiv 0$. We argue by contradiction assuming that $\mu_{\text{sing}}(\Sigma) > 0$. By GMT, we can find a Borel subset $\widetilde{\Sigma} \subseteq \Sigma$ such that $\mathcal{H}^{n-2s}(\Sigma \setminus \widetilde{\Sigma}) = 0$ and

$$\lim_{r\downarrow 0} \frac{1}{r^{n-2s}} \mathbf{E}\big(u_*, B_r^+(\mathbf{x}_0)\big) = 0 \quad \text{for every } \mathbf{x}_0 \in \widetilde{\Sigma}.$$

Then $\mu_{\text{sing}}(\widetilde{\Sigma}) = \mu_{\text{sing}}(\Sigma) > 0$. Moreover, by our choice of $\widetilde{\Sigma}$, the density

$$\Theta^{n-2s}(\mu_{\text{sing}}, \mathbf{x}_0) := \lim_{r\downarrow 0} \frac{\mu_{\text{sing}}(B_r(\mathbf{x}_0))}{\omega_{n-2s} r^{n-2s}}$$

exists at every $\mathbf{x}_0 \in \widetilde{\Sigma}$, and

$$\Theta^{n-2s}(\mu_{\text{sing}}, \mathbf{x}_0) = \Theta^{n-2s}(\mu, \mathbf{x}_0) \in (0, \infty).$$

By Marstrand's Theorem (see the Appendix for a precise statement and the proof), it implies that $(n-2s)$ is an integer, which is an obvious contradiction. Hence $\mu_{\text{sing}} \equiv 0$.

Step 2: Uniform convergence. Let us define

$$E^+ := \Big\{ \mathbf{x} = (x,0) \in \partial^0 G : u_* = 1 \text{ a.e. on } D_r(x) \text{ for some } r \in (0, \operatorname{dist}(\mathbf{x}, \partial^+ G)) \Big\},$$

and

$$E^- := \Big\{ \mathbf{x} = (x,0) \in \partial^0 G : u_* = -1 \text{ a.e. on } D_r(x) \text{ for some } r \in (0, \operatorname{dist}(\mathbf{x}, \partial^+ G)) \Big\}.$$

By construction, E^+ and E^- are disjoint relatively open subsets of $\partial^0 G$.

We claim that $E^\pm \cap \Sigma = \emptyset$. Since $\mathcal{L}^n(\Sigma) = 0$, it implies in particular that

$$u_* = \chi_{E^+} - \chi_{\partial^0 G \setminus E^+} \text{ on } \partial^0 G.$$

We claim that $u_k \to \pm 1$ locally uniformly in $E^\pm$ (respectively). Fix an arbitrary compact set $K \subseteq E^+$. By Lemma 13, we can find a radius $r_K \leqslant \min\big\{\operatorname{dist}(K, \partial E^+), \mathbf{R}_{b,T}\big\}$ such that

$$\mathbf{E}\big(u_*, B^+_{r_K}(\bar{\mathbf{x}})\big) < \theta_{b,T} r_K^{n-2s}$$

for every $\bar{\mathbf{x}} \in K \times \{0\}$. Then we deduce from Step 1 that

$$\lim_{k\to\infty} \mathbf{E}_{\varepsilon_k}\big(u_k, B^+_{r_K}(\bar{\mathbf{x}})\big) < \theta_{b,T} r_K^{n-2s}$$

for every $\bar{\mathbf{x}} \in K \times \{0\}$. By regularity and a standard covering argument, it implies that $u_k \to u_* = 1$ uniformly on K.

Step 3: Convergence of level sets. We fix $t \in (-1,1)$, a compact set $K \subseteq \partial^0 G$, and a radius $r > 0$. First, from (iii) we deduce that $|u_k| \to 1$ uniformly on $K \setminus \mathcal{T}_r(\Sigma)$. Therefore, $L^t_k \cap K \subseteq \mathcal{T}_r(\Sigma)$ for k large enough. Then we consider a covering of $\Sigma \cap K$ made by finitely many discs $D_{r/2}(x_1), \dots, D_{r/2}(x_J)$ (included in $\partial^0 G$, choosing a smaller radius if necessary). Then, for each j we can find a point $x_j^+ \in D_{r/2}(x_j) \cap E^+$ and a point $x_j^- \in D_{r/2}(x_j) \cap E^-$. From (ii) we infer that for k large enough,

$$u_k(x_j^+) \geqslant 1/2(1+t) \quad \text{and} \quad u_k(x_j^-) \leqslant 1/2(-1+t) \qquad \forall j \in \{1, \dots, J\}.$$

Then, by the mean value theorem, for k large enough we can find for each j a point $x_j^k \in [x_j^-, x_j] \cup [x_j, x_j^+] \subseteq D_{r/2}(x_j)$ such that $u_k(x_j^k) = t$. Now, if x is an arbitrary point in $\Sigma \cap K$, then $x \in D_{r/2}(x_{j_x})$ for some $j_x \in \{1, \dots, J\}$, and thus $|x - x_{j_x}^k| \leqslant |x - x_{j_x}| + |x_{j_x} - x_{j_x}^k| < r$. Hence $\Sigma \cap K \subseteq \mathcal{T}_r(L^t_k)$ whenever k is sufficiently large.

Step 4: Proof of (vi). Let $\mathbf{X} = (X, \mathbf{X}_{n+1}) \in C^1(\overline{G}; \mathbb{R}^{n+1})$ be a compactly supported vector field in $G \cup \partial^0 G$ such that $\mathbf{X}_{n+1} = 0$ on $\partial^0 G$. We have

$$\delta \mathbf{E}\big(u_k, G \cup \partial^0 G\big)[\mathbf{X}] + \frac{1}{\varepsilon_k^{2s}} \int_{\partial^0 G} W(u_k) \operatorname{div} X \, \mathrm{d}x = \int_{\partial^0 G} u_k \operatorname{div}(f_k X) \, \mathrm{d}x.$$

From formula established for the first variation and the convergences established in Step 1, we can pass to the limit $k \to \infty$ in this identity to infer that

$$\delta\mathbf{E}\big(u_*, G \cup \partial^0 G\big)[\mathbf{X}] = \int_{\partial^0 G} u_* \mathrm{div}(fX)\,\mathrm{d}x\,,$$

and the proof is complete.

□

4.7. Asymptotics for the fractional Allen-Cahn equation

The object of this section is to prove a general convergence result as $\varepsilon \downarrow 0$ for the fractional equation (4.26). As we already explained, we rely on the previous results obtained for the degenerate equation with boundary reaction.

THEOREM 20. *Let Ω be a smooth bounded open set, and $\varepsilon_k \downarrow 0$ a given sequence. Let $\{g_k\}_{k\in\mathbb{N}} \subseteq C^{0,1}_{\mathrm{loc}}(\mathbb{R}^n)$ be such that $\sup_k \|g_k\|_{L^\infty(\mathbb{R}^n\setminus\Omega)} < \infty$ and $g_k \to g$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n \setminus \Omega)$ for a function g satisfying $|g| = 1$ a.e. in $\mathbb{R}^n \setminus \Omega$. Let $\{f_k\}_{k\in\mathbb{N}} \subseteq C^{0,1}(\Omega)$ satisfying*

$$\sup_k \big(\varepsilon_k^{2s}\|f_k\|_{L^\infty(\Omega)} + \|f_k\|_{W^{1,q}(\Omega)}\big) < \infty \quad \textit{for some } n/(1+2s) < q < n\,,$$

and such that $f_k \rightharpoonup f$ weakly in $W^{1,q}(\Omega)$. Let $\{v_k\}_{k\in\mathbb{N}} \subseteq H^s_{g_k}(\Omega) \cap L^p(\Omega)$ be a sequence such that v_k weakly solves

$$\begin{cases} (-\Delta)^s v_k + \dfrac{1}{\varepsilon_k^{2s}} W'(v_k) = f_k & \textit{in } \Omega\,, \\ v_k = g_k & \textit{in } \mathbb{R}^n \setminus \Omega\,. \end{cases} \tag{4.26}$$

If $\sup_k \mathcal{F}_{\varepsilon_k}(v_k, \Omega) < \infty$, then there exist a (not relabeled) subsequence and a Borel set $E_* \subseteq \mathbb{R}^n$ of finite $2s$-perimeter in Ω such that $v_k \to v_* := \chi_{E_*} - \chi_{\mathbb{R}^n\setminus E_*}$ strongly in $H^s_{\mathrm{loc}}(\Omega)$ and $L^2_{\mathrm{loc}}(\mathbb{R}^n)$. Moreover, $E_* \cap \Omega$ is an open set, and

$$\delta P_{2s}(E_*, \Omega)[X] = \frac{1}{\gamma_{n,s}} \int_{E_*\cap\Omega} \mathrm{div}(fX)\,\mathrm{d}x \quad \textit{for every } X \in C^1_c(\Omega; \mathbb{R}^n)\,. \tag{4.27}$$

In addition, for every smooth open subset $\Omega' \subseteq \Omega$ such that $\overline{\Omega'} \subseteq \Omega$,

(i) $\mathcal{E}(v_k, \Omega') \to 2\gamma_{n,s} P_{2s}(E_*, \Omega')$;

(ii) $\dfrac{1}{\varepsilon_k^{2s}} W(v_k) \to 0$ *in* $L^1(\Omega')$;

(iii) $f_k(x) - \dfrac{1}{\varepsilon_k^{2s}} W'(v_k(x)) \to \left(\dfrac{\gamma_{n,s}}{2} \displaystyle\int_{\mathbb{R}^n} \frac{|v_*(x) - v_*(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}y\right) v_*(x)$ *strongly in* $H^{-s}(\Omega')$;

(iv) $v_k \to v_*$ *in* $C^0_{\mathrm{loc}}(\Omega \setminus \partial E_*)$*;*

(v) *if* $\sup_k \|f_k\|_{L^\infty(\Omega)} < \infty$*, then* $v_k \to v_*$ *in* $C^{0,\alpha}_{\mathrm{loc}}(\Omega \setminus \partial E_*)$ *for every* $\alpha \in (0, \beta_*)$ *with* β_* *given by Lemma 10;*

(vi) *if* $\sup_k \|f_k\|_{C^{0,1}(\Omega)} < \infty$*, then* $v_k \to v_*$ *in* $C^{1,\alpha}_{\mathrm{loc}}(\Omega \setminus \partial E_*)$ *for every* $\alpha \in (0, \beta_*)$*;*

(vii) *for each* $\delta \in (-1,1)$*, the level set* $L^\delta_k := \{v_k = \delta\}$ *converges locally uniformly in* Ω *to* $\partial E_* \cap \Omega$*, i.e., for every compact set* $K \subseteq \Omega$ *and every* $r > 0$*,*

$$L^\delta_k \cap K \subseteq \mathscr{T}_r(\partial E_* \cap \Omega) \quad \textit{and} \quad \partial E_* \cap K \subseteq \mathscr{T}_r(L^\delta_k \cap \Omega)$$

whenever k *is large enough.*

Proof. Step 1. First we recall that, under the assumptions of the theorem, we have that $v_k \in C^{1,\beta_*}_{\mathrm{loc}}(\Omega) \cap C^0(\mathbb{R}^n)$ and $\sup_k \|v_k\|_{L^\infty(\mathbb{R}^n)} < \infty$. Then the assumption $\sup_k \mathcal{F}_{\varepsilon_k}(v_k, \Omega) < \infty$ clearly implies $\sup_k \mathcal{E}_{\varepsilon_k}(v_k, \Omega) < \infty$. In turn, Lemma 1 shows that the sequence $\{v_k\}_{k\in\mathbb{N}}$ is bounded in $L^2(\mathbb{R}^n, \mathfrak{m})$, where the measure $\mathfrak{m}$ is defined in (3.33). Therefore, we can find a (not relabeled) subsequence and $v_* \in L^2(\mathbb{R}^n, \mathfrak{m})$ such that $v_k \rightharpoonup v_*$ weakly in $L^2(\mathbb{R}^n, \mathfrak{m})$. In particular, $v_k \rightharpoonup v_*$ weakly in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$. On the other hand, the uniform energy bound shows that $|v_k| \to 1$ in $L^1(\Omega)$, and $\{v_k\}_{k\in\mathbb{N}}$ is bounded in $H^s(\Omega)$. Hence $v_k \rightharpoonup v_*$ weakly in $H^s(\Omega)$, and from the compact embedding $H^s(\Omega) \hookrightarrow L^2(\Omega)$, it implies that $v_k \to v_*$ strongly in $L^2(\Omega)$. By assumption we have $g_k \to g$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n \setminus \Omega)$ and $\sup_k \|g_k\|_{L^\infty(\mathbb{R}^n\setminus\Omega)} < \infty$, so that $g_k \to g$ in $L^2_{\mathrm{loc}}(\mathbb{R}^n \setminus \Omega)$. Since $v_k = g_k$ in $\mathbb{R}^n \setminus \Omega$, we conclude that $v_* = g$ in $\mathbb{R}^n \setminus \Omega$ and $v_k \to v_*$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$. Extracting a further subsequence if necessary, we may assume that $v_k \to v_*$ a.e. in $\mathbb{R}^n$. Since $|g| = 1$ a.e. in $\mathbb{R}^n$, we derive that $|v_*| = 1$ a.e. in $\mathbb{R}^n$. Hence we can find a Borel set $F \subseteq \mathbb{R}^n$ such that

$$v_* = \chi_F - \chi_{\mathbb{R}^n\setminus F} \text{ a.e. in } \mathbb{R}^n.$$

Moreover, we easily infer from Fatou's lemma that

$$\mathcal{E}(v_*, \Omega) \leqslant \liminf_{k\to\infty} \mathcal{E}(v_k, \Omega) < \infty. \tag{4.28}$$

We end this first step showing that $v^{\mathrm{e}}_k \rightharpoonup v^{\mathrm{e}}_*$ weakly in $H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+ \cup \Omega, |z|^a \mathrm{d}\mathbf{x})$. Indeed, we start deducing from continuity of the trace of extension operator that $v^{\mathrm{e}}_k \rightharpoonup v^{\mathrm{e}}_*$ weakly in $L^2_{\mathrm{loc}}(\overline{\mathbb{R}^{n+1}_+}, |z|^a \mathrm{d}\mathbf{x})$. On the other hand, the uniform energy bound together with Lemma 8 and standard elliptic estimates shows that $\{v^{\mathrm{e}}_k\}_{k\in\mathbb{N}}$ is bounded in $H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+ \cup \Omega, |z|^a \mathrm{d}\mathbf{x})$, whence the announced weak convergence.

Step 2. Let us now consider an increasing sequence $\{G_l\}_{l\in\mathbb{N}}$ of bounded admissible open sets such that $\overline{\partial^0 G_l} \subseteq \Omega$ for every $l \in \mathbb{N}$, $\cup_l G_l = \mathbb{R}^{n+1}_+$, and $\cup_l \partial^0 G_l = \Omega$. By (3.36), Step 1, and the results in Section 4.3, $v^{\mathrm{e}}_k \in H^1(G_l, |z|^a \mathrm{d}\mathbf{x}) \cap L^\infty(G_l)$ satisfies

$\sup_k \|v_k^{\mathrm{e}}\|_{L^\infty(G_l)} \leqslant \sup_k \|v_k\|_{L^\infty(\mathbb{R}^n)} < \infty$, and each v_k^{e} solves

$$\begin{cases} \operatorname{div}(z^a \nabla v_k^{\mathrm{e}}) = 0 & \text{in } G_l\,, \\ d_s \partial_z^{(2s)} v_k^{\mathrm{e}} = \dfrac{1}{\varepsilon_k^{2s}} W'(v_k^{\mathrm{e}}) - f_k & \text{on } \partial^0 G_l\,, \end{cases}$$

for every $l \in \mathbb{N}$. In addition, $\sup_k \mathbf{E}_{\varepsilon_k}(v_k^{\mathrm{e}}, G_l) < \infty$ for every $l \in \mathbb{N}$, still by Step 1. Therefore, we can find a further subsequence such that the conclusions of Theorem 12 hold in every G_l, and v_*^{e} is the limiting function in each G_l by Step 1. In particular, $v_k^{\mathrm{e}} \to v_*^{\mathrm{e}}$ strongly in $H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+ \cup \Omega, |z|^a \mathrm{d}\mathbf{x})$.

For each $l \in \mathbb{N}$, denote by E_l the limiting open subset of $\partial^0 G_l$ provided by Theorem 12, and observe that $E_l = E_{l+1} \cap \partial^0 G_l$ for every $l \in \mathbb{N}$ (see the proof of Theorem 12, Step 2). Then we define $E_\Omega := \cup_l E_l$, so that E_Ω is an open subset of Ω, $E_l = E_\Omega \cap \partial^0 G_l$ for every $l \in \mathbb{N}$, and $v_* = \chi_{E_\Omega} - \chi_{\Omega \setminus E_\Omega}$ a.e. in Ω. Setting

$$E_* := (F \setminus \Omega) \cup E_\Omega\,,$$

it follows that $v_* = \chi_{E_*} - \chi_{\mathbb{R}^n \setminus E_*}$ a.e. in $\mathbb{R}^n$. In particular, E_* has finite $2s$-perimeter in Ω since

$$\mathcal{E}(v_*, \Omega) = 2\gamma_{n,s} P_{2s}(E_*, \Omega)\,.$$

Finally, the conclusions of Theorem 12 in each G_l clearly imply the announced results stated in (ii), (iv), (v), (vi), and (vii).

Step 3. Now we show items (i), (iii), and the strong convergence of v_k in $H^s_{\mathrm{loc}}(\Omega)$. To this purpose, we fix a smooth open set $\Omega' \subseteq \Omega$ such that $\overline{\Omega'} \subseteq \Omega$. Setting for an arbitrary function $v \in \widehat{H}^s(\Omega)$,

$$\mathrm{e}_s\big(v(x), \Omega\big) := \frac{\gamma_{n,s}}{2} \int_\Omega \frac{|v(x) - v(y)|^2}{|x-y|^{n+2s}}\, \mathrm{d}y + \gamma_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \frac{|v(x) - v(y)|^2}{|x-y|^{n+2s}}\, \mathrm{d}y\,,$$

we claim that

$$\mathrm{e}_s(v_k, \Omega)\, \mathscr{L}^n \llcorner \Omega \rightharpoonup \mathrm{e}_s(v_*, \Omega)\, \mathscr{L}^n \llcorner \Omega$$

weakly* as Radon measures on Ω. Indeed, by the uniform energy bound, we can extract a subsequence such that $\mathrm{e}_s(v_k, \Omega)\, \mathscr{L}^n \llcorner \Omega \overset{*}{\rightharpoonup} \nu$ for some finite Radon measure ν on Ω. Then we fix $\varphi \in \mathscr{D}(\Omega)$ arbitrary. Notice that

$$\begin{aligned} \int_\Omega \mathrm{e}_s(v_k, \Omega) \varphi\, \mathrm{d}x = {} & \big\langle (-\Delta)^s v_k, \varphi v_k \big\rangle_\Omega \\ & - \frac{\gamma_{n,s}}{2} \iint_{\Omega \times \Omega} \frac{(v_k(x) - v_k(y)) v_k(y) (\varphi(x) - \varphi(y))}{|x-y|^{n+2s}}\, \mathrm{d}x\mathrm{d}y \\ & - \gamma_{n,s} \iint_{\Omega \times \Omega^c} \frac{(v_k(x) - v_k(y)) v_k(y) \varphi(x)}{|x-y|^{n+2s}}\, \mathrm{d}x\mathrm{d}y \\ =: {} & I_k - II_k - III_k\,. \end{aligned}$$

We consider a function $\Phi \in C^\infty(\overline{\mathbb{R}^{n+1}_+})$ compactly supported in $G \cup \partial^0 G$ for some bounded admissible open set $G \subseteq \mathbb{R}^{n+1}_+$ such that $\overline{\partial^0 G} \subseteq \Omega$ and $\Phi_{|\mathbb{R}^n} = \varphi$. Since $\varphi v_k \in H^s_{00}(\Omega)$ and $\Phi v_k^{\mathrm{e}} \in H^1(G, |z|^a \mathrm{d}\mathbf{x})$ is compactly supported in $G \cup \partial^0 G$, Lemma 4 yields

$$\big\langle (-\Delta)^s v_k, \varphi v_k \big\rangle_\Omega = d_s \int_G z^a |\nabla v_k^{\mathrm{e}}|^2 \Phi \, \mathrm{d}\mathbf{x} + d_s \int_G z^a \nabla v_k^{\mathrm{e}} \cdot (v_k^{\mathrm{e}} \nabla \Phi) \, \mathrm{d}\mathbf{x}.$$

Since $v_k^{\mathrm{e}} \to v_*^{\mathrm{e}}$ strongly in $H^1(G, |z|^a \mathrm{d}\mathbf{x})$, we obtain

$$\begin{aligned}\big\langle (-\Delta)^s v_k, \varphi v_k \big\rangle_\Omega \underset{k\to\infty}{\longrightarrow} d_s \int_G z^a |\nabla v_*^{\mathrm{e}}|^2 \Phi \, \mathrm{d}\mathbf{x} + d_s \int_G z^a \nabla v_*^{\mathrm{e}} \cdot (v_*^{\mathrm{e}} \nabla \Phi) \, \mathrm{d}\mathbf{x} \\ = d_s \int_{\mathbb{R}^{n+1}_+} z^a \nabla v_*^{\mathrm{e}} \cdot \nabla (\Phi v_*^{\mathrm{e}}) \, \mathrm{d}\mathbf{x}.\end{aligned}$$

By Lemma 4 again, we have thus proved that

$$\big\langle (-\Delta)^s v_k, \varphi v_k \big\rangle_\Omega \underset{k\to\infty}{\longrightarrow} \big\langle (-\Delta)^s v_*, \varphi v_* \big\rangle_\Omega . \tag{4.29}$$

On the other hand, we easily deduce by dominated convergence that

$$II_k \to \frac{\gamma_{n,s}}{2} \iint_{\Omega\times\Omega} \frac{(v_*(x) - v_*(y)) v_*(y) (\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} \, \mathrm{d}x\mathrm{d}y \tag{4.30}$$

and

$$III_k \to \gamma_{n,s} \iint_{\Omega\times\Omega^c} \frac{(v_*(x) - v_*(y)) v_*(y) \varphi(x)}{|x-y|^{n+2s}} \, \mathrm{d}x\mathrm{d}y \tag{4.31}$$

as $k \to \infty$. Gathering (4.29), (4.30), and (4.31) leads to

$$\int_\Omega \mathrm{e}_s(v_k, \Omega) \varphi \, \mathrm{d}x \underset{k\to\infty}{\longrightarrow} \int_\Omega \mathrm{e}_s(v_*, \Omega) \varphi \, \mathrm{d}x,$$

and thus $\nu = \mathrm{e}_s(v_*, \Omega) \mathscr{L}^n \llcorner \Omega$ by the arbitrariness of φ.

Since $\nu(\partial\Omega') = 0$, we now derive that

$$\int_{\Omega'} \mathrm{e}_s(v_k, \Omega) \, \mathrm{d}x \to \int_{\Omega'} \mathrm{e}_s(v_*, \Omega) \, \mathrm{d}x. \tag{4.32}$$

Then, since Ω' is smooth and bounded, it has finite $2s$-perimeter in $\mathbb{R}^n$, and thus

$$\int_{\Omega'} \int_{\Omega\setminus\Omega'} \frac{1}{|x-y|^{n+2s}} \, \mathrm{d}x\mathrm{d}y \leqslant \int_{\Omega'} \int_{\mathbb{R}^n\setminus\Omega'} \frac{1}{|x-y|^{n+2s}} \, \mathrm{d}x\mathrm{d}y = P_{2s}(\Omega', \mathbb{R}^n) < \infty. \tag{4.33}$$

It now follows by dominated convergence and (4.32) that

$$\begin{aligned}\mathcal{E}(v_k, \Omega') &= \frac{1}{2} \int_{\Omega'} \mathrm{e}_s(v_k, \Omega) \, \mathrm{d}x + \frac{\gamma_{n,s}}{4} \int_{\Omega'} \int_{\Omega\setminus\Omega'} \frac{|v_k(x) - v_k(y)|^2}{|x-y|^{n+2s}} \, \mathrm{d}x\mathrm{d}y \\ &\underset{k\to\infty}{\longrightarrow} \frac{1}{2} \int_{\Omega'} \mathrm{e}_s(v_*, \Omega) \, \mathrm{d}x + \frac{\gamma_{n,s}}{4} \int_{\Omega'} \int_{\Omega\setminus\Omega'} \frac{|v_*(x) - v_*(y)|^2}{|x-y|^{n+2s}} \, \mathrm{d}x\mathrm{d}y \\ &= \mathcal{E}(v_*, \Omega') = 2\gamma_{n,s} P_{2s}(E_*, \Omega').\end{aligned} \tag{4.34}$$

Using (4.32) again, the same argument shows that

$$[v_k]^2_{H^s(\Omega')} \to [v_*]^2_{H^s(\Omega')},$$

and thus $v_k \to v_*$ strongly in $H^s(\Omega')$, since we already know that $v_k \rightharpoonup v_*$ weakly in $H^s(\Omega')$. In turn, the strong convergence in $H^s(\Omega')$ and (4.33) easily imply $\big\langle(-\Delta)^s v_k, v_*\big\rangle_{\Omega'} \to \big\langle(-\Delta)^s v_*, v_*\big\rangle_{\Omega} = 2\mathcal{E}(v_*,\Omega')$ by dominated convergence. Consequently,

$$\mathcal{E}(v_k - v_*,\Omega') = \mathcal{E}(v_k,\Omega') + \mathcal{E}(v_*,\Omega') - \big\langle(-\Delta)^s v_k, v_*\big\rangle_{\Omega'} \longrightarrow 0.$$

Next we infer from (3.3) that $(-\Delta)^s v_k \to (-\Delta)^s v_*$ strongly in $H^{-s}(\Omega')$.

Then, fix some $\varphi \in \mathscr{D}(\Omega')$. Since $v_*^2 = 1$, we have the identity

$$\big(v_*(x) - v_*(y)\big)\big(\varphi(x) - \varphi(y)\big) = \frac{1}{2}|v_*(x) - v_*(y)|^2\big(v_*(x)\varphi(x) + v_*(y)\varphi(y)\big), \tag{4.35}$$

that we may insert in (3.2) to obtain

$$\big\langle(-\Delta)^s v_*, \varphi\big\rangle_{\Omega'} = \int_{\Omega'}\left(\frac{\gamma_{n,s}}{2}\int_{\mathbb{R}^n}\frac{|v_*(x) - v_*(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}y\right)v_*(x)\varphi(x)\,\mathrm{d}x. \tag{4.36}$$

Using this equation and (4.26), item (iii) follows.

Step 4. Now it only remains to show that E_* satisfies (4.27). Let $X \in C^1(\mathbb{R}^n;\mathbb{R}^n)$ compactly supported in Ω, and $\mathbf{X} = (\mathbf{X}_1,\dots,\mathbf{X}_{n+1}) \in C^1(\overline{\mathbb{R}^{n+1}_+};\mathbb{R}^{n+1})$ compactly supported in $\mathbb{R}^{n+1}_+ \cup \Omega$ satisfying $\mathbf{X} = (X,0)$ on Ω. Setting $\{\phi_t\}_{t\in\mathbb{R}}$ to be the flow on $\mathbb{R}^n$ generated by X, we notice that

$$P_{2s}\big(\phi_t(E_*),\Omega\big) = \frac{1}{2\gamma_{n,s}}\mathcal{E}(v_*\circ\phi_{-t},\Omega). \tag{4.37}$$

Since the support of $\mathbf{X}$ is contained in $G_l \cup \partial^0 G_l$ for l large enough, we can apply (vi) in Theorem 12. In view of Remark 7 and (4.37), we obtain

$$\begin{aligned}\delta P_{2s}(E_*,\Omega)[X] &= \frac{1}{2\gamma_{n,s}}\delta\mathcal{E}(v_*,\Omega)[X]\\ &= \frac{1}{2\gamma_{n,s}}\delta\mathbf{E}\big(v_*^{\mathrm{e}}, G_l\cup\partial^0 G_l\big)[\mathbf{X}] = \frac{1}{2\gamma_{n,s}}\int_\Omega v_*\,\mathrm{div}(fX)\,\mathrm{d}x\\ &= \frac{1}{\gamma_{n,s}}\int_{E_*\cap\Omega}\mathrm{div}(fX)\,\mathrm{d}x,\end{aligned}$$

by the divergence theorem, and the proof is complete. □

5. Appendix: Marstrand's Theorem

Main goal of this short appendix is to present a compact proof of Marstrand's Theorem by using the results derived by Kirchheim and Preiss in [31]. We begin by recalling the following definition.

DEFINITION 11. *A measure μ over $\mathbb{R}^n$ is said to be uniformly distributed if there is a function $f_\mu : (0,\infty) \to [0,\infty]$ such that*

(i) $\mu(B(x,r)) = f_\mu(r)$ *for every* $x \in \mathrm{spt}(\mu)$ *and* $r > 0$*, and*

(ii) $f_\mu(r) < \infty$ *for some* $r > 0$.

THEOREM 21 (Marstrand's Theorem). *Let be μ be a metric outer measure in $\mathbb{R}^n$, $n \geqslant 1$. Assume that $s \in \mathbb{R}$ is nonnegative and $B \subseteq A \subseteq \mathbb{R}^n$ satisfy*

(i) $\mu(B) > 0$*, and*

(ii) $\lim_{r\downarrow 0} \dfrac{\mu(B(x,r)\cap A)}{r^s}$ *exists and is positive and finite for any* $x \in B$.

Then s is an integer.

Proof. Using standard regularization methods, we infer the measurability of both

$$x \mapsto \limsup_{r\downarrow 0} \frac{\mu(B(x,r)\cap A)}{r^s} \text{ and } x \mapsto \liminf_{r\downarrow 0} \frac{\mu(B(x,r)\cap A)}{r^s},$$

and in addition to (ii) we may assume that $\tilde{\mu} = \mu \llcorner A$ is Borel regular and locally finite in some open superset U of A and that B is a compact set on which the functions $x \mapsto k^s \tilde{\mu}\left(B(x,\frac{1}{k})\right)$ uniformly converge, as $k \to \infty$, to a continuous function $d : B \to (0,\infty)$. The Besicovitch Differentiation theorem ensures [20, 2.9.7] ensure the existence of an $x_0 \in B$ with

$$\lim_{r\downarrow 0} \frac{\tilde{\mu}(B(x_0,r)\setminus B)}{\tilde{\mu}(B(x_0,r))} = 0.$$

For $k \geqslant 1$ we define

$$\tilde{\mu}_k(M) := k^s \tilde{\mu}\left(x_0 + \frac{1}{k}M\right)$$

and $B_k := k\cdot(B - x_0)$. Since $\lim_{k\to\infty}\tilde{\mu}_k(B(0,i)) = d(x_0)i^s$ for any integer $i \geqslant 0$, we infer from well-known weak compactness results for measure, (see [32, Theorem 1.23]), Proposition 1.12 of [31], that there is a sequence $k_i \uparrow \infty$ of integers and a Radon measure ϕ over $\mathbb{R}^n$ such that $\phi(C) \geqslant \limsup_{i\to\infty}\tilde{\mu}_{k_i}(C)$ for any compact set and $\phi(G) \leqslant \liminf_{i\to\infty}\tilde{\mu}_k(G)$ for any open set G. We show that ϕ is a nonzero uniformly distributed measure with $f_\phi(r) = r^s d(x_0)$. Then it easily follows from [31, Theorem 1.4 (ii)] that s is an integer. This will finish the proof.

Since $\phi(B(0,1)) \geqslant d(x_0) > 0$, we have $\mathrm{spt}(\phi) \neq \emptyset$. Let $x \in \mathrm{spt}(\phi)$, $R > 0$ and fix $\varepsilon < R$. Then

$$0 < \phi(B(x,\varepsilon/2)) \leqslant \liminf_{i\to\infty} \tilde{\mu}_{k_i}(B(x,\varepsilon)).$$

According to the choice of x_0 we have $\liminf_{i\to\infty}\tilde{\mu}_{k_i}(B(0,||x||+2\varepsilon)\setminus B_{k_i}) = 0$ and therefore for any sufficiently large we may find a point $y_i \in B(x,\varepsilon)\cap B_{k_i}$. Consequently,

we can estimate

$$\begin{aligned}\phi\left(B(x,R)\right) &\geqslant \limsup_{i\to\infty} \tilde{\mu}_{k_i}\left(B(x,R)\right) \geqslant \limsup_{i\to\infty} \tilde{\mu}_{k_i}\left(B(y_i,R-\varepsilon)\right)\\ &\geqslant \limsup_{i\to\infty}\ k_i^s\,\tilde{\mu}\left(B\left(x_0+\frac{y_i}{k_i},\frac{R-\varepsilon}{k_i}\right)\right) \geqslant d(x_0)\left(R-\varepsilon\right)^s\end{aligned}$$

since all (x_0+y_i/k_i) belong to B and the "average density" uniformly converges on B. Similarly one can estimate that the ϕ-measure of the interior of the ball $B\left(x,R+\varepsilon\right)$ is at most $d(x_0)(R+2\varepsilon)^s$ and, tending ε to 0, we infer that $\phi\left(B(x,R)\right)=d(x_0)R^s$ as required. □

6. Acknowledgements

Y.S. would like to thank his collaborators V.Millot, K. Wang and H. Yu with whom he investigated in several works the problems presented in this survey. He would like to thank also T. De Pauw for pointing out the reference about the alternative proof of Marstrand's theorem by Kirchheim and Preiss.

Appendix

Table 7.1: Notations for functions spaces

$BV(\Omega;X)$	The space of all functions of Bounded Variations on Ω taking values in the set X.
$L^p(\Omega;X,wdx)$	The space of all $X-$ valued weighted L^p functions on Ω with weight w.
$L^p\left(\bigwedge^1\mathbb{R}^n\right)$	The L^p space of 1-forms on $\mathbb{R}^n$.
$\mathcal{H}^1(\mathbb{R}^n)$	The Hardy space.
$H^{1/2}_{00}(\omega)$	The space of functions in $H^{1/2}(\mathbb{R}^n)$ vanishing outside ω .
$H^{-1/2}(\omega)$	The topological dual of $H^{1/2}_{00}(\omega)$.
$H^{1/2}_{00}(\omega;X)$	The space of $X-$ valued $H^{1/2}_{00}$ functions.
$H^{-s}(\omega)$	The topological dual of $H^s(\omega)$, for $0<s<1/2$.
$H^s(\omega;X)$	The space of $X-$ valued H^s functions.
$\widehat{H}^s(\Omega)$	The space of all $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ functions v satisfying $\mathcal{E}(v,\Omega):=\left\langle(-\Delta)^s v,v\right\rangle_\Omega<\infty$.
$\widehat{H}^s(\Omega;X)$	The space of all $X-$ valued $\widehat{H}^s$ functions.
$H^1(\Omega;X,wdx)$	The space of all $X-$ valued weighted H^1 functions on Ω with weight w.
$\mathcal{M}(\mathbb{R}^n)$	The set of all functions measurable with respect to the Lebesgue measure dx.
$\mathcal{M}(\bigwedge^1_{od}\mathbb{R}^n)$	The space of functions $F:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ measurable with respect to the measure $\frac{dxdy}{\lvert x-y\rvert^n}$.
$H^{1/2}_g(\omega;X)$	This space is precisely $g+H^{1/2}_{00}(\omega;X)$.
$H^s_g(\omega;X)$	For $0<s<1/2$, this space is defined by $g+H^s(\omega;X)$.

References

[1] Abatangelo, N. and Valdinoci, E. , *A notion of nonlocal curvature*, Numer. Funct. Anal. Optim. **35** (2014), 793–815.

[2] AMBROSIO, L., FUSCO, N., PALLARA, D. , *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford University Press, New York (2000).

[3] AMBROSIO, L. AND SONER, H.-M. , *A measure-theoretic approach to higher codimension mean curvature flows*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **25** (1997), 27–49.

[4] BALDES, A. , *Harmonic mappings with a partially free boundary*, Manuscripta Math. **40** (1982), 255–275.

[5] BARRIOS BARRERA, B., FIGALLI, A., VALDINOCI, E. , *Bootstrap regularity for integro-differential operators, and its application to nonlocal minimal surfaces*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. Vol. XIII (2014), 609–639.

[6] CABRÉ, X., FALL, M.M., SOLÀ-MORALES, J., WETH, T. , *Curves and surfaces with constant nonlocal mean curvature: meeting Alexandrov and Delaunay,* to appear in *J. Reine Angew. Math. (Crelle's Journal)*, preprint `arXiv:1503.00469`.

[7] CABRÉ, X., FALL, M.M., WETH T., *Delaunay hypersurfaces with constant nonlocal mean curvature*, preprint `arXiv:1602.02623`.

[8] CABRÉ, X. AND SIRE, Y. , *Nonlinear equations for fractional Laplacians I: Regularity, maximum principles, and Hamiltonian estimates*, Ann. Inst. H. Poincaré Anal. Non Linéaire **31** (2014), 23–53.

[9] CABRÉ, X. AND SOLÀ-MORALES, J., *Layer solutions in a halfspace for boundary reactions*, Comm. Pure Appl. Math. **58** (2005), 1678–1732.

[10] CAFFARELLI, L.A., ROQUEJOFFRE, J.M. , SAVIN, O. , *Nonlocal minimal surfaces*, Comm. Pure Appl. Math. **63** (2010), 1111–1144.

[11] CHEEGER, J. AND NABER, A. , *Quantitative stratification and the regularity of harmonic maps and minimal currents*, Comm. Pure Appl. Math. **66** (2013), 965–990.

[12] CHEN, Y.-M. AND STRUWE, M., *Existence and partial regularity results for the heat flow for harmonic maps*, Math. Z. **201** (1989), 83–103.

[13] COIFMAN, R, LIONS, P. L., MEYER, Y., SEMMES, S, *Compensated compactness and Hardy spaces*, J. Math. Pures Appl. **72** (1993), 247–286.

[14] DA LIO, F. AND RIVIÈRE, T. , *Three-term commutator estimates and the regularity of $\frac{1}{2}$-harmonic maps into spheres*, Anal. PDE **4** (2011), 149–190.

[15] DÁVILA, J., DEL PINO, M., WEI, J. , *Nonlocal s-minimal surfaces and Lawson cones*, to appear in J. Differential Geom., preprint `arXiv:1402.4173`.

[16] DE GIORGI, E., *Su una teoria generale della misura $(r-1)$- dimensionale in uno spazio ad r dimensioni*, Ann. Mat. Pura Appl. **4** 36 (1954), 191–213.

[17] DUZAAR, F. AND STEFFEN, K., *A partial regularity theorem for harmonic maps at a free boundary*, Asymptotic Anal. **2** (1989), 299–343.

[18] DUZAAR, F. AND STEFFEN, K., *An optimal estimate for the singular set of a harmonic map in the free boundary*, J. Reine Angew. Math. **401** (1989), 157–187.

[19] EVANS, L.-C. AND GARIEPY, R.-F. , *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton FL (1992).

[20] FEDERER, H. , *Geometric Measure Theory*, Grundlehren Math. Wiss. 153 (1969).

[21] FIGALLI, A., FUSCO, N., MAGGI, F., MILLOT, V., MORINI, M., *Isoperimetry and stability properties of balls with respect to nonlocal energies*, Comm. Math. Phys. **336** (2015), 441–507.

[22] FIGALLI, A. AND VALDINOCI, E. , *Regularity and Bernstein-type results for nonlocal minimal surfaces*, to appear in J. Reine Angew. Math. (Crelle's Journal), preprint `arXiv:1307.0234`.

[23] FOCARDI, M. , MARCHESE, A., SPADARO, E. , *Improved estimate of the singular set of Dir-minimizing Q-valued functions via an abstract regularity result*, J. Funct. Anal. **268** (2015), 3290-3325.

[24] GILBARG, D. AND TRUDINGER, N.-S. , *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics, Springer-Verlag, Berlin (2001).

[25] GIUSTI E., *Minimal surfaces and functions of bounded variation.*, Monographs in Mathematics, 80. Birkhäuser Verlag, Basel, 1984.

[26] GULLIVER, R. AND JOST, J. , *Harmonic maps which solve a free-boundary problem*, J. Reine Angew. Math. **381** (1987), 61–89.

[27] HAMILTON, R.-S. , *Harmonic maps of manifolds with boundary*, Lecture Notes in Mathematics, Springer-Verlag, Berlin (1975).

[28] HARDT, R. AND LIN, F.-H., *Partially constrained boundary conditions with energy minimizing mappings*, Comm. Pure Appl. Math. **42** (1989), 309–334.

[29] HÉLEIN, F., *Régularité des applications faiblement harmoniques entre une surface et une variété riemannienne*, C. R. Acad. Sci. Paris Sér. I Math. **312** (1991), 591–596.

[30] HUTCHINSON, J.E. AND TONEGAWA, Y, *Convergence of phase interfaces in the van der Waals-Cahn-Hilliard theory*, Calc. Var. Partial Differential Equations **10** (2000), 49–84.

[31] KIRCHHEIM, B AND PREISS, D, *Uniformly distributed measures in Euclidean spaces*, Math. Scand. **90** (2002), 152–160.

[32] MATTILA, P., *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press (1995).

[33] MAZOWIECKA, K. AND SCHIKORRA, A, *Fractional div-curl quantities and applications to nonlocal geometric equations*, J. Funct. Anal. **275** (2018), 1–44.

[34] LIN, F.-H. , *Gradient estimates and blow-up analysis for stationary harmonic maps*, Annals of Math. **149** (1999), 785–829.

[35] LIN, F.H. AND WANG, C.Y. , *Harmonic and quasi-harmonic spheres*, Comm. Anal. Geom. **7** (1999), 397–429.

[36] LIN, F.H. AND WANG, C.Y. , *Harmonic and quasi-harmonic spheres. II*, Comm. Anal. Geom. **10** (2002), 341–375.

[37] LIN, F.-H. AND WANG, C.-Y. , *Harmonic and quasi-harmonic spheres. III. Rectifiability of the parabolic defect measure and generalized varifold flows*, Ann. Inst. H. Poincaré Anal. Non Linéaire **19** (2002), 209–259.

[38] MATTILA, P. , *Geometry of sets and measures in Euclidean spaces*, Cambridge Studies in Advanced Mathematics **44**, Cambridge University Press (1995).

[39] MEEKS, WILLIAM H. AND PÉREZ, J., *A Survey on classical minimal surface theory*, University Lecture Series, 60. American Mathematical Society, Providence 2012.

[40] MILLOT, V. AND SIRE, Y. , *On a fractional Ginzburg-Landau equation and* $1/2$*-harmonic maps into spheres*, Arch. Rational Mech. Anal. **215** (2015), 125–210.

[41] MILLOT, V., SIRE, Y., WANG, K , *Asymptotics for the Fractional Allen?Cahn Equation and Stationary Nonlocal Minimal Surfaces*, Arch. Rational Mech. Anal. **231** (2019), 1129?1216.

[42] MILLOT, V., SIRE, Y., YU, H. , *Minimizing fractional harmonic maps on the real line in the supercritical regime*, eprint arXiv:1710.04754 (2017).

[43] MODICA, L., *The gradient theory of phase transitions and the minimal interface criterion*, Arch. Rational Mech. Anal. **98** (1987), 123–142.

[44] MODICA, L AND MORTOLA, S., *Un esempio di* Γ*-convergenza*, Boll. Un. Mat. Ital. B **14** (1977), 285–299.

[45] PREISS, D., *Geometry of measures in* $\mathbb{R}^n$*: distributions, rectifiability, and densities*, Ann. of Math. **125** (1987), 537–643.

[46] RIVIÈRE, T., *Everywhere discontinuous harmonic maps into spheres*, Acta Math. **175** (1995), 197–226.

[47] SAVARÉ, G., *Regularity and perturbation results for mixed second order elliptic problems*, Comm. Partial Differential Equations **22** (1997), 869–899.

[48] SAVIN O., *Minimal Surfaces and Minimizers of the Ginzburg-landau Energy*, 2009.

[49] SAVIN, O. AND VALDINOCI, E., *Regularity of nonlocal minimal cones in dimension 2*, Calc. Var. Partial Differential Equations **48** (2013), 33–39.

[50] SAVIN, O. AND VALDINOCI, E. , Γ*-convergence for nonlocal phase transitions*, Ann. Inst. H. Poincaré Anal. Non Linéaire **29** (2012), 479–500.

[51] SCHEVEN, C., *Partial regularity for stationary harmonic maps at a free boundary*, Math. Z. **253** (2006), 135–157.

[52] SCHOEN, R. AND UHLENBECK, K., *A regularity theory for harmonic maps*, J. Differ. Geom. **17** (1982), 307–335.

[53] SEMMES, S., *A primer on Hardy spaces, and some remarks on a theorem of Evans and Müller*, Comm. Partial Differential Equations **19** (1994), 277–319.

[54] SHAMIR, E., *Regularization of mixed second-order elliptic problems*, Isr. J. of Math. **6** (1968), 150–168.

[55] SHATAH, J., *Weak solutions and development of singularities of the* $SU(2)$ σ*-model,*, Comm. Pure Appl. Math. **41** (1988), 459–469.

[56] STAMPACCHIA, G., *Problemi al contorno ellitici, con dati discontinui, dotati di soluzionie hölderiane*, Ann. Mat. Pura Appl. (4) **51** (1960), 1–37.

[57] STAMPACCHIA, G. , *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Inst. Fourier (Grenoble) **15** (1965), 189–258.

[58] STERNBERG, P., *The effect of a singular perturbation on nonconvex variational problems*, Arch. Rational Mech. Anal. **101** (1988), 209–260.

[59] STRUWE, M, *On a free boundary problem for minimal surfaces*, Invent. Math. **75** (1984), 547–560.

[60] WANG, C.Y. , *Limits of solutions to the generalized Ginzburg-Landau functional*, Comm. Partial Differential Equations **27** (2002), 877–906.

[61] WENTE, H.C. , *An existence theorem for surfaces of constant mean curvature*, J. Math. Anal. Appl **26** (1969), 318–344.

[62] ZIEMER, W.-P., *Weakly differentiable functions*, Graduate Texts in Mathematics, Springer-Verlag, New York (1989).

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