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## RECENT RESULTS ON NON-CONVEX FUNCTIONALS PENALIZING OBLIQUE OSCILLATIONS

**Abstract.** The aim of this note is to review some recent results on a family of functionals penalizing oblique oscillations. These functionals naturally appeared in some variational problem related to pattern formation and are somewhat reminiscent of those introduced by Bourgain, Brezis and Mironescu to characterize Sobolev functions. We obtain both qualitative and quantitative results for functions of finite energy. It turns out that this problem naturally leads to the study of various differential inclusions and has connections with branched transportation models.

We review in this paper some recent results obtained in [GM19a, GM19b, GM19c] on non-convex functionals penalizing oblique oscillations. We are mainly interested in both qualitative and quantitative rigidity results for functions with finite energy. We also obtain concentration and rectifiability properties of the corresponding ‘defect’ measures. We will focus here on the most important results and sacrifice generality for clarity. In particular, we will restrict ourselves to a periodic setting to avoid boundary effects.

### 1. The energy

For  $n_1, n_2 \geq 1$  and  $n = n_1 + n_2$ , we decompose  $\mathbb{R}^n = X_1 \oplus X_2$  with  $X_1 \perp X_2$  and  $n_i = \dim X_i$  and consider the  $n$  dimensional torus  $\mathbb{T}^n := (\mathbb{R}/\mathbb{Z})^n = \mathbb{T}^{n_1} \oplus \mathbb{T}^{n_2}$ . For  $x \in \mathbb{R}^n$  we write  $x = x_1 + x_2$  its decomposition in  $X_1 \oplus X_2$ . For  $(x, z) \in \mathbb{T}^n \times \mathbb{R}^n$ , we introduce the notation

$$Du(x, z) := u(x + z) - u(x)$$

for the discrete derivative. We also fix a radial non-negative kernel\*

$$\rho \in L^1(\mathbb{R}^n, \mathbb{R}_+) \quad \text{with} \quad \int_{\mathbb{R}^n} \rho \, dx = 1, \quad \text{and} \quad \text{supp } \rho \subset B_1.$$

As usual, for  $\varepsilon > 0$  we introduce the rescaled kernel  $\rho_\varepsilon := \varepsilon^{-n} \rho(\varepsilon^{-1} \cdot)$  so that  $\{\rho_\varepsilon\}_{\varepsilon > 0}$  forms a family of radial mollifiers. We introduce three real parameters  $p, \theta_1, \theta_2 > 0$  and define for any measurable function  $u : \mathbb{T}^n \rightarrow \mathbb{R}$  and any  $\varepsilon > 0$ , the quantity

$$(1.1) \quad \mathcal{E}_{\varepsilon, p}^{\theta_1, \theta_2}(u) := \int_{\mathbb{R}^n} \rho_\varepsilon(z) \int_{\mathbb{T}^n} \frac{|Du(x, z_1)|^{\theta_1} |Du(x, z_2)|^{\theta_2}}{|z|^p} \, dx \, dz.$$

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\*we denote by  $B_1$  the unit ball of  $\mathbb{R}^n$ .

Eventually, we send  $\varepsilon$  to 0 and define the functional

$$\mathcal{E}_p^{\theta_1, \theta_2}(u) := \liminf_{\varepsilon \downarrow 0} \mathcal{E}_{\varepsilon, p}^{\theta_1, \theta_2}(u).$$

Most of the time, we omit the dependency on the parameters  $\theta_1, \theta_2$  and note

$$\mathcal{E}_{\varepsilon, p}(u) := \mathcal{E}_{\varepsilon, p}^{\theta_1, \theta_2}(u), \quad \mathcal{E}_p(u) := \mathcal{E}_p^{\theta_1, \theta_2}(u).$$

Let  $S(\mathbb{T}^n)$  be the non-convex set of functions  $u : \mathbb{T}^n \mapsto \mathbb{R}$  which depend only on the first  $n_1$  coordinates or only on the last  $n_2$  coordinates. That is

$$S(\mathbb{T}^n) := \{u : u(x) = u_l(x_l) \text{ in } \mathbb{T}^n \text{ with } l = 1 \text{ or } l = 2\}.$$

The functional  $\mathcal{E}_p$  vanishes on  $S(\mathbb{T}^n)$  and our main question is to understand when the converse implication

$$(1.2) \quad \mathcal{E}_p(u) = 0 \implies u \in S(\mathbb{T}^n)$$

is also true. It turns out that the critical exponent  $p$  depends on  $\theta_1$  and  $\theta_2$  but also on the regularity of  $u$ . Once this question is settled, it is natural to investigate the properties of functions with finite energy in the critical case. This leads in particular to quantitative versions of (1.2). Throughout the paper we will use the notation

$$\theta := \theta_1 + \theta_2.$$

## 2. Motivations and examples

The main motivation for studying the functionals  $\mathcal{E}_p$  comes from the study of pattern formation in some variational models involving competition between a perimeter term and a non-local repulsive one. We refer the reader to [GR19, DR19] where energies related to  $\mathcal{E}_p$  are used to show that some sets are union of stripes. These functionals may also be seen as variants of those introduced by Bourgain, Brezis and Mironescu in [BBM01] to characterize Sobolev spaces. We recall for instance that it is proven in [Bre02, DMMS08] that for  $p > \theta \geq 1$ , if

$$\mathcal{F}_p(u) := \liminf_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n} \rho_\varepsilon(z) \int_{\mathbb{T}^n} \frac{|Du(x, z)|^\theta}{|z|^p} dx dz < \infty$$

then  $u$  is constant. Let us however point out that the functional  $\mathcal{F}_p$  is convex, which makes its analysis much easier compared to the one of  $\mathcal{E}_p$ .

Let us observe that if  $\mathcal{E}_p(u) < \infty$  and  $p > \theta$ , then at almost every point of differentiability of  $u$  we have<sup>†</sup>

$$(2.1) \quad |\nabla_1 u(x)| |\nabla_2 u(x)| = 0$$

<sup>†</sup>we denote by  $\nabla_l$  the gradient with respect to  $x_l$ .

and one can wonder whether this differential inclusion (this may be rewritten as  $\nabla u \in K$  where  $K := X_1 \cup X_2$ ) is already rigid enough to imply  $u \in S(\mathbb{T}^n)$ . While this is the case for  $C^1$  functions, since the convex hull of  $K$  is  $\mathbb{R}^n$ , it is already not rigid in the class of Lipschitz functions (see [Dac08, Theorem 10.18]). This may serve as another motivation for studying functions of finite energy  $\mathcal{E}_p(u)$  rather than (2.1) in order to characterize functions in  $S(\mathbb{T}^n)$ .

If the threshold for rigidity is given by  $p = \theta$  in the case of  $C^1$  functions, it is larger for more general functions. Indeed, the main two examples of functions with finite energy but which are not in  $S(\mathbb{T}^n)$  are the ‘roof’ and ‘corner’ functions. We present the constructions in  $\mathbb{R}^2$  since they can be both easily extended to higher dimension and to the periodic setting. The ‘roof’ function is the Lipschitz function defined by  $u(x) := \min(x_1, x_2)$ . After localization, it satisfies  $0 < \mathcal{E}_{1+\theta}(u) < \infty$  for every  $\theta > 0$ . The ‘corner’ function is defined as  $u = 1_{(0, \infty)^2}$  and it is not hard to check that  $0 < \mathcal{E}_2(u) < \infty$ .

### 3. Qualitative and quantitative rigidity estimates

The results from this section were obtained in [GM19a].

The previous examples of the ‘roof’ and ‘corner’ functions show that in full generality, implication (1.2) can only hold for  $p \geq \max(1 + \theta, 2)$ . Our first main result is that this estimate is essentially sharp. Define

$$(3.1) \quad P(\theta_1, \theta_2) := \begin{cases} 2 & \text{if } \theta \leq 1 \\ 1 + \theta & \text{if } \theta \geq 1 \text{ and } \min(\theta_1, \theta_2) \leq 1 \\ \min(\theta_1, \theta_2) + \theta & \text{otherwise.} \end{cases}$$

From now on we use the notation

$$\mathcal{E}(u) := \mathcal{E}_{P(\theta_1, \theta_2)}(u).$$

**THEOREM 1.** *For every measurable function  $u$ , if  $\mathcal{E}(u) = 0$  then  $u \in S(\mathbb{T}^n)$ .*

*Sketch of proof.* Let us give a sketch of proof in the simplest case  $n = 2$ ,  $\theta = 1$  (and thus  $P(\theta_1, \theta_2) = 2$ ). Substituting  $u$  by  $\arctan u$ , we may assume that  $\|u\|_\infty \leq 1$ .

Making the changes of variables  $(\tilde{x}, \tilde{z}) = (x + z_1, -z_1 + z_2)$ ,  $(\tilde{x}, \tilde{z}) = (x + z_2, z_1 - z_2)$  and  $(\tilde{x}, \tilde{z}) = (x + z_1 + z_2, -z_1 - z_2)$  in the definition of  $\mathcal{E}(u)$ , we see that if  $\mathcal{E}(u) = 0$  then also

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^2} \rho_\varepsilon(z) \int_{\mathbb{T}^2} \frac{(|Du(x + z_2, z_1)| + |Du(x, z_1)|)^{\theta_1} (|Du(x + z_1, z_2)| + |Du(x, z_2)|)^{\theta_2}}{|z|^2} dx dz \\ = 0. \end{aligned}$$

Since by triangle inequality, we have

$$\begin{aligned} |Du(x+z_2, z_1)| + |Du(x, z_1)| &\geq |u(x+z_1+z_2) - u(x+z_2) - u(x+z_1) + u(x)| \\ &= |D[Du(\cdot, z_2)](x, z_1)| \end{aligned}$$

and the same lower bound for  $|Du(x+z_2, z_1)| + |Du(x, z_1)|$ , we obtain (recall that  $\theta = 1$ )

$$\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^2} \rho_\varepsilon(z) \int_{\mathbb{T}^2} \frac{|D[Du(\cdot, z_2)](x, z_1)|}{|z|^2} dx dz = 0.$$

This implies that we can find a sequence  $z^k \rightarrow 0$  with  $|z_1^k| \sim |z_2^k|$  for which

$$(3.2) \quad \lim_{k \rightarrow \infty} \int_{\mathbb{T}^2} \frac{|D[Du(\cdot, z_2^k)](x, z_1^k)|}{|z^k|^2} dx = 0.$$

From this we may deduce that in the sense of distributions,

$$(3.3) \quad \partial_1 \partial_2 u = 0.$$

We can thus write  $u(x) = u_1(x_1) + u_2(x_2)$ . Plugging this back into the definition of  $\mathcal{E}(u)$ , we conclude that either  $u_1 = 0$  or  $u_2 = 0$ . □

If this result is sharp in the first two cases of (3.1) as seen from the ‘roof’ and ‘corner’ functions, we believe that it is not in the last case.

**Conjecture:** For every  $\theta \geq 1$ , Theorem 1 holds with  $P(\theta_1, \theta_2) = 1 + \theta$ .

The main insight in the proof of Theorem 1 is that the energy  $\mathcal{E}(u)$  does not only control first derivatives as seen from (2.1) but also the mixed second order derivatives (see (3.3))

$$\mu[u] := \nabla_1 \nabla_2 u.$$

With a more careful proof, it is possible to show that the energy gives a quantitative control on the defect measure  $\mu[u]$ .

**PROPOSITION 1.** *If  $\|u\|_\infty \leq 1$  and  $\theta \leq 1$ , then for every<sup>‡</sup>  $\varphi \in C^\infty(\mathbb{T}^n, \mathbb{R}^{n_1 \times n_2})$ ,*

$$(3.4) \quad \langle \mu[u], \varphi \rangle \lesssim \mathcal{E}(u) \|\varphi\|_\infty.$$

In particular this means that in the case  $\theta \leq 1$ , if  $\mathcal{E}(u) < \infty$  then  $\mu[u]$  is a Radon measure. If  $\theta > 1$  we can also obtain control on  $\mu[u]$  by  $\mathcal{E}(u)$  in some Sobolev spaces with negative regularity index. However, for every  $\theta > 1$  we can construct Lipschitz functions with finite energy for which  $\mu[u]$  is not a Radon measure. Since all the

<sup>‡</sup>By convention  $a \lesssim b$  means that there exists a non-negative constant  $C$  which may only depend on  $\theta_1, \theta_2, n$  or on the kernel  $\rho$  such that  $a \leq Cb$ .

subsequent results build on (3.4), we only consider from now on the case  $\theta \leq 1$ .

In light of (3.4), one can wonder if a quantitative version of Theorem 1 holds: is it true that  $\mathcal{E}(u)$  controls the distance of  $u$  to  $S(\mathbb{T}^n)$  in some norm? The strongest result we obtained is for  $n = 2$ .

**THEOREM 2.** *Assume that  $n_1 = n_2 = 1$  and  $\theta \leq 1$ . Then for every function  $u$  with  $\|u\|_\infty \leq 1$ , there exists  $\bar{u} \in S(\mathbb{T}^2)$  such that  $u - \bar{u} \in BV(\mathbb{T}^2) \cap L^\infty(\mathbb{T}^2)$  with*

$$(3.5) \quad \|u - \bar{u}\|_\infty + |\nabla[u - \bar{u}]|(\mathbb{T}^2) \lesssim \mathcal{E}(u) + \mathcal{E}(u)^{\frac{1}{2}}.$$

The idea of the proof is to first decompose  $u$  as  $u(x) = u_1(x_1) + u_2(x_2) + w(x)$  where  $w$  satisfies  $\partial_1 \partial_2 w = \mu[u] = \partial_1 \partial_2 u$  and  $\|w\|_\infty + |\nabla w|(\mathbb{T}^2) \lesssim \mathcal{E}(u)$ . Using this (and in particular the  $L^\infty$  bound on  $w$ ), we can quantify how much the integration with respect to  $x_1$  and  $x_2$  in the definition of  $\mathcal{E}(u)$  decouples. In higher dimension, the failure of the Sobolev embedding  $BV(\mathbb{T}^{n_i}) \subset L^\infty(\mathbb{T}^{n_i})$  makes the situation more complex (and in particular the energy does not control the corresponding  $w$  in  $L^\infty$ ) and we were not able to obtain a  $BV$  estimate. Nevertheless, we have,

**THEOREM 3.** *Assume that  $\hat{n} := \max(n_1, n_2) \geq 2$  and that  $\theta \leq 1$ . Then for every function  $u$  with  $\|u\|_\infty \leq 1$ , there exists  $\bar{u} \in S(\mathbb{T}^n)$  such that  $u - \bar{u} \in L^{\frac{\hat{n}}{\hat{n}-1}}(\mathbb{T}^n)$  with*

$$\|u - \bar{u}\|_{L^{\frac{\hat{n}}{\hat{n}-1}}(\mathbb{T}^n)} \lesssim \mathcal{E}(u) + \mathcal{E}(u)^{\frac{1}{2}}.$$

#### 4. Structure of the defect measure

The results from this section can be found in [GM19b] with the exception of Theorem 8 which is part of [GM19c].

A natural question to investigate is the structure of the defect measure  $\mu[u]$ . Since the ‘corner’ function, whose defect measure is a Dirac mass, is optimal for  $\theta \leq 1$  while the ‘roof’ function, whose defect measure is concentrated on the line  $x_1 = x_2$ , is optimal only for  $\theta = 1$ , we can expect that this structure will depend on the value of  $\theta$ .

##### 4.1. The case $\theta < 1$

We start with the case  $n_1 = n_2 = 1$ .

**THEOREM 4.** *For  $n = 2$  and  $\theta < 1$ , if  $u$  is such that  $\|u\|_\infty \leq 1$  and  $\mathcal{E}(u) < \infty$ , then there exist  $\sigma^k \in \mathbb{R}$  and  $x^k \in \mathbb{T}^2$  such that*

$$\mu[u] = \sum_k \sigma^k \delta_{x^k}.$$

Moreover,

$$(4.1) \quad \sum_k |\sigma^k|^\theta \lesssim \mathcal{E}(u).$$

*Sketch of proof.* The proof of this result is much easier if  $u$  is assumed to be a characteristic function. Indeed, in this case the differential inclusion<sup>§</sup>

$$(4.2) \quad \partial_1 \partial_2 u \in \mathcal{M}(\mathbb{T}^2) \quad \text{and } u \in \{0, 1\}$$

is quite rigid. If  $Q$  is a rectangle with vertices  $\{a, b, c, d\}$  (with  $a$  the top left vertex and using then clockwise enumeration), and if  $u$  satisfies (4.2) then

$$\mu[u](Q) = u(b) + u(d) - u(a) - u(c) \in \pm\{0, 1, 2\},$$

which forces  $\mu[u]$  to be atomic.

Let us now outline the proof in the case of a generic function  $u$ . For  $(x, z) \in \mathbb{T}^2 \times \mathbb{R}^2$ , we define  $Q_{x,z} := x + [0, z_1] \times [0, z_2]$  and observe that

$$\mu[u](Q_{x,z}) = D[Du(\cdot, z_2)](x, z_1).$$

Arguing as for (3.2), we may find a sequence  $z^k \rightarrow 0$  with  $|z_1^k| \sim |z_2^k|$  and such that

$$\overline{\lim}_{k \rightarrow \infty} \int_{\mathbb{T}^2} \frac{|\mu[u](Q_{x,z^k})|^\theta}{|z^k|^2} dx \lesssim \mathcal{E}(u).$$

We then prove that every measure  $\mu$  for which the left-hand side is finite must be atomic.  $\square$

Using this result, one can improve Theorem 2. First, if  $u$  is a characteristic function and  $\mathcal{E}(u)$  is small enough, then (4.1) implies that  $\mu[u] = 0$  (since in this case  $\sigma^k \in \pm\{1, 2\}$ ) from which we obtain that  $u \in \mathcal{S}(\mathbb{T}^n)$ . Second, for generic functions  $u$ , we can prove that the function  $v = u - \bar{u}$ , which appears in (3.5) has essentially the structure of the ‘corner’ function: its distributional derivative is purely concentrated on a jump part. That is  $\nabla v = \sigma \mathcal{H}^1 \llcorner J_v$ , with  $J_v$  a 1-rectifiable set and moreover

$$\int_{J_v} |\sigma|^\theta d\mathcal{H}^1 \lesssim \mathcal{E}(u) + \mathcal{E}^{\frac{1}{2}}(u).$$

In higher dimension, we can prove the following rectifiability result.

**THEOREM 5.** *Let  $\theta < 1$  and  $n \geq 2$ . If  $u$  is such that  $\|u\|_\infty \leq 1$  and  $\mathcal{E}(u) < \infty$ , then  $\mu$  is  $(n-2)$ -rectifiable i.e.*

$$\mu[u] = \sigma \nu_1 \otimes \nu_2 \mathcal{H}^{n-2} \llcorner \Sigma,$$

<sup>§</sup>here  $\mathcal{M}(\mathbb{T}^n)$  denotes the set of Radon measures on  $\mathbb{T}^n$ .

where  $\Sigma \subset \mathbb{T}^n$  is  $(n-2)$ -rectifiable,  $\sigma : \Sigma \mapsto \mathbb{R}$  is a multiplicity and  $(\mathbf{v}_1, \mathbf{v}_2)$  are normals to  $\Sigma$  with  $\mathbf{v}_l \in X_l$ . Moreover, if we let  $h(\sigma) := |\sigma| + |\sigma|^\theta$ ,

$$(4.3) \quad \mathbb{M}_h(\mu[u]) := \int_{\Sigma} h(\sigma) d\mathcal{H}^{n-2} \lesssim \mathcal{E}(u).$$

*Sketch of proof.* To prove this result we first identify the measure  $\mu[u]$  with an  $(n-2)$ -current (see [Fed69]). If we slice the corresponding current by 2-dimensional planes of the form  $V = \text{span}(\xi_1, \xi_2)$  with  $\xi_l \in X_l$ , we obtain 0-rectifiable currents applying Theorem 4 to the trace of  $u$  on the slice. We then conclude using White's rectifiability criterion [Whi99]. Estimate (4.3) is a consequence of (4.1) and the co-area formula [Fed69, Theorem 3.2.22].  $\square$

Let us point out that the quantity  $\mathbb{M}_h$  appearing in (4.3) is often called the  $h$ -mass. The fact that it is finite implies that  $\mu$  has to be quite concentrated since  $\theta < 1$ . This type of energies appears naturally in branched transportation models and has been the subject of intense recent work (see for instance [BCM09, CFM18, CRM19, BW18]).

Let us also observe that since  $\mu[u] = \nabla_1 \nabla_2 u$ , we have  $\nabla_1 \times \mu[u] = 0$  and  $\nabla_2 \times \mu[u] = 0$ . From [ARDPHR19] we could have thus directly obtained that  $\mu[u]$  is concentrated on a set with Hausdorff dimension at least  $n-2$  and that its  $(n-2)$ -part is rectifiable. Here we obtain a stronger result since we rule out more diffuse parts of the measure.

In the case when  $n_1 = 1$  or  $n_2 = 1$ , we can say more and prove that  $\Sigma$  is tensorized. For definiteness let us state the result in the case  $n_2 = 1$ .

**PROPOSITION 2.** *Assume that  $n_1 > 1$ ,  $n_2 = 1$  and  $\theta < 1$ . If  $u$  is such that  $\|u\|_{\infty} \leq 1$  and  $\mathcal{E}(u) < \infty$ , then we may write  $\mu[u] = \sum_k \mu^k$  where for every  $k$ ,*

$$\mu^k = \sigma^k \mathbf{v}^k \mathcal{H}^{n_1-1} \llcorner (\Sigma^k \times \{x_2^k\})$$

with  $\Sigma^k \subset \mathbb{T}^{n_1}$  an  $(n_1-1)$ -rectifiable set with normal  $\mathbf{v}^k$  and such that

$$\mathbb{M}_h(\mu[u]) = \sum_k \mathbb{M}_h(\mu^k).$$

The main ingredient in the proof of Proposition 2 is the extension to rectifiable currents with finite  $h$ -mass of the classical decomposition in indecomposable components of integral currents (see [Fed69, 4.2.25]).

**LEMMA 1.** *Let  $T$  be an  $m$ -rectifiable current in  $\mathbb{R}^n$  with<sup>¶</sup>  $\mathbb{M}_h(T) < \infty$  and  $\partial T = 0$ . Then,  $T$  may be decomposed into indecomposable components  $T^k$  in the sense that*

$$T = \sum_k T^k \quad \text{and} \quad \mathbb{M}_h(T) = \sum_k \mathbb{M}_h(T^k)$$

<sup>¶</sup>The definition of the  $h$ -mass for a rectifiable current is analogous to (4.3).

and the currents  $T^k$  cannot be further decomposed.

It is tempting to conjecture that a similar tensorized structure also holds for arbitrary dimension. It is however possible to construct an example in  $\mathbb{R}^4$  where it is impossible to write  $\mu[u]$  as  $\mu[u] = \sum_k \mu_1^k \otimes \mu_2^k$  with

$$\mu_l^k = \sigma_l^k \nu_l^k \mathcal{H}^{n_l-1} \llcorner \Sigma_l^k$$

where  $\Sigma_l^k \subset \mathbb{T}^{n_l}$  are  $(n_l - 1)$ -rectifiable sets and such that

$$\mathbb{M}_h(\mu[u]) = \sum_k \mathbb{M}_h(\mu_1^k) \mathbb{M}_h(\mu_2^k).$$

However, we can combine Proposition 2 with slicing to improve the conclusion of Theorem 5. Indeed, if we see  $\mu[u]$  as a  $(n_2 - 1)$ -flat chain in  $\mathbb{T}^{n_2}$  with values in the group of  $(n_1 - 1)$ -rectifiable currents in  $\mathbb{T}^{n_1}$ , Proposition 2 tells us that every slice with respect to  $(n_1 + 1)$ -spaces of the form  $V = \text{span}(X_1, \xi_2)$  where  $\xi_2 \in X_2$ , is rectifiable. Hence, another application of White's rectifiability criterion yields rectifiability of  $\mu[u]$  (as a flat chain with values in the space of currents). Exchanging the roles played by  $n_1$  and  $n_2$  we obtain our final result which improves Theorem 5.

**THEOREM 6.** *Let  $\theta < 1$  and  $n \geq 2$ . If  $u$  is such that  $\|u\|_\infty \leq 1$  and  $\mathcal{E}(u) < \infty$ , then  $\mu$  is  $(n - 2)$ -rectifiable with  $(n_1 - 1) \times (n_2 - 1)$  tensor structure i.e.*

$$(4.4) \quad \mu[u] = \sigma \nu_1 \otimes \nu_2 \mathcal{H}^{n-2} \llcorner (\Sigma_1 \times \Sigma_2),$$

where  $\Sigma_l \subset \mathbb{T}^{n_l}$  are  $(n_l - 1)$ -rectifiable,  $\sigma : \Sigma_1 \times \Sigma_2 \mapsto \mathbb{R}$  is a multiplicity and  $\nu_l$  is normal to  $\Sigma_l$ .

Notice that in (4.4), the multiplicity  $\sigma$  is a priori *not* tensorized (and is actually equal to zero at many points of  $\Sigma_1 \times \Sigma_2$ ).

## 4.2. The case $\theta = 1$

In the case  $\theta = 1$ , both the 'roof' and the 'corner' functions have finite energy. Therefore,  $\mu[u]$  can contain parts of different dimensions and the situation is more delicate to study. For this reason we were only able to obtain results in the case  $n = 2$ .

Our first result is that  $\mu[u]$  is concentrated on a set of Hausdorff dimension at most one. The proof has a similar flavor to the proof of Theorem 4 but is much more involved.

**THEOREM 7.** *Let  $n = 2$  and  $\theta = 1$ . If  $u$  is such that  $\mathcal{E}(u) < \infty$  then there exists a Borel set  $\Sigma \subset \mathbb{T}^2$  such that  $\mathcal{H}^1 \llcorner \Sigma$  is  $\sigma$ -finite and  $|\mu[u]|(\mathbb{T}^2 \setminus \Sigma) = 0$ .*

In order to study the case when  $\mu[u]$  is expected to concentrate on one-dimensional objects, it is natural to consider the class of Lipschitz functions of finite



energy. We can actually prove that for this problem the energy plays no role. Letting  $v = \nabla u$ , and recalling that  $K = X_1 \cup X_2$ , (2.1) and (3.4), we are thus interested in the differential inclusion

$$(4.5) \quad \|v\|_\infty \leq 1, \quad \partial_2 v_1 = \partial_1 v_2 \in \mathcal{M}(\mathbb{T}^2) \quad \text{and} \quad v \in K \text{ a.e. .}$$

Notice that (4.5) is reminiscent of the entropy solutions of the Eikonal equation studied by De Lellis and Otto in [DLO03] (where  $K$  is replaced by  $\mathbb{S}^1$ ). As in [DLO03], we can prove that if  $v$  satisfies (4.5), then it has a BV type structure.

**THEOREM 8.** *If  $v$  satisfies (4.5) then there exists a 1-rectifiable set  $\Sigma$  such that letting  $\nu$  be a normal to  $\Sigma$  and  $B_r^\pm(x) := \{y \in B_r(x) : \pm y \cdot \nu > 0\}$ , we have*

(i) *for  $\mathcal{H}^1$  a.e.  $x \in \Sigma$ ,  $v$  has traces  $v^\pm(x) \in K$  on  $\Sigma$  in the sense that*

$$\lim_{r \downarrow 0} \frac{1}{r^2} \int_{B_r^\pm(x)} |v(y) - v^\pm(x)| dy = 0;$$

(ii) *for  $\mathcal{H}^1$  a.e.  $x \in \Sigma$  and  $l = 1, 2$  we have the implication  $v_l^\pm(x) = 0 \implies v_l^\mp(x) \neq 0$ ;*

(iii) *the defect measure  $\mu := \partial_2 v_1 - \partial_1 v_2$  is concentrated on  $\Sigma$  and*

$$\mu = (v_1^+ - v_1^-) \nu_2 \mathcal{H}^1 \llcorner \Sigma = (v_2^+ - v_2^-) \nu_1 \mathcal{H}^1 \llcorner \Sigma;$$

(iv)  *$\mathcal{H}^1$  a.e.  $x \in \mathbb{T}^2 \setminus \Sigma$  is either a Lebesgue point of  $v_1$  with  $v_1(x) = 0$  or a Lebesgue point of  $v_2$  with  $v_2(x) = 0$ .*

Notice that in comparison with [DLO03], we obtain a stronger result since we prove here that  $\mu$  is exactly concentrated on a one-dimensional set while for the Eikonal equation it is still an open problem to exclude concentration of the energy on more diffused sets.

*Sketch of proof.* As in [DLO03], the main point is to prove the rectifiability of  $\mu$ , that is (iii). While one could argue along the lines of [DLO03], our proof relies on the study of the level sets of the function  $u$  such that  $v = \nabla u$ .

For  $t \in \mathbb{R}$ , we define  $E_t := \{u < t\}$  and  $\Gamma_t := \{u = t\}$  so that for a.e.  $t$ ,  $\Gamma_t = \partial E_t$ . Using the co-area formula, we can decompose the measure  $\mu$  on the level sets of  $u$  and prove that for every  $\varphi \in C^\infty(\mathbb{T}^2)$  and every  $\psi \in C^0(\mathbb{R})$ ,

$$\int_{\mathbb{T}^2} \varphi(x) \psi(u(x)) d\mu(x) = - \int_{\mathbb{R}} \psi(t) \left[ \int_{\Gamma_t} \partial_1 \varphi \nu_2 d\mathcal{H}^1 \right] dt.$$

Therefore, for a.e.  $t$ , the distribution

$$\langle \kappa_t, \varphi \rangle := \int_{\Gamma_t} \partial_1 \varphi \nu_2 d\mathcal{H}^1$$

is a Radon measure with

$$\int_{\mathbb{R}} |\kappa_t|(\mathbb{T}^2) dt = |\mu|(\mathbb{T}^2).$$

Notice that since  $v_1 v_2 = 0$  a.e.,  $\kappa_t$  coincides with the mean curvature of  $\partial E_t$  in the sense of varifolds (see [Sim83]). By the divergence theorem,

$$\langle \kappa_t, \varphi \rangle = \langle \partial_1 \partial_2 1_{E_t}, \varphi \rangle$$

and thus if  $\kappa_t$  is a Radon measure, then  $1_{E_t}$  satisfies (4.2). By a variant of Theorem 4,  $E_t$  is a finite union of polygons with sides parallel to the coordinate axes. The measure  $\kappa_t$  is then a sum of Dirac masses located at the corners of  $E_t$ .

Now, fix a point  $\bar{x}$  around which  $\mu$  is concentrated. Generically, it is a corner of  $\Gamma_{\bar{t}}$  for some  $\bar{t} \in \mathbb{R}$ , which is also a Lebesgue point of  $t \mapsto |\kappa_t|(\Gamma_t)$  (which counts the number of vertices of  $\Gamma_t$ ). For  $t$  close to  $\bar{t}$  most  $\Gamma_t$  have the same number of vertices as  $\Gamma_{\bar{t}}$ . In particular, for such  $t$  there is exactly one corner  $x_t$  of  $\Gamma_t$  which is close to  $\bar{x}$  and we can locally parameterize the set  $\Sigma$  by  $x_t$ . The proof is then concluded by proving that  $t \mapsto x_t$  has a derivative in  $\bar{t}$ . In turn, this follows from the fact that the distance between  $\Gamma_t$  and  $\Gamma_{\bar{t}}$  is essentially given by the difference of the areas of  $E_t$  and  $E_{\bar{t}}$  which is itself closely related to  $\nabla u$ . In particular, because of (4.5), the normal to  $\Sigma$  is determined by  $v^\pm$  through the condition

$$(v_1^+ - v_1^-)v_2 = (v_2^+ - v_2^-)v_1.$$

□

### Acknowledgments

M. Goldman is partially supported by the ANR SHAPO. B. Merlet is partially supported by the INRIA team RAPSODI and the Labex CEMPI (ANR-11-LABX-0007-01).

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**AMS Subject Classification: 26B99, 35J50, 35J60**

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*Lavoro pervenuto in redazione il 08.10.2019.*