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COURANT-SHARP ROBIN EIGENVALUES FOR THE SQUARE: A REVIEW

Abstract. This is a review article based on a talk delivered by the author during the workshop "Analysis and Applications: Contributions from young researchers" at Politecnico di Torino on 8 April 2019. The subject of this talk, based on joint work with Bernard Helffer [18, 19], was the Courant-sharp eigenvalues of the Robin realisation of the Laplacian on the square.

1. Introduction

Let $\Omega \subset \mathbb{R}^m$, $m \geq 2$, be a bounded, connected, open set with Lipschitz boundary. For $h \in \mathbb{R}$, $h \geq 0$, let $-\Delta_h$ be the h-Robin realisation of the Laplacian acting on $L^2(\Omega)$. We consider the solutions $\lambda_h(\Omega) \in \mathbb{R}$, $u \in H^1(\Omega)$ of the problem:

$$(1) \qquad \begin{cases} -\Delta_h u(x) = \lambda_h(\Omega) u(x) & x \in \Omega, \\ \frac{\partial u(x)}{\partial \vec{n}} + h u(x) = 0 & x \in \partial\Omega, \end{cases}$$

where $\vec{n}$ is the outward-pointing unit normal vector along $\partial\Omega$. Under the aforementioned assumptions on Ω, $-\Delta_h$ has discrete spectrum. Moreover, the Robin eigenvalues can be written in a non-decreasing sequence:

$$\lambda_{1,h}(\Omega) \leq \lambda_{2,h}(\Omega) \leq \cdots \leq \lambda_{k,h}(\Omega) \leq \ldots,$$

counted with multiplicities, and there exists a corresponding basis of eigenfunctions $u_k \in H^1(\Omega)$ that are orthogonal in $L^2(\Omega)$.

When $h = 0$, we observe that (1) corresponds to the Neumann eigenvalue problem. On the other hand, in the limit $h \to +\infty$, (1) corresponds to the Dirichlet eigenvalue problem (formally, we search for a corresponding eigenfunction in $H_0^1(\Omega)$). We denote the Neumann, respectively Dirichlet, eigenvalues by $\mu_k(\Omega)$, respectively $\lambda_k(\Omega)$. We recall that if $\Omega \subset \mathbb{R}^m$ is a bounded, connected, open set with Lipschitz boundary, then the spectrum of the Neumann Laplacian is discrete. For discreteness of the Dirichlet Laplacian on Ω, we require only that Ω has finite Lebesgue measure $|\Omega| < +\infty$.

We recall that the Robin eigenvalues satisfy the min-max characterisation:

$$\lambda_{k,h}(\Omega) = \inf_{K \subset H^1(\Omega), \dim(K)=k} \sup_{0 \neq u \in K} \frac{\int_\Omega |\nabla u|^2 \, dx + \int_{\partial\Omega} h u^2 \, dS}{\int_\Omega u^2 \, dx},$$

from which we see that for $h \geq 0$, the Robin eigenvalues interpolate between the Neumann and Dirichlet eigenvalues:

$$\mu_k(\Omega) \leq \lambda_{k,h}(\Omega) \leq \lambda_k(\Omega).$$

The lower bound (Neumann) follows by taking $h = 0$ and the upper bound (Dirichlet) follows by restricting the function space to $H_0^1(\Omega)$. We refer to [11] and references therein for further details about the Robin problem including the asymptotic behaviour of and estimates for the Robin eigenvalues. We also refer to [28] and references therein for conjectures involving the first and second Robin eigenvalues and for corresponding results for rectangular boxes.

Let u be an eigenfunction of the Laplacian on Ω with either a Dirichlet, Neumann or Robin boundary condition.

- The nodal set of u is the set $\overline{\{x \in \Omega : u(x) = 0\}} \subset \overline{\Omega}$.
- The nodal domains of u are the components of $\Omega \setminus \overline{\{x \in \Omega : u(x) = 0\}}$.

The nodal set of u partitions Ω into a union of nodal domains which we refer to as a nodal partition of Ω. The nodal sets and the nodal domains of Laplacian eigenfunctions have received a great deal of attention throughout the last 2 centuries in Mathematics and Physics alike (see, for example, [22, 27]). We also refer to [27] for a review of related problems regarding quantum billiards.

The celebrated Courant Nodal Domain Theorem asserts the following.

THEOREM 1 (Courant's Nodal Domain Theorem). *Any function in the eigenspace associated with the k–th eigenvalue has at most k nodal domains.*

A proof of Courant's Nodal Domain Theorem was first given in [14] (see also VI §6 of [15]). Moreover, in Appendix D of [5], Courant's Nodal Domain Theorem is proved for the Dirichlet Laplacian on smooth, compact Riemannian manifolds of dimension $m \geq 2$.

The objects of interest of this review article are the Courant-sharp eigenvalues which are defined below.

DEFINITION 1. *If an eigenfunction u associated with the k–th eigenvalue has exactly k nodal domains, then we call it a Courant-sharp eigenfunction. In this case, we call the corresponding eigenvalue a Courant-sharp eigenvalue.*

In particular, the k–th eigenvalue is Courant-sharp if it has a corresponding eigenfunction with k nodal domains.

We remark that by Courant's Nodal Domain Theorem and the orthogonality of any second eigenfunction u_2 with the non-vanishing first eigenfunction u_1, the eigenfunctions u_1 and u_2 are Courant-sharp.

In the Courant-sharp case, a key motivation for the study of the partition of Ω into nodal domains comes from its links with spectral minimal partitions of Ω (see [9] and references therein).

In Section 2, we discuss Pleijel's Theorem, which shows that there are finitely many Courant-sharp Dirichlet eigenvalues, and associated questions. We then focus our attention on the particular case where $\Omega \subset \mathbb{R}^2$ is a square in Section 3.

Acknowledgements

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2. Pleijel’s Theorem and related questions

Let $\Omega \subset \mathbb{R}^m$ be an open, connected set of finite measure. First of all, we consider the Courant-sharp eigenvalues of the Dirichlet Laplacian on Ω. Let u_k be an eigenfunction associated with the eigenvalue $\lambda_k(\Omega)$ and let $\nu(u_k)$ denote the number of nodal domains of u_k.

THEOREM 2 (Pleijel’s Theorem). *Let Ω be an open, connected set in $\mathbb{R}^m$, $m \geq 2$, with finite Lebesgue measure. Then*

$$\limsup_{k\to\infty} \frac{\nu(u_k)}{k} \leq \gamma_m, \tag{2}$$

where

$$\gamma_m = \frac{(2\pi)^m}{\omega_m^2}\big(\lambda_1(\mathcal{B}_m)\big)^{-m/2} < 1,$$

$\mathcal{B}_m$ is a ball of radius 1 in $\mathbb{R}^m$ and ω_m is the volume of $\mathcal{B}_m$.

Pleijel’s Theorem gives that there are finitely many Courant-sharp eigenvalues of the Dirichlet Laplacian on Ω.

The original statement and proof of this theorem is due to Pleijel [32] for bounded, open sets in $\mathbb{R}^2$. An analogue of Pleijel’s Theorem in a Riemannian manifold setting was given by Peetre [31], and was later generalised by Bérard and Meyer [5].

If u_k is Courant-sharp, then $\nu(u_k) = k$ and, since each eigenfunction u_k satisfies the Dirichlet problem ((1) with $h \to +\infty$) on each nodal domain Ω_i, $i = 1,2,\ldots,k$, and does not vanish in Ω_i, $\lambda_k(\Omega) = \lambda_1(\Omega_i)$. A key step in the proof of Pleijel’s Theorem is to apply the Faber–Krahn inequality

$$\lambda_1(\Omega_i) \geq \lambda_1(B),$$

where $B \subset \mathbb{R}^m$ is a ball with $|B| = |\Omega_i|$, for each nodal domain. This amounts to replacing each nodal domain by a ball of the same volume.

It was remarked by Polterovich [34] that, as the nodal domains of u_k cannot all be balls, the inequality in (2) is not sharp. In recent years, several improvements of this inequality have been obtained by invoking refinements of the Faber–Krahn inequality, see [10, 17, 35]. We define the Pleijel constant of Ω as

$$\mathcal{P}(\Omega) := \limsup_{k\to+\infty} \frac{\nu(u_k)}{k}.$$

The Pleijel constant has been computed in [7] for certain domains, including the disc for which the value obtained is significantly smaller than γ_2.

For the 2-dimensional case, Polterovich [34] conjectures that the optimal Pleijel constant (that is, the minimal constant among all regular, bounded, planar domains) is $\frac{2}{\pi}$. The latter value is the Pleijel constant for a rectangle in $\mathbb{R}^2$ where the square of the ratio of the side-lengths is irrational. The determination of the optimal Pleijel constant is an interesting open problem.

We now turn our attention to the Courant-sharp eigenvalues of the Neumann Laplacian on Ω. For the Neumann problem, Pleijel [32] showed that the square in $\mathbb{R}^2$ has finitely many Courant-sharp Neumann eigenvalues. We note that in contrast to the Dirichlet problem, one cannot apply the Faber–Krahn inequality to boundary nodal domains, that is nodal domains whose boundaries intersect the boundary of $\partial\Omega$ in a non-trivial segment. But, as there are explicit formulae for the eigenfunctions of the Neumann Laplacian on the square in terms of trigonometric functions, one can obtain an upper bound for the number of such nodal domains of a given eigenfunction (see [32]).

In [34], Polterovich proved that a bounded, connected domain in $\mathbb{R}^2$ with piecewise analytic boundary has finitely many Courant-sharp Neumann eigenvalues. In particular, that the analogue of (2) holds in this case. The proof uses an estimate due to Toth–Zelditch [37] for the number of boundary zeros of a Neumann eigenfunction on a planar domain with piecewise analytic boundary.

Recently, Léna [29] proved that a bounded, connected, open set in $\mathbb{R}^m$, $m \geq 2$, with $C^{1,1}$ boundary has finitely many Courant-sharp Neumann and Robin eigenvalues. More precisely, that the analogue of (2) holds in this case. The idea of the proof is to locally straighten the boundary and reflect the boundary nodal domain in the straightened boundary of Ω to obtain a domain that satisfies a Dirichlet boundary condition almost everywhere. One can then apply the Pleijel-type argument mentioned above. In [29], the restriction to sets with $C^{1,1}$ boundary is for technical reasons. We expect that the analogue of (2) holds for bounded, connected, open sets with Lipschitz boundary in the Neumann and Robin cases.

With the result of Pleijel's Theorem in hand, it is natural to ask how many Courant-sharp eigenvalues are there and how big are they?

We first discuss the latter question. The following theorem is due to Bérard and Helffer [2].

THEOREM 3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, connected domain. Then, there exists a constant $\beta(\Omega) > 0$ depending only on the geometry of Ω, such that any Courant-sharp Dirichlet eigenvalue $\lambda_k(\Omega)$ satisfies*

$$k\frac{\lambda_1(\mathbb{D}_1)}{|\Omega|} \leq \lambda_k(\Omega) \leq \beta(\Omega),$$

where $\mathbb{D}_1$ is the disc of unit area.

If Ω is regular, the constant $\beta(\Omega)$ can be computed in terms of the area and the perimeter of Ω, bounds on the curvature of $\partial\Omega$ and on the cut-distance to $\partial\Omega$.

In [21], we obtain corresponding upper bounds for the Courant-sharp Neumann and Robin eigenvalues of the Laplacian on an open, bounded, connected set in $\mathbb{R}^m$, $m \geq 2$, with C^2 boundary. Under the additional assumption that the set is convex, the bounds that we obtain depend upon the volume of the set, the isoperimetric ratio of the set and the cut-distance to the boundary of the set.

We recall that Pleijel's Theorem only requires the set Ω to be open and connected with finite measure. As in [6], for $\varepsilon \geq 0$ and $|\Omega| < \infty$, we define

$$\mu_\Omega(\varepsilon) = |\{x \in \Omega : d(x, \partial\Omega) < \varepsilon\}|,$$

and

$$\varepsilon(\Omega) = \inf\{\varepsilon : \mu_\Omega(\varepsilon) \geq 2^{-1}(1-\gamma_m)|\Omega|\}.$$

We have the following theorem, [6].

THEOREM 4. *Let Ω be an open, connected set in $\mathbb{R}^m$ with finite Lebesgue measure. If $\lambda_k(\Omega)$ is a Courant-sharp Dirichlet eigenvalue, then*

(i)

$$\lambda_k(\Omega) \leq \left(\frac{2\pi m^2}{(1-\gamma_m)\varepsilon(\Omega)}\right)^2.$$

(ii)

$$k \leq \frac{\omega_m}{(1-\gamma_m)^m}\left(m^3(m+2)\right)^{m/2}\frac{|\Omega|}{\varepsilon(\Omega)^m}.$$

Moreover, if $k \in \mathbb{N}$, $k > \frac{\omega_m}{(1-\gamma_m)^m}\left(m^3(m+2)\right)^{m/2}\frac{|\Omega|}{\varepsilon(\Omega)^m}$, then $\lambda_k(\Omega)$ is not Courant-sharp.

If we suppose that Ω is a long, thin rectangle in $\mathbb{R}^2$ such that $\varepsilon(\Omega)$ is very small, then this gives rise to a large upper bound in Theorem 4(ii). On the other hand, $(0,\varepsilon) \times (0,L)$ for L sufficiently large has a large number of Courant-sharp Dirichlet eigenvalues (see [9] for example). In some sense, this bound is the best that we can obtain in such generality.

In Example 1 of [6], Theorem 4(ii) is used to obtain an upper bound for the number of Courant-sharp Dirichlet eigenvalues of an open, bounded, convex set in $\mathbb{R}^m$. By using this example, we get that the number of Courant-sharp Dirichlet eigenvalues of a disc, respectively a square, in $\mathbb{R}^2$ is bounded from above by 558325, respectively 710881. These upper bounds are very crude. Indeed, it was shown in [23], respectively [1], that the number of Courant-sharp Dirichlet eigenvalues of the disc, respectively the square, in $\mathbb{R}^2$ is 3. In particular, the Courant-sharp Dirichlet eigenvalues of the disc are λ_1, λ_2 and λ_4.

In [3], it was shown that the Courant-sharp Dirichlet eigenvalues of the right-angled isosceles triangle and of the hemi-equilateral triangle are λ_1 and λ_2. So λ_4 is not necessarily Courant-sharp even for planar domains.

In addition, for the Neumann problem, it was shown in [26] that the Courant-sharp Neumann eigenvalues of the disc in $\mathbb{R}^2$ are μ_1, μ_2, μ_4. There, it was also shown that only the first and second Dirichlet, respectively Neumann, eigenvalues of the ball in $\mathbb{R}^m$, $m \geq 3$ are Courant-sharp.

We refer to [9] and references therein for examples of other domains whose Courant-sharp eigenvalues have been identified, as well as other problems related to Courant-sharp eigenvalues.

We now restrict our attention to the case where $\Omega \subset \mathbb{R}^2$ is a square. In the pioneering paper [32], Pleijel asserts the following theorem.

THEOREM 5. *The Courant-sharp Dirichlet eigenvalues of the square in* $\mathbb{R}^2$ *are* λ_1, λ_2 *and* λ_4.

A complete proof of this theorem was given in [1]. For the Neumann case the following result was obtained in [25].

THEOREM 6. *The Courant-sharp Neumann eigenvalues of the square in* $\mathbb{R}^2$ *are* μ_1, μ_2, μ_4, μ_5 *and* μ_9.

We recall from Section 1 that the Robin eigenvalues interpolate between the Neumann and the Dirichlet eigenvalues. The results of Theorem 5 and Theorem 6 beg the question: is it possible to follow the Courant-sharp eigenvalues from $h = 0$ to $h = +\infty$? This question is the focus of the following section, and was the catalyst for our work [18, 19].

3. Courant-sharp Robin eigenvalues for the square

3.1. Recap of Dirichlet and Neumann eigenfunctions for the square

We first recall the formulae for the Dirichlet, respectively Neumann, eigenvalues and eigenfunctions of the Laplacian on $S := (-\frac{\pi}{2}, \frac{\pi}{2})^2$ (see, for example, [22]).

The Dirichlet eigenvalues of S are

$$\lambda_k(S) = \lambda_{m,n}(S) = m^2 + n^2,$$

where $m, n \in \mathbb{N}^*$ (the set of positive integers), and a corresponding orthogonal basis of eigenfunctions is given by

$$u_{m,n}(x,y) = \sin\left(\frac{m\pi x}{\pi} + \frac{m\pi}{2}\right)\sin\left(\frac{n\pi y}{\pi} + \frac{n\pi}{2}\right).$$

The Neumann eigenvalues of S are

$$\mu_k(S) = \mu_{m,n}(S) = m^2 + n^2,$$

where $m, n \in \mathbb{N}$, and a corresponding orthogonal basis of eigenfunctions is given by

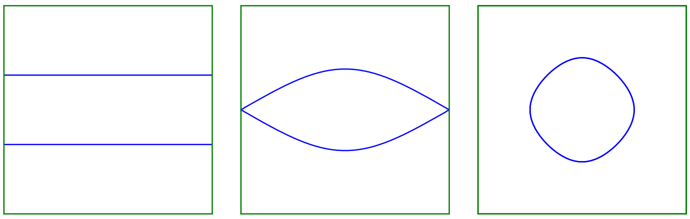
$$u_{m,n}(x,y) = \cos\left(\frac{m\pi x}{\pi} + \frac{m\pi}{2}\right)\cos\left(\frac{n\pi y}{\pi} + \frac{n\pi}{2}\right).$$

We see immediately that not all the Dirichlet eigenvalues are simple. If $\lambda_{k-1} = \lambda_k$, then λ_k is not Courant-sharp, otherwise it would give a contradiction to Courant's Nodal Domain Theorem. This observation also holds in the Neumann and Robin cases.

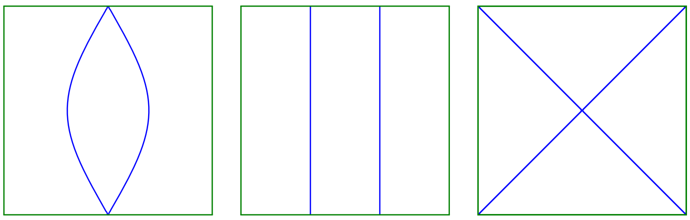
In addition, if λ_k has multiplicity at least 2, then the corresponding eigenspace consists of linear combinations of $u_{m,n}(x,y)$. In order to determine whether λ_k is Courant-sharp or not, one must either find a linear combination with k nodal domains or show that no such linear combination exists. In addition, any two functions belonging to this eigenspace need not have the same number of nodal domains. For example, the fifth Dirichlet eigenvalue $\lambda_5(S)$ corresponds to the pair $(1,3)$, so we consider

$$\cos\theta\, u_{1,3}(x,y) + \sin\theta\, u_{3,1}(x,y),$$

where $\theta \in [0,\pi)$. From Figure 1, we see that the number of nodal domains of the fifth Dirichlet eigenfunction changes from 3 to 2 to 3 to 4 and back to 3 (see [1]).



(a) From left to right: $\theta = 0$, $\theta = \arctan(1/3)$, $\theta = \frac{\pi}{4}$.



(b) From left to right: $\theta = \frac{\pi}{2} - \arctan(1/3)$, $\theta = \frac{\pi}{2}$, $\theta = \frac{3\pi}{4}$.

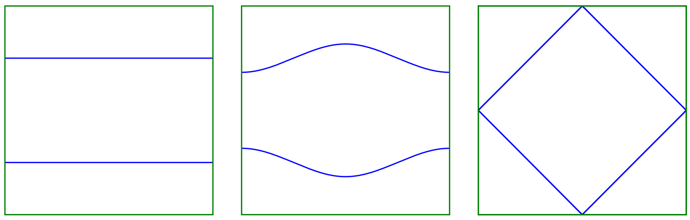
Figure 1: The nodal set of the Dirichlet eigenfunction corresponding to $\lambda_5(S)$ for various values of θ.

Similarly, the fifth Neumann eigenvalue $\mu_5(S)$ corresponds to the pair $(0,2)$, so we consider

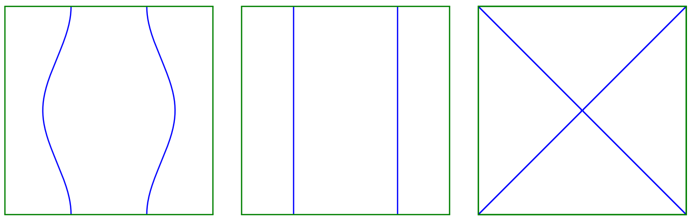
$$\cos\theta\, u_{0,2}(x,y) + \sin\theta\, u_{2,0}(x,y),$$

where $\theta \in [0,\pi)$. From Figure 2, we have that the number of nodal domains of the fifth Neumann eigenfunction changes from 3 to 5 to 3 to 4 and back to 3 (see [25]).

We keep the Dirichlet and Neumann problems in mind in what follows. Our goal is to see the extent to which our understanding of these cases carries over to the Robin problem in the particular context of the Courant-sharp eigenvalues for the square.



(a) From left to right: $\theta = 0$, $\theta = \frac{\pi}{8}$, $\theta = \frac{\pi}{4}$.



(b) From left to right: $\theta = \frac{3\pi}{8}$, $\theta = \frac{\pi}{2}$, $\theta = \frac{3\pi}{4}$.

Figure 2: The nodal set of the Neumann eigenfunction corresponding to $\mu_5(S)$ for various values of θ.

3.2. Eigenvalues and eigenfunctions of the Robin Laplacian on the square

We recall from [18] that an orthogonal basis of eigenfunctions of the Robin realisation of the Laplacian on $S := (-\frac{\pi}{2}, \frac{\pi}{2})^2$ can be given in the form

$$u_{m,n}(x,y) = u_m(x)u_n(y),$$

$m, n \in \mathbb{N}$, where for $p \in \mathbb{N}$ even,

$$u_p(x) = \frac{1}{\sin \frac{\alpha_p(h)}{2}} \cos\left(\frac{\alpha_p(h)x}{\pi}\right),$$

and for $p \in \mathbb{N}^*$ odd,

$$u_p(x) = \frac{1}{\cos \frac{\alpha_p(h)}{2}} \sin\left(\frac{\alpha_p(h)x}{\pi}\right),$$

and $\alpha_p := \alpha_p(h)$ is the unique solution in $[p\pi, (p+1)\pi)$ of

$$\frac{2\alpha_p}{h\pi}\cos\alpha_p + \left(1 - \frac{(\alpha_p)^2}{h^2\pi^2}\right)\sin\alpha_p = 0\,.$$

We note that $\alpha_p(0) = p\pi$, respectively $\lim_{h\to+\infty}\alpha_p(h) = (p+1)\pi$. In this way, we recover the Neumann, respectively Dirichlet, eigenfunctions of S (see [18]).

The Robin eigenvalues of S are thus equal to:

$$\lambda_{k,h}(S) = \lambda_{m,n,h}(S) = \pi^{-2}(\alpha_m(h)^2 + \alpha_n(h)^2).$$

Each pair (m,n) gives rise to a function of h, $\lambda_{m,n,h}(S)$, which we refer to as an eigencurve.

As above, for the Robin eigenvalues of multiplicity 2, we consider

$$\Phi_{m,n,h,\theta}(x,y) = \cos\theta\, u_{m,n}(x,y) + \sin\theta\, u_{m,n}(x,y)$$

for $\theta \in [0,\pi)$.

3.3. Cases $k = 1,2,3,4,9$

We recall from Section 1 that $\lambda_{1,h}(S)$ and $\lambda_{2,h}(S)$ are Courant-sharp for $h \geq 0$. Since $\lambda_{3,h}(S) = \lambda_{2,h}(S)$, $\lambda_{3,h}(S)$ is not Courant-sharp for any $h \geq 0$.

In addition, $\lambda_{4,h}(S)$ corresponds to the pair $(m,n) = (1,1)$. As

$$u_{1,1}(x,y) = \frac{1}{\cos^2 \frac{\alpha_1}{2}} \sin\left(\frac{\alpha_1 x}{\pi}\right) \sin\left(\frac{\alpha_1 y}{\pi}\right)$$

has nodal lines $x = 0$ and $y = 0$, $\lambda_{4,h}(S)$ is Courant-sharp for all $h \geq 0$, [18].

We now turn our attention to one of the cases which motivated our investigation [18], namely $\lambda_{9,h}(S)$. Following the discussion of Section 2, we are interested in the following questions:

(i) Does there exist $\underline{h}_9^*$ such that $\lambda_{9,h}(S)$ is Courant-sharp for $h < \underline{h}_9^*$?

(ii) Does there exist $\overline{h}_9^*$ such that $\lambda_{9,h}(S)$ is not Courant-sharp for $h > \overline{h}_9^*$?

(iii) Is $\underline{h}_9^* = \overline{h}_9^*$?

In [18], we proved the following proposition.

PROPOSITION 1. *There exists $h_9^* > 0$ such that $\lambda_{9,h}$ is Courant-sharp for $0 \leq h \leq h_9^*$ and is not Courant-sharp for $h > h_9^*$.*

We refer to [18] for the proof. We note that the ninth Neumann eigenvalue of S, $\mu_9(S)$, corresponds to the pair $(m,n) = (2,2)$ and $2\pi^{-2}\alpha_2(+\infty)^2 = 18$. On the other hand, the ninth Dirichlet eigenvalue of S, $\lambda_9(S) = 17$, corresponds to the pair $(m,n) = (1,4)$. So, as the Robin eigenvalues interpolate between the Neumann and Dirichlet eigenvalues, the eigencurves $\lambda_{2,2,h}(S)$ and $\lambda_{0,3,h}(S)$ must cross (see Figure 3). In [18], we proved that any two eigencurves $\lambda_{m,n,h}(S)$ and $\lambda_{m',n',h}(S)$ with $m < m' \leq n' < n$ have at most one point of intersection. Numerically we obtained that $h_9^* \approx 1.6967$.

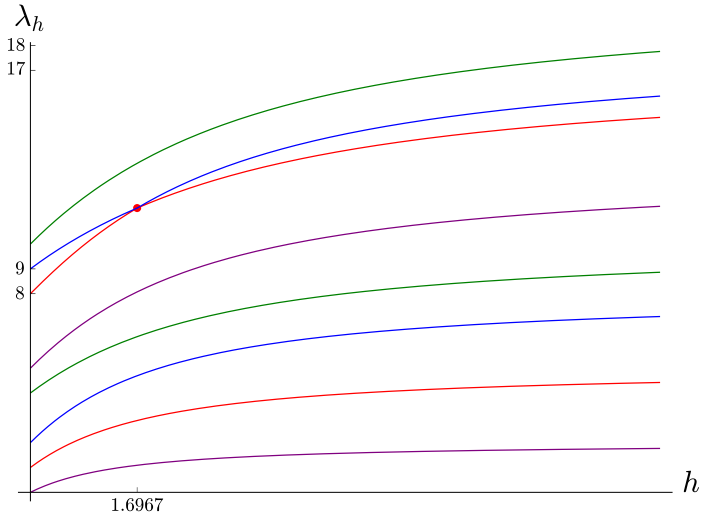


Figure 3: The first 13 Robin eigenvalues (as functions of h) for $h \leq 10$ corresponding to the pairs $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$, $(0,2)$, $(2,0)$, $(1,2)$, $(2,1)$, $(2,2)$, $(0,3)$, $(3,0)$, $(1,3)$, $(3,1)$.

3.4. Reducing the number of remaining cases

In [32], the first step of the proof of Theorem 5 is to show that λ_k with $k \geq 10$ is not Courant-sharp by invoking the Faber–Krahn inequality and estimates for the Dirichlet counting function for a square. For the cases $5 \leq k \leq 9$, Pleijel refers to [33] (see also [15]). The case $k = 3$ is eliminated by multiplicities.

A complete proof is given in [1]. The cases $k = 3, 6, 8$ are ruled out due to multiplicities. The fact that the cases $k = 7, 9$ are not Courant-sharp can be seen by considering the symmetry properties of the corresponding eigenfunctions. For example, any Dirichlet eigenfunction on $(-\frac{\pi}{2}, \frac{\pi}{2})^2$ corresponding to λ_7 has the form

$$u_7(x,y) = \cos\theta \sin(2x)\cos(3y) + \sin\theta \cos(3x)\sin(2y)$$

where $\theta \in [0,\pi)$. We see that $u_7(-x,-y) = -u_7(x,y)$ so any eigenfunction corresponding to λ_7 has an even number of nodal domains, and hence λ_7 is not Courant-sharp.

The case $k = 5$ is dealt with by analysing the nodal set of an arbitrary function Φ in the eigenspace corresponding to λ_5. This analysis rests upon a result due to Leydold [30] that the number of nodal domains can only change if there are interior critical points of Φ or if the number of boundary points of Φ changes.

The strategy described above was successfully extended to treat the Courant-sharp eigenvalues of the Neumann Laplacian on the square in [25] with some modifications to deal with the Neumann boundary condition:

(I) The Pleijel-type argument ([32, 1]) can be applied to the nodal domains whose boundaries intersect the boundary of the square in at most finitely many points. It is possible to estimate the number of remaining nodal domains by using the explicit formulae for the eigenfunctions (see [32, 25]) or Sturm's Theorem (see [36, 4]). It was then shown in [25] that if $k \geq 209$ then μ_k is not Courant-sharp.

(II) For $k \leq 208$, many cases were eliminated by considering the symmetry properties of the corresponding Neumann eigenfunctions (see [25, 9]).

(III) For the outstanding cases, the nodal partitions of the eigenfunctions were analysed keeping in mind the result of Leydold mentioned above.

In [18], we employ analogous arguments to those of [25] to obtain a corresponding result to item (I) for the Courant-sharp Robin eigenvalues of the square.

THEOREM 7. *Let $h > 0$. If $\lambda_{k,h}(S)$ is an eigenvalue of the Robin Laplacian on S with parameter $h > 0$ and $k \geq 520$, then it is not Courant-sharp.*

We see that there are a lot of remaining cases to check! As the Robin eigenvalues are continuous with respect to h, in [18, 19] we investigate whether the results described above for the Dirichlet and Neumann cases hold in the large h (close to Dirichlet) and small h (close to Neumann) regimes.

3.5. The case h large

In this subsection, we show what can be gained by restricting to the large h regime. To that end, we first recall the Pleijel-type argument (see, for example, [32, 1, 25, 18, 19]).

Pleijel-type argument

Let $\Omega \subset \mathbb{R}^2$ be a bounded, open set with Lipschitz boundary. Let $(u_k, \lambda_{k,h})$ be a Courant-sharp eigenpair of the Robin realisation of the Laplacian on Ω. Let Ω_j be a nodal domain of u_k such that $|\Omega_j| \leq |\Omega|/k$.

Firstly, suppose that $\partial\Omega_j \cap \partial S$ consists of finitely many points. Then, by the Faber–Krahn inequality, we have

$$\lambda_{k,h} = \lambda_1(\Omega_j) \geq \frac{\lambda_1(\mathbb{D}_1)}{|\Omega_j|},$$

where $\mathbb{D}_1$ is a disc of unit area. So if there exists such an Ω_j, then

$$\frac{|\Omega|}{k\lambda_1(\mathbb{D}_1)} \geq \frac{1}{\lambda_{k,h}}. \tag{3}$$

By [32], we have the following bound on the Dirichlet counting function for the square:

$$N^D(\lambda_{k,h}) = \#\{j \in \mathbb{N} : \lambda_j < \lambda_{k,h}\} > \frac{\pi}{4}\lambda_{k,h} - 2\sqrt{\lambda_{k,h}} + 2.$$

Since $\lambda_{k,h} \leq \lambda_k$, $N^{R,h}(\lambda_{k,h}) = \#\{j \in \mathbb{N} : \lambda_{j,h} < \lambda_{k,h}\} \geq N^D(\lambda_{k,h})$ and, as $\lambda_{k,h}$ is Courant-sharp, we obtain

$$k > \frac{\pi}{4}\lambda_{k,h} - 2\sqrt{\lambda_{k,h}} + 2,$$

which, together with (3), gives $\lambda_{k,h} \leq 50$ as in the Dirichlet case [32].

Otherwise, Ω_j is a nodal domain whose boundary intersects ∂S in at least a non-trivial segment. By the min-max characterisation, we have that $\lambda_{k,h} \geq \lambda_{1,h}(\Omega_j)$.

The first Robin eigenvalue satisfies the following Robin Faber–Krahn inequality [13] (see also [12, 16, 8]).

THEOREM 8. *Let $\Omega \subset \mathbb{R}^2$ be an open set with finite Lebesgue measure, and rectifiable boundary $\partial\Omega$ with finite* 1*-dimensional Hausdorff measure. Then*

$$\lambda_{1,h}(\Omega) \geq \lambda_{1,h}(D),$$

where $D \subset \mathbb{R}^2$ is a disc with $|D| = |\Omega|$.

Re-scaling D to obtain $\mathbb{D}_1$ gives

$$\lambda_{1,h}(D) = \lambda_{1,h|\Omega|^{1/2}}(\mathbb{D}_1)|\Omega|^{-1}.$$

Since $\lambda_{k,h}$ is Courant-sharp, $k \leq 520$ and we thus show in [18] that there exists a constant $c > 0$ such that $|\Omega_j| \geq c$. So for h large enough, $h|\Omega|^{1/2}$ is large enough that, by invoking the Robin Faber–Krahn inequality instead of the Faber–Krahn inequality, we obtain $\lambda_{k,h} \leq 50$ as above.

By continuity of the Robin eigenvalues with respect to h, for h large enough, this leaves the remaining candidates $k = 1, 2, 4, 5, 7, 9$ as in the Dirichlet case.

We already discussed the cases $k = 1, 2, 4, 9$ in Subsection 3.3. The case $k = 7$ is not Courant-sharp and can be excluded by symmetry arguments analogous to those described above for the Dirichlet case. The fact that $\lambda_{5,h}$ is not Courant-sharp can be shown by analysing the possible nodal partitions of the functions in the corresponding eigenspace.

In [18] we prove the following theorem.

THEOREM 9. *There exists $h_1 > 0$ such that for $h \geq h_1$, the Courant-sharp cases for the Robin problem are the same as those for $h = +\infty$.*

In fact, starting from the Dirichlet case with $h = +\infty$, we show in [18] that an eigenvalue which is not Courant-sharp for $h = +\infty$ is not Courant-sharp under a small perturbation of h.

The proof of this fact rests upon the result of Leydold [30] mentioned above, and the following proposition applied in a neighbourhood of the interior critical points, respectively boundary points, of the eigenfunction under consideration.

PROPOSITION 2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, connected set with piecewise $C^{2,\alpha}$ boundary $(\alpha > 0)$. Let $h_0 > 0$ and $M > 0$. For $h \in I \subset [h_0, +\infty)$ and $\theta \in [0, \pi)$, let*

$\Phi_{h,\theta}$ denote a smooth family of eigenfunctions for $-\Delta_h$, the h-Robin realisation of the Laplacian on Ω, associated with $\lambda_h(\Omega) \leq M$. Then there exists $\varepsilon_0 > 0$ such that no nodal domain of an eigenfunction in this family has area less than ε_0.

We glean from this proposition that in a small neighbourhood of an interior critical point, respectively a boundary point, it is not possible to create additional nodal domains via a small perturbation of h. Hence, due to [30], the number of nodal domains either stays the same or decreases. See [18] for full details.

To conclude this section on the large h regime, we remark that not all properties that are enjoyed by the nodal sets of the Dirichlet eigenfunctions hold for the corresponding nodal sets of the Robin eigenfunctions.

We consider the twenty-fifth Dirichlet eigenvalue of S, $\lambda_{25}(S)$, which corresponds to the pair $(2,6)$. We recall that $\lambda_5(S)$ corresponds to the pair $(1,3)$, so

$$\cos\theta\, u_{2,6}(x/2,y/2) + \sin\theta\, u_{6,2}(x/2,y/2) = \cos\theta\, u_{1,3}(x,y) + \sin\theta\, u_{3,1}(x,y).$$

Hence the nodal partition of the twenty-fifth Dirichlet eigenfunction Φ_{25} consists of 4 scaled copies of the nodal partition of the fifth Dirichlet eigenfunction, and can be obtained by restricting Φ_{25} to $(0, \frac{\pi}{2})^2$ and then reflecting in the x and y axes. In [25] this is referred to as a folding procedure.

In Figure 4, we plot the fifth Robin eigenfunction for $h = 20$, and in Figure 5 we plot the twenty-fifth Robin eigenfunction for $h = 20$. We note that by comparing Figure 1 and Figure 4, we see that $h = 20$ is large enough so that the asymptotic structure for the fifth eigenfunction is preserved.

By comparing Figure 4 and Figure 5, we observe that the nodal partitions of the twenty-fifth Robin eigenfunction with $h = 20$ are not always obtained from the nodal partitions of the fifth Robin eigenfunction with $h = 20$ via the folding procedure mentioned above (for example, when $\theta = \frac{3\pi}{4}$). We refer to [18] (arXiv version) for further details.

3.6. The case h small

In this subsection, we first discuss the extent to which the methods described in Items (I), (II), (III), Subsection 3.4, for the Neumann case can be employed in the small h regime.

Analogously to [25], in [19] we show that for $k \geq 209$ and h small enough, $\lambda_{k,h}$ is not Courant-sharp.

We then rule out many of the cases $\lambda_{k,h}$ with $k \leq 208$ by considering the symmetry properties of the corresponding eigenfunctions (see also [25, 9]). However, in comparison to the results for the Neumann case, there are 4 eigenvalues for which we cannot conclude via such symmetry arguments.

Starting from a nodal partition for the Neumann case, in [19] for small h we obtain an analogous perturbation result to that of [18] for large h.

For critical points that are sufficiently far from the boundary ∂S, the perturbation

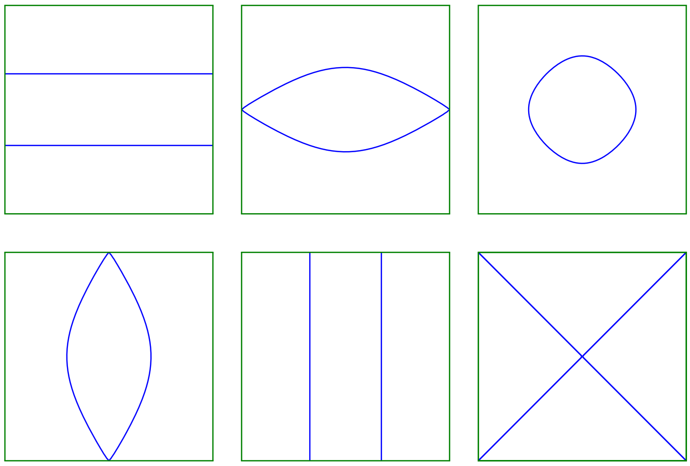


Figure 4: The nodal set of the Robin eigenfunction with $h = 20$ corresponding to $\lambda_{5,20}(S)$ for various values of θ. Top from left to right: $\theta = 0$, $\theta = 0.3245$, $\theta = \frac{\pi}{4}$. Bottom from left to right: $\theta = 1.2463$, $\theta = \frac{\pi}{2}$, $\theta = \frac{3\pi}{4}$.

result described in the previous subsection for large h also holds for small h. Indeed, Proposition 2 applies as the Robin Faber–Krahn inequality is satisfied in this case.

However, for critical points that are sufficiently close to the boundary Proposition 2 does not apply. In [19], under additional assumptions on the Neumann eigenfunctions, we obtain that in the neighbourhood of a critical point, the number of nodal domains does not increase under a small perturbation of h (see [19] for full details).

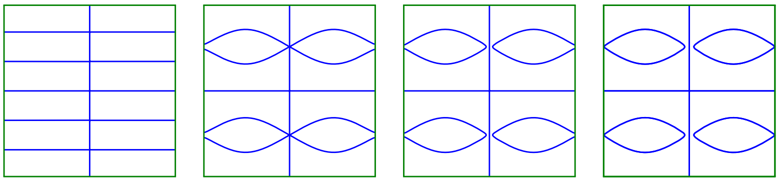
Since the number of nodal domains cannot increase under a small perturbation of h, if μ_k is not Courant-sharp, then $\lambda_{k,h}$ is not Courant-sharp for h small enough.

There is one case $\lambda_{116,h}$ for which neither the symmetry argument nor the perturbation argument applies. For this case, we show that under a small perturbation of h, the number of nodal domains cannot increase too much.

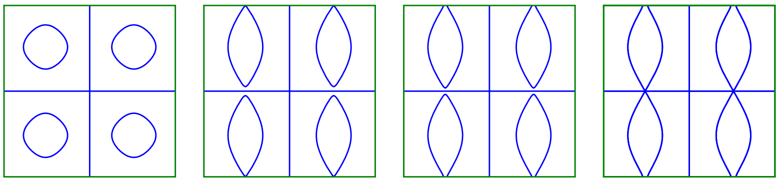
The final eigenvalue to consider is $\lambda_{5,h}$ which is Courant-sharp when $h = 0$ so the perturbation argument does not allow us to determine whether $\lambda_{5,h}$ is Courant-sharp or not for $h > 0$. In [19], we analyse the nodal structure of an eigenfunction which belongs to the eigenspace corresponding to $\lambda_{5,h}$ and show that for $h > 0$ small enough, such an eigenfunction has either 2, 3 or 4 nodal domains (see Figure 6). Therefore, in [19], we obtain the following theorem.

THEOREM 10. *There exists $h_0 > 0$ such that for $0 < h \leq h_0$, the Courant-sharp cases for the Robin problem are the same, except the fifth one, as those for $h = 0$.*

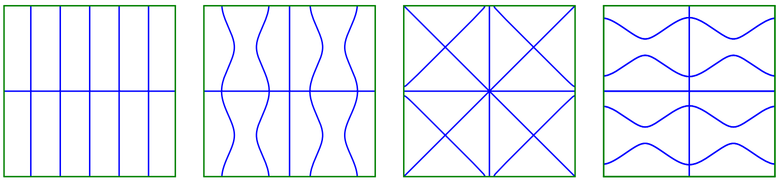
Indeed, $\lambda_{5,h}$ is not Courant-sharp for $0 < h \leq h_0$.



(a) From left to right: $\theta = 0$, $\theta = 0.3215$, $\theta = 0.3270$, $\theta = 0.3325$.



(b) From left to right: $\theta = \frac{\pi}{4}$, $\theta = 1.2383$, $\theta = 1.2438$, $\theta = 1.2493$.



(c) From left to right: $\theta = \frac{\pi}{2}$, $\theta = \frac{5\pi}{8}$, $\theta = \frac{3\pi}{4}$, $\theta = \frac{13\pi}{16}$.

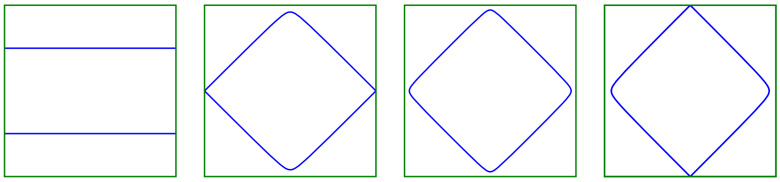
Figure 5: The nodal set of the Robin eigenfunction with $h = 20$ corresponding to $\lambda_{25,20}(S)$ for various values of θ.

4. Concluding remarks and questions

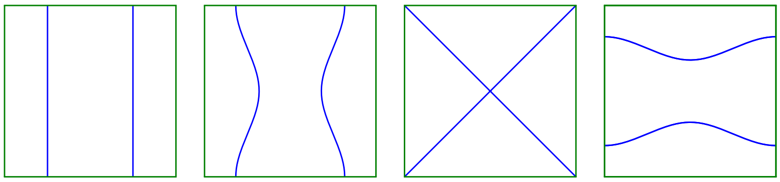
For the square, we remark that in Theorem 9, respectively Theorem 10, we treat the case $h \geq h_1$, respectively $h \leq h_0$. For $k \neq 1,2,3,4,9$ and $k \leq 519$, we do not address the question of whether $\lambda_{k,h}(S)$ is Courant-sharp for $h \in (h_0, h_1)$. Another interesting direction is to consider the case where $h < 0$, see [20].

For other domains, it would also be interesting to investigate the Courant-sharp Robin eigenvalues in the same spirit. That is, is it possible to follow the Courant-sharp Neumann eigenvalues to Courant-sharp Dirichlet eigenvalues for other domains? We remark that in the proofs of Theorems 9 and 10, for certain eigenvalues, we must analyse the corresponding eigenfunctions by making use of the explicit formulae for the eigenfunctions on a square. This is a drawback of the above method. On the other hand, the perturbation result established in Proposition 2 holds for a larger class of domains and not only for the square.

Another challenging open problem is to determine the Courant-sharp Neumann



(a) From left to right: $\theta = 0$, $\theta = 0.7776$, $\theta = \frac{\pi}{4}$, $\theta = 0.7932$.



(b) From left to right: $\theta = \frac{\pi}{2}$, $\theta = \frac{5\pi}{8}$, $\theta = \frac{3\pi}{4}$, $\theta = \frac{7\pi}{8}$.

Figure 6: The nodal set of the Robin eigenfunction with $h = 0.02$ corresponding to $\lambda_{5,0.02}(S)$ for various values of θ.

and Robin eigenvalues of a cube in $\mathbb{R}^3$. A key difficulty being that one must understand the structure of the nodal set on each face of the boundary. The Courant-sharp Dirichlet eigenvalues of the cube were obtained in [24].

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