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ON THE SYMMETRY BREAKING AND STRUCTURE OF THE MINIMIZERS FOR A FAMILY OF LOCAL/NONLOCAL INTERACTION FUNCTIONALS

Abstract. In this paper we review and present some results on the one-dimensionality and periodicity of minimizers for two families of local/nonlocal interaction functionals arising in generalized antiferromagnetic models or in models for colloidal suspensions. The local term is given in both cases by the 1-perimeter, penalizing interfaces. The nonlocal term instead favours oscillations: the interactions between the set and its complementary are modulated through a kernel of power law type for generalized antiferromagnetic models and through the Yukawa (or screened Coulomb) kernel in models for colloidal suspensions. Though the functionals are symmetric w.r.t. permutation of coordinates, we show that in suitable regimes the competition between the two terms causes symmetry breaking and global minimizers are periodic stripes, in any space dimension.

1. Introduction

In these notes we consider the following two functionals. For $d \geq 1$, $L > 0$, $E \subset \mathbb{R}^d$ $[0,L)^d$-periodic, let

$$\tilde{\mathcal{F}}^A_{J,L}(E) = \frac{1}{L^d}\Big(J\mathrm{Per}_1(E,[0,L)^d) - \int_{[0,L)^d}\int_{\mathbb{R}^d} |\chi_E(x) - \chi_E(y)| K^A_1(x-y)\,\mathrm{d}y\,\mathrm{d}x\Big), \tag{1}$$

where J is a positive constant,

$$\mathrm{Per}_1(E,[0,L)^d) := \int_{\partial E\cap[0,L)^d} |\nu^E(x)|_1 \, d\mathcal{H}^{d-1}(x), \quad |z|_1 = \sum_{i=1}^d |z_i|,$$

with $\nu^E(x)$ exterior normal to E in x, is the 1-perimeter of E and

$$K^A_1(\zeta) = \frac{1}{(|\zeta|_1+1)^p}, \qquad p \geq d+2. \tag{2}$$

Then, let

$$\tilde{\mathcal{F}}^Y_{J,L}(E) = \frac{1}{L^d}\Big(J\mathrm{Per}_1(E,[0,L)^d) - \int_{[0,L)^d}\int_{\mathbb{R}^d} |\chi_E(x) - \chi_E(y)| K^Y_1(x-y)\,\mathrm{d}y\,\mathrm{d}x\Big), \tag{3}$$

where

$$K^Y_1(\zeta) := \frac{e^{-|\zeta|_1}}{|\zeta|_1^{d-2}}.$$

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In these notes we are concerned with the structure of global minimizers of (1) and (3). In order to make our problem well-posed we impose periodic boundary conditions, namely we restrict the functional to $[0,L)^d$-periodic sets. However, the results on the structure of minimizers will be independent of L provided L is sufficiently large.

In both functionals there is a short-range attracting force (the perimeter) and a long-range repulsive force (nonlocal term) which are in competition. Moreover, there exists a critical constant, respectively given by J_c^A and J_c^Y, such that if $J > J_c^A$ ($J > J_c^Y$) then the global minimizers of the two functionals are trivial, namely either empty or the whole domain.

The critical constant $J_c^{(\cdot)}$ is

$$J_c^{(\cdot)} := \int_{\mathbb{R}^d} |\zeta_1| K_1^{(\cdot)}(\zeta)\,\mathrm{d}\zeta.$$

When J is strictly smaller than $J_c^{(\cdot)}$ but close to its value, it has been conjectured that the minimizers of the two models should be periodic unions of stripes of optimal period (in a range which is independent of L, for L large).

In order to define what we mean by unions of stripes, let us fix a canonical basis $\{e_i\}_{i=1}^d$. A union of stripes in the continuous setting is a $[0,L)^d$-periodic set which is, up to Lebesgue null sets, of the form $V_i^\perp + \hat{E}e_i$ for some $i \in \{1,\dots,d\}$, where $V_i^\perp$ is the $(d-1)$-dimensional subspace orthogonal to e_i and $\hat{E} \subset \mathbb{R}$ with $\hat{E} \cap [0,L) = \cup_{k=1}^N (s_i,t_i)$. A union of stripes is periodic if $\exists\, h > 0$, $\nu \in \mathbb{R}$ s.t. $\hat{E} \cap [0,L) = \cup_{k=0}^N (2kh+\nu,(2k+1)h+\nu)$. In the following, we will also sometimes call unions of stripes simply stripes.

We consider then J in a range of values slightly below J_c.

The motivation for studying these functionals comes from the will to understand spontaneous pattern formation, namely the ability of matter to organize itself into periodic structures. This phenomenon is of fundamental importance in Science, Technology, Engineering and Mathematics and it is often caused by the interaction between local attractive and nonlocal repulsive forces. (see e.g. [11, 21, 24, 4, 33]).

The ability of block copolymers to spontaneously organize themselves in periodic patterns has for example important applications in the production of nanosized micelles for drug delivery in the human body or in the production of nanometer memory cells. The most famous and studied model for diblock copolymers is due to Ohta and Kawasaki [32] and corresponds to the functional $\mathcal{F}_{J,L}^A$ with the 1-norm substituted by the Euclidean norm and with the exponent $p = d-2$.

Another important instance of spontaneous pattern formation at a mesoscopic scale is that showed by certain suspensions of charged colloids and polymers and also by protein solutions, when the attractive and repulsive forces compete at some strength ratio [35]. In particular, one can observe gathering of the particles in lamellas (stripes) or bubbles (clusters) according to the different regimes between the two mutual interactions. These self-assembly processes play a crucial role in applications such as therapeutic monoclonal antibodies, nanolithography or gelation processes.

For colloidal systems, the long-range repulsive forces have been shown on theoretical grounds to be represented by the Yukawa (or *screened Coulomb*) potential (1)

[12, 36] (the so-called DLVO Theory).

In dimension $d = 1$, there are many instances in which pattern formation is rigorously shown, either using convexity methods or the reflection positivity technique (see e.g. [31, 5, 14]).

However, in dimension $d \geq 2$, showing pattern formation is a rather difficult problem, due to the phenomenon of symmetry breaking.

Concerning the Ohta-Kawasaki model, which is the most famous and well-studied model close to the type (1) (see e.g. [6, 8, 26, 1, 33, 31, 7, 19, 27, 30]), even though periodic pattern formation is expected due to physical experiments and numerical simulations (e.g. [34, 6]), the problem is still open.

In order to state our results concerning minimizers of (1) and (3), it is convenient to suitably rescale the functionals so that the stripes with optimal period are of width and energy of order $O(1)$ as we approach the critical constant.

In the first model, setting $\tau = J_c - J > 0$, we obtain the functional

(4)
$$\mathcal{F}^A_{\tau,L}(E) = \frac{1}{L^d}\Big(-\mathrm{Per}_1(E,[0,L)^d) + \int_{\mathbb{R}^d} K^A_\tau(\zeta)\Big[\int_{\partial E\cap[0,L)^d}\sum_{i=1}^d |\nu^E_i(x)||\zeta_i|\,\mathrm{d}\mathcal{H}^{d-1}(x) - \int_{[0,L)^d}|\chi_E(x)-\chi_E(x+\zeta)|\,\mathrm{d}x\Big]\,\mathrm{d}\zeta\Big),$$

where $K^A_\tau(\zeta) = \frac{1}{(|z|_1+\tau^{1/\beta})^p}$.

In the second model, setting for M large but finite

$$J = \tilde{J}_M = \int_{\mathbb{R}^{d-1}}\int_{-M}^{M}|\zeta_1|K^Y_1(\zeta)\,\mathrm{d}\zeta < \int_{\mathbb{R}^{d-1}}\int_{-\infty}^{+\infty}|\zeta_1|K^Y_1(\zeta)\,\mathrm{d}\zeta = J^Y_c,$$

$J_M = \tilde{J}_M M^{-3}(e^*_M)^{-1}$, one gets the rescaled functional

(5)
$$\mathcal{F}^Y_{M,L}(E) = \frac{M^2}{L^d}\Big(J_M\mathrm{Per}_1(E,[0,L)^d) - \int_{[0,L)^d}\int_{\mathbb{R}^d}|\chi_E(x+\zeta)-\chi_E(x)|\bar{K}^Y_M(\zeta)\,\mathrm{d}x\,\mathrm{d}\zeta\Big)$$

where

$$\bar{K}^Y_M(\zeta) = \frac{-e^{-M|\zeta|_1}}{e^*_M|\zeta|_1^{d-2}}$$

and e^*_M is defined in the following way. For fixed $M > 0$, consider first for all $L > 0$ the minimal value obtained by $\mathcal{F}^Y_{M,L}$ on $[0,L)^d$-periodic unions of stripes and then the minimal among these values as L varies in $(0,+\infty)$. We will denote this value by e^*_M. In particular, $e^*_M \geq -e^{-\alpha_M M}$, with $\alpha_M \leq 1$, $\alpha_M \to 1$ as $M \to +\infty$.

For the functional (1), our main theorems (see [9]) are the following.

For fixed $\tau > 0$, consider first for all $L > 0$ the minimal value obtained by $\mathcal{F}^A_{\tau,L}$ on $[0,L)^d$-periodic stripes and then the minimal among these values as L varies in $(0,+\infty)$.

We will denote this value by C_τ^*. By the reflection positivity technique, this value is attained on periodic stripes of width and distance $h_\tau^* > 0$.

THEOREM 1. *Let $d \geq 1$, $p \geq d+2$. Then there exists $\hat{\tau} > 0$ s.t. whenever $0 < \tau < \hat{\tau}$, h_τ^* is unique.*

In the next two theorems, we deal with the occurrence of pattern formation for $\mathcal{F}_{\tau,L}^A$.

THEOREM 2. *Let $d \geq 1$, $p \geq d+2$, $L > 0$. Then there exists $\bar{\tau} > 0$ such that $\forall 0 < \tau \leq \bar{\tau}$ there exists $h_{\tau,L}$ such that the minimizers of $\mathcal{F}_{\tau,L}^A$ are periodic stripes of width and distance $h_{\tau,L}$.*

The next theorem shows that $h_{\tau,L}$ is close to h_τ^* whenever L is large.

THEOREM 3. *There exists a constant C such that for every $0 < \tau \leq \bar{\tau}$, one has that the width $h_{\tau,L}$ of a minimizer of $\mathcal{F}_{\tau,L}^A$ satisfies*

$$|h_\tau^* - h_{\tau,L}| \leq \frac{C}{L}.$$

In Theorem 2 the constant $\bar{\tau}$ depends on L. One expects $\bar{\tau}$ to be independent on L. In this respect, when L is of the form $L = 2kh_\tau^*$, the independence is shown in Theorem 4, namely τ_0 does not depend on L if $L = 2kh_\tau^*$.

THEOREM 4. *Let $d \geq 1$, $p \geq d+2$ and h_τ^* be the optimal stripes' width for fixed τ. Then there exists τ_0, such that for every $\tau < \tau_0$, one has that for every $k \in \mathbb{N}$ and $L = 2kh_\tau^*$, the minimizers E_τ of $\mathcal{F}_{\tau,L}$ are optimal stripes of width h_τ^*.*

Notice that the periodic boundary conditions were imposed in order to give sense to the functional which is otherwise not well-defined. If one is interested to show that optimal periodic stripes of width and distance h_τ^* are "optimal" if one varies also the periodicity, then it is not difficult to see that Theorem 1.3 is sufficient. This corresponds to the "thermodynamic limit" and is relevant in physics.

Let us now focus on the functional (3).

By the reflection positivity technique, the value e_M^* is attained on periodic stripes of width and distance $h_M^* > 0$.

Our main theorems (see [10]) are the following:

THEOREM 5. *There exists $\tilde{M} > 0$ such that $\forall M > \tilde{M}$, there exists a unique optimal period h_M^*.*

THEOREM 6. *There exists a constant M_0 such that for every $M > M_0$ and $L = 2kh_M^*$ for some $k \in \mathbb{N}$, then the minimizer of $\mathcal{F}_{M,L}^Y$ are optimal stripes of width and distance h_M^*.*

In Theorem 6 notice that M_0 is independent of L.

As in [9] for the functional (1), one can provide a characterization of minimizers of $\mathcal{F}^Y_{M,L}$ also for arbitrary L, but this time with M larger than a constant depending on L. Namely, one has the following

THEOREM 7. *Let $L > 0$. Then there exists $\bar{M} > 0$ such that $\forall M \geq \bar{M}$ there exists $h_{M,L}$ such that the minimizers of $\mathcal{F}^Y_{M,L}$ are periodic stripes of width and distance $h_{M,L}$.*

According to the next theorem, when L is large then $h_{M,L}$ is close to h^*_M.

THEOREM 8. *There exists $C > 0$ and $\hat{M} > 0$ such that for every $M > \hat{M}$ the width and distance $h_{M,L}$ of minimizers of $\mathcal{F}^Y_{M,L}$ satisfies*

$$\left| h_{M,L} - h^*_M \right| \leq \frac{C}{L}.$$

In the continuous setting, to our knowledge these are the first examples of models with local/nonlocal terms in competition such that the functional is invariant under permutation of coordinates and the minimizers can be rigorously shown to display a pattern formation which is one-dimensional.

For a discrete version of (1) and the smaller range of exponents $p > 2d$ such a result was already shown in [17]. In [9], we recover also the results of [17] in the discrete setting for the larger set of exponents $p \geq d+2$. Our improvement on the exponents can be viewed as progress towards the aim of proving pattern formation for the more “physical” exponents. Among them we recall the case of thin magnetic films ($p = d+1$ see e.g. [34, 14]), 3D micromagnetics ($p = d$ see e.g. [28, 22, 25]) and diblock copolymers ($p = d-2$ see e.g. [32]). Our proof, though sharing some broad similarities with the one in [17], like the use of a two-scale approach in order to identify regions where a set resembles more stripes and reflection positivity, is substantially different. Indeed, while in [17] discrete concepts like “angles” and “holes” are used in order measure deviations from being a stripe, in the continuous setting such a classification of the defects is not available. In particular, a new rigidity estimate for finite perimeter sets of finite $\mathcal{F}^A_{0,L}$ energy has to be implemented which guarantees closeness of minimizers to stripes in the limit as $\tau \to 0$. In [20], for a smaller range of exponents ($p > 2d$ instead of $p \geq d+2$) a rigidity estimate was shown, leading to prove that minimizers of $\mathcal{F}^A_{\tau,L}$ converge in L^1 to periodic stripes as $\tau \downarrow 0$. In [9] we show that pattern formation really appears not only for τ tending to 0 (as was done in [20]) but for a positive fixed τ, in the range $p \geq d+2$. Moreover, we show that in case L is an even multiple of the optimal period h^*_τ, such τ does not depend on how big L is.

While for the power-like potential K^A_1 the physical exponents $p = d+1$ (thin magnetic films), $p = d$ (3D-micromagnetics) and $p = d-2$ (Coulomb potential) remain excluded by the above-mentioned results, for the functional (3) we were first able to prove pattern formation for a physical model. As discussed in [2, 3, 23, 18], one of the possible models used to show gelification in charged colloids and pattern formation is to consider both as attractive and as repulsive term the Yukawa potential, with different signs and appropriate rescaling, namely the functional

$$
\begin{aligned}
\tilde{\mathcal{E}}^{Y}_{\beta,J,L}(E) := \frac{1}{L^d}\Big(& JC_{\beta,L}\int_{[0,L)^d}\int_{\mathbb{R}^d}|\chi_E(x+\zeta)-\chi_E(x)|K^Y_\beta(\zeta)\,\mathrm{d}\zeta\,\mathrm{d}x \\
(6)\qquad & -\int_{[0,L)^d}\int_{\mathbb{R}^d}|\chi_E(x+\zeta)-\chi_E(x)|K^Y_1(\zeta)\,\mathrm{d}\zeta\,\mathrm{d}x\Big),
\end{aligned}
$$

where

$$
K^Y_\beta(\zeta) := \frac{e^{-\beta|\zeta|_1}}{|\zeta|_1^{d-2}},
$$

$\beta > 1$ and $C_{\beta,L}$ is a positive normalization constant depending on β and L.

Therefore, the following Γ-convergence result connects our analytical results with the ones obtained in the above cited experiments and simulations for the functional (6).

THEOREM 9. *The functionals $\tilde{\mathcal{E}}^{Y}_{\beta,J,L}$ defined in* (6) *Γ-converge in the L^1 topology as $\beta \to +\infty$ and up to subsequences to the functional $\tilde{\mathcal{F}}^{Y}_{J,L}$ defined in* (1).

In these notes we will focus on the main ideas of the proofs of Theorems 2 and 7, giving some details for the proof of Theorem 7. Indeed, they already contain most of the ingredients of the proofs of Theorems 4 and 6, where a delicate and technical averaging and localization argument is used in order to make the results independent of L.

2. General strategy for the proof of striped pattern formation

The proofs of Theorems 2 and 7 for both functionals (1) and (3), although exploiting different estimates, share a common strategy, which we try to summarize below.

1. **Splitting** Decompose the functional as a sum of $2d$ functionals, where the first d of them are affected only by oscillations of the set/function in a direction e_i, $i \in \{1,\dots,d\}$ (**one-dimensional terms**) and the remaining d are positive and result each from the interaction between oscillations in a direction e_i and oscillations in any of the orthogonal directions (**cross interaction terms**);

2. **One-dimensional estimates** Develop estimates in direction e_i for the corresponding one-dimensional term of the splitting, showing that such a term is strictly positive in case the slices of the characteristic functions of a finite perimeter set in direction e_i oscillate too wildly;

3. **Rigidity estimate in the limit to the critical constant** Via Γ-convergence as the parameters approach the critical constant and rigidity estimates for the limit functionals deduce closeness in L^1 to unions of stripes for values of the

parameters close to the critical constant;

4. **Stability argument** By 3. minimizers are, for values of the parameters close to the critical constant, L^1-close to stripes, w.l.o.g. in direction e_1.

   For any e_i, $i \neq 1$, take the i-th one-dimensional functional plus the i-th cross interaction term and write them as integrals of one-dimensional functionals of the slices of the set in direction e_i.

   The aim is to show that the value of the sum of the i-th functionals on such slices is strictly positive unless the value of the set on the slice is constant. If this holds, then minimizers are one-dimensional.

   By 2., wild oscillations between 0 and 1 are excluded. Therefore, whenever there is a boundary point in the slice, the set has to assume values close to 0 to the left and close to 1 to the right (or viceversa) for some interval of given length.

   Finally, i-th cross interaction term penalizes the case of at least one of such oscillations. Indeed, in most of closer slices the function will assume either values close to 1 or values close to 0, thus producing a chessboard effect, which is exactly penalized by the i-th cross interaction term;

5. **One-dimensional minimization** Once we know that minimizers are one-dimensional, we show via reflection positivity that minimizers of the one-dimensional functional in direction e_1 are periodic.

3. Elements of the proof of Theorem 7

Below we point out some key details of the proof of Theorem 7, regarding points $1.-4.$ above.

In the following, we let $\mathbb{N} = \{1,2,\dots\}$, $d \geq 1$. On $\mathbb{R}^d$, we let $\langle \cdot, \cdot \rangle$ be the Euclidean scalar product and $|\cdot|$ be the Euclidean norm. We let $(e_1,\dots,e_d)$ be the canonical basis in $\mathbb{R}^d$ and for $y \in \mathbb{R}^d$ we let $y_i = \langle y, e_i \rangle e_i$ and $y_i^\perp := y - y_i$. For $y \in \mathbb{R}^d$, let $|y|_1 = \sum_{i=1}^d |y_i|$ be its 1-norm and $|y|_\infty = \max_i |y_i|$ its ∞-norm.

For $i \in \{1,\dots,d\}$, let $x_i^\perp$ be a point in the subspace orthogonal to e_i. We define the one-dimensional slices of $E \subset \mathbb{R}^d$ by

$$E_{x_i^\perp} := \left\{ s \in [0,L) : s e_i + x_i^\perp \in E \right\}.$$

Notice that in the above definition there is an abuse of notation as the information on the direction of the slice is contained in the index $x_i^\perp$. As it would be always clear from the context which is the direction of the slicing, we hope this will not cause confusion to the reader.

Given a set of locally finite perimeter E, for a.e. $x_i^\perp$ its slice $E_{x_i^\perp}$ is a set of locally finite perimeter in $\mathbb{R}$ and the following slicing formula (see [29]) holds for every $i \in \{1, \dots, d\}$

$$\mathrm{Per}_{1i}(E, [0,L)^d) = \int_{\partial E \cap [0,L)^d} |\nu_i^E(x)| \, \mathrm{d}\mathcal{H}^{d-1}(x) = \int_{[0,L)^{d-1}} \mathrm{Per}_1(E_{x_i^\perp}, [0,L)) \, \mathrm{d}x_i^\perp.$$

3.1. Splitting

Using the equality

$$\begin{aligned} |\chi_E(x) - \chi_E(x+\zeta)| = & |\chi_E(x) - \chi_E(x+\zeta_i)| + |\chi_E(x+\zeta) - \chi_E(x+\zeta_i)| \\ & - 2|\chi_E(x) - \chi_E(x+\zeta_i)||\chi_E(x+\zeta) - \chi_E(x+\zeta_i)| \end{aligned}$$

one splits the nonlocal term of (3) getting the following lower bound:

$$\begin{aligned} \mathcal{F}_{M,L}^Y(E) \geq & \frac{M^2}{L^d} \sum_{i=1}^d \Big[\int_{[0,L)^d \cap \partial E} \int_{\mathbb{R}^{d-1}} \int_{-1}^1 |\nu_i^E(x)||\zeta_i| \bar{K}_M^Y(\zeta) \, \mathrm{d}\zeta \, \mathrm{d}\mathcal{H}^{d-1}(x) \\ & - \int_{[0,L)^d} \int_{\mathbb{R}^d} |\chi_E(x+\zeta_i) - \chi_E(x)| \bar{K}_M^Y(\zeta) \, \mathrm{d}\zeta \, \mathrm{d}x \Big] \\ & + \frac{2}{d} \frac{M^2}{L^d} \sum_{i=1}^d \int_{[0,L)^d} \int_{\mathbb{R}^d} |\chi_E(x+\zeta_i) - \chi_E(x)||\chi_E(x+\zeta_i^\perp) - \chi_E(x)| \bar{K}_M^Y(\zeta) \, \mathrm{d}\zeta \, \mathrm{d}x \\ = & \frac{M^2}{L^d} \Big(\sum_{i=1}^d \mathcal{G}_{M,L}^{i,Y}(E) + \sum_{i=1}^d I_{M,L}^{i,Y}(E) \Big), \end{aligned} \tag{7}$$

where

$$\begin{aligned} \mathcal{G}_{M,L}^{i,Y}(E) : \; = \; & \int_{[0,L)^d \cap \partial E} \int_{\mathbb{R}^{d-1}} \int_{-1}^1 |\nu_i^E(x)||\zeta_i| \bar{K}_M^Y(\zeta) \, \mathrm{d}\zeta \, \mathrm{d}\mathcal{H}^{d-1}(x) \\ - \; & \int_{[0,L)^d} \int_{\mathbb{R}^d} |\chi_E(x+\zeta_i) - \chi_E(x)| \bar{K}_M^Y(\zeta) \, \mathrm{d}\zeta \, \mathrm{d}x \end{aligned}$$

and

$$I_{M,L}^{i,Y}(E) := \frac{2}{d} \int_{[0,L)^d} \int_{\mathbb{R}^d} |\chi_E(x+\zeta_i) - \chi_E(x)||\chi_E(x+\zeta_i^\perp) - \chi_E(x)| \bar{K}_M^Y(\zeta) \, \mathrm{d}\zeta \, \mathrm{d}x.$$

Moreover, let

$$I_{M,L}^Y(E) := \sum_{i=1}^d I_{M,L}^{i,Y}(E).$$

One can further express $\mathcal{G}_{M,L}^{i,Y}(E)$ as a sum of contributions obtained by first slicing and then considering interactions with neighbouring points on the slice lying on ∂E, namely

$$\mathcal{G}_{M,L}^{i,Y}(E) = \int_{[0,L)^{d-1}} \sum_{s \in \partial E_{t_i^\perp} \cap [0,L)} r_{i,M}(E, t_i^\perp, s) \, \mathrm{d}t_i^\perp \tag{8}$$

where for $s \in \partial E_{t_i^\perp}$

$$r_{i,M}(E,t_i^\perp,s) := \int_{-1}^{1} |\zeta_i| \widehat{\bar{K}}_M^Y(\zeta_i)\,\mathrm{d}\zeta_i - \int_{s^-}^{s} \int_0^{+\infty} |\chi_{E_{t_i^\perp}}(u+\rho) - \chi_{E_{t_i^\perp}}(u)| \widehat{\bar{K}}_M^Y(\rho)\,\mathrm{d}\rho\,\mathrm{d}u$$

$$(9) \qquad - \int_{s}^{s^+} \int_{-\infty}^{0} |\chi_{E_{t_i^\perp}}(u+\rho) - \chi_{E_{t_i^\perp}}(u)| \widehat{\bar{K}}_M^Y(\rho)\,\mathrm{d}\rho,\,\mathrm{d}u$$

$$\widehat{\bar{K}}_M^Y(t) := \int_{\mathbb{R}^{d-1}} \bar{K}_M^Y(t,\zeta_2,\dots,\zeta_d)\,\mathrm{d}\zeta_2 \cdots \mathrm{d}\zeta_d.$$

and

$$(10) \qquad \begin{aligned} s^+ &= \inf\{t' \in \partial E_{t_i^\perp}, \text{with } t' > s\} \\ s^- &= \sup\{t' \in \partial E_{t_i^\perp}, \text{with } t' < s\}. \end{aligned}$$

The functional $\mathcal{G}_{M,L}^{i,Y}(E)$ is the above-mentioned one-dimensional term in direction e_i and $I_{M,L}^{i,Y}(E)$ the cross-interaction term in direction e_i.

3.2. One-dimensional estimates

Estimating the optimal energy of stripes for the functional (3) with $J = \tilde{J}_M$, one finds that there exists a constant $1 > \gamma_M > 0$ such that

$$(11) \qquad \bar{K}_M^Y(\zeta) \geq \frac{1}{|\zeta|_1^{d-2}} e^{-M(|\zeta|_1 - \gamma_M)}$$

with $\gamma_M \to 1$ as $M \to +\infty$.

As a consequence, one can prove the following:

LEMMA 1. *There exists a function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that, for all $E \subset \mathbb{R}^d$ of locally finite perimeter, $t_i^\perp \in [0,L)^{d-1}$, $s \in \partial E_{t_i^\perp}$*

$$(12) \qquad r_{i,M}(E,t_i^\perp,s) \geq g\big((\gamma_M - \min(|s-s^-|,|s-s^+|),M\big)$$

with γ_M defined in (11). The function g satisfies the following: $g(v,M) \geq g(v',M)$ whenever $v > v'$, $g(v,M) \geq -e^{-cM}$ for some $c > 0$ and $g(v,M) \to +\infty$ as $M \to +\infty$ provided $v > 0$.

In particular, for every $0 < \eta_0 < 1$ there exists M_0 such that, for all $M > M_0$ if $\min\{|s-s^-|,|s-s^+|\} < \eta_0$ then $r_{i,M}(E,t_i^\perp,s) > 0$.

Such an estimate penalizes strongly boundary points on the slices which have mutual distance strictly less than 1, for M large enough.

3.3. Γ-convergence

The main Theorem of this section, namely the rigidity estimate for the limit functional as $M \to \infty$ is the following:

THEOREM 10. *Let $L > 0$. Then one has that $\mathcal{F}^Y_{M,L}$ Γ-converge in the L^1-topology as $M \to \infty$ to a functional $\mathcal{F}^Y_{\infty,L}$ which is invariant under permutation of coordinates and finite on sets (up to permutation of coordinates) of the form $E = F \times \mathbb{R}^{d-1}$, where $F \subset \mathbb{R}$ is L-periodic with $\#\{\partial F \cap [0,L)\} < \infty$.*

On sets of the form $E = F \times \mathbb{R}^{d-1}$ the functional is defined by

$$\mathcal{F}^Y_{\infty,L}(E) = \frac{1}{L}\Big(-\#\{\partial F \cap [0,L)\} + \int_{\mathbb{R}^d} \bar{K}^Y_\infty(z)\Big[\sum_{s \in \partial F \cap [0,L)} |\zeta_1| - \int_0^L |\chi_F(s) - \chi_F(s+\zeta_1)|\,\mathrm{d}s\Big]\,\mathrm{d}\zeta\Big), \tag{13}$$

where

$$\bar{K}^Y_\infty(\zeta) := \liminf_{M \to +\infty} \bar{K}^Y_M(\zeta), \quad \bar{K}^Y_\infty \geq 0 \quad \text{and} \quad \bar{K}^Y_\infty(\zeta) = +\infty \text{ whenever } |\zeta| < 1. \tag{14}$$

Moreover, let $\{E_M\}$ be a family of $[0,L)^d$-periodic subsets of $\mathbb{R}^d$ such that there exists C such that for every M one has that $\mathcal{F}^Y_{M,L}(E_M) \leq C$, then, up to a permutation of coordinates, one has that there is a subsequence which converges in $L^1([0,L)^d)$ to some set $E = F \times \mathbb{R}^{d-1}$ with $\#\{\partial F \cap [0,L)\} < \infty$.

As for the one-dimensional term, one can make the following observation.

REMARK 1. From Lemma 1, since $\gamma_M \to 1$ as $M \to +\infty$, it follows as well that the function

$$r_{i,\infty}(E, t_i^\perp, s) := \liminf_{M \to +\infty} r_{i,M}(E, t_i^\perp, s)$$

satisfies

$$r_{i,\infty}(E, t_i^\perp, s) = +\infty \quad \text{whenever} \quad \min(|s - s^-|, |s - s^+|) < 1. \tag{15}$$

In particular, if $\{E_M\}_{M>0} \subset \mathbb{R}^d$ is a family of sets of locally finite perimeter with $\sup_M \mathcal{F}^Y_{M,L}(E_M) \leq C$, then for a.e. $t_i^\perp \in Q_l^\perp(z_i^\perp)$ and for every $I \subset \mathbb{R}$ open interval,

$$\liminf_{M \to +\infty} \min\big\{|s_i^M - s_{i+1}^M| : \partial E_{M,t_i^\perp} \cap I = \{s_i^M\}_{i=1}^{m(M)}\big\} \geq 1 \tag{16}$$

In particular, E_M converges in L^1_{loc} to a set E_∞ of locally finite perimeter such that

$$\min\big\{|s_i^\infty - s_j^\infty| : \partial E_{\infty,t_i^\perp} \cap I = \{s_k^\infty\}_{k=1}^{m(\infty)}\big\} \geq 1. \tag{17}$$

As for the cross interaction term, setting

$$f_E(t_i^\perp, t_i, t_i'^\perp, t_i') := |\chi_E(t_i^\perp + t_i + t_i') - \chi_E(t_i + t_i^\perp)||\chi_E(t_i^\perp + t_i + t_i'^\perp) - \chi_E(t_i + t_i^\perp)|, \tag{18}$$

for the limit set E one has that

$$(19)\qquad \begin{aligned} I^{i,Y}_{\infty,L}(E) &= \frac{2}{d}\int_{[0,L)^d}\int_{\mathbb{R}^d} f_E(t_i^{\perp},t_i,\zeta_i^{\perp},\zeta_i)\bar{K}^Y_{\infty}(\zeta)\,\mathrm{d}\zeta \\ \mathrm{d}t &= \frac{2}{d}\int_{[0,L)^{d-1}}\int_{[0,L)^{d-1}} \mathrm{Int}(t_i^{\perp},t_i'^{\perp})\,\mathrm{d}t_i^{\perp},\,\mathrm{d}t_i'^{\perp} \lesssim C<+\infty. \end{aligned}$$

where

$$(20)\qquad \mathrm{Int}(t_i^{\perp},t_i'^{\perp}) := \int_0^L\int_0^L f_E(t_i^{\perp},t_i,t_i'^{\perp}-t_i^{\perp},t_i'-t_i)\bar{K}^Y_{\infty}(t-t')\,\mathrm{d}t_i\,\mathrm{d}t_i'.$$

Now given $t_i^{\perp}\in[0,L)^{d-1}$ we denote by

$$(21)\qquad \begin{aligned} r^i(u,t_i^{\perp}) &:= \min\{|u-s|:\ s\in\partial E_{t_i^{\perp}}\} \\ r^i_o(t_i^{\perp}) &:= \inf_{s\in\partial E_{t_i^{\perp}}}\min(s^+-s,s-s^-), \end{aligned}$$

where s^+,s^- are defined in (10). Notice that by Remark 1 $r^i_o(t_i^{\perp})\geq 1$.

Notice that the map $r^i_{\lambda}(\cdot,t_i^{\perp})$ is well-defined for almost every $t_i^{\perp}$ and measurable.

Suppose that, for every u, one has that $r^i(u,\cdot)$ is constant almost everywhere: if this holds for every i, then it is not difficult to see that E is (up to null sets) either a union of stripes or a checkerboards, where by checkerboards we mean any set whose boundary is the union of affine subspace orthogonal to coordinate axes, and there are at least two of these directions.

The checkerboards however can be ruled out via an energetic argument.

In order to obtain that $r^i_{\lambda}(u,\cdot)$ is constant almost everywhere we proceed in the following way.

First we give the following lower bound for the interaction term.

LEMMA 2. *Let $t_i'^{\perp},t_i^{\perp}\in[0,L)^{d-1}$, $t_i^{\perp}\neq t_i'^{\perp}$ be such that $\min(r^i_o(t_i^{\perp}),r^i_o(t_i'^{\perp}))>|t_i'^{\perp}-t_i^{\perp}|$ and $|t_i'^{\perp}-t_i^{\perp}|\leq 1/2$. Then for every $u\in[0,L)$ it holds*

$$(22)\qquad \mathrm{Int}(t_i'^{\perp},t_i^{\perp})\geq \not{K}^{\infty}_{\{(t_i'^{\perp},t_i^{\perp}):\,r^i(u,t_i'^{\perp})\neq r^i(u,t_i^{\perp})\}}(t_i'^{\perp},t_i^{\perp}).$$

In (22), we set

$$(23)\qquad \not{K}^{\infty}_A(x)=\begin{cases} +\infty & \text{if } x\in A \\ 0 & \text{if } x\in\mathbb{R}^d\setminus A \end{cases}$$

Sketch of the proof of Lemma 2: W.l.o.g. we can assume that $|u - s_o| = r^i(u, t_i^\perp) < r^i(u, t_i'^\perp)$. Denoting by $\delta = |t_i'^\perp - t_i^\perp|$ and by $r = |r^i(u, t_i^\perp) - r^i(u, t_i'^\perp)|$ we can also assume that (since $r_o(t_i^\perp) > \delta$)

$$(s_o - \delta, s_o + \delta) \cap E_{\infty, t_i^\perp} = (s_o, s_o + \delta) \tag{24}$$

and moreover $r > \delta/2$.

For every $a \in (s_o - \delta/2, s_o)$ and $a' \in (s_o, s_o + \delta/2)$, one has that

$$f_E(t_i^\perp, a, t_i'^\perp - t_i^\perp, a' - a) = 1.$$

Given that $r^i(u, t_i^\perp) \leq L$, one has that

$$\begin{aligned}
\mathrm{Int}(t_i^\perp, t_i'^\perp) &= \int_0^L \int_0^L f_E(t_i^\perp, t_i, t_i'^\perp - t_i^\perp, t_i' - t_i) \bar{K}_\infty^Y(t' - t)\, \mathrm{d}t_i\, \mathrm{d}t_i' \\
&\geq \int_{s_o - \delta/2}^{s_o} \int_{s_o}^{s_o + \delta/2} f_E(t_i^\perp, t_i, t_i'^\perp - t_i^\perp, t_i' - t_i) \bar{K}_\infty^Y(t' - t)\, \mathrm{d}t_i'\, \mathrm{d}t_i \\
&\geq \int_{s_o - \delta/2}^{s_o} \int_{s_o}^{s_o + \delta/2} \bar{K}_\infty^Y(t' - t)\, \mathrm{d}t_i'\, \mathrm{d}t_i = +\infty,
\end{aligned}$$

since $|t - t'| < 1$ if $\max(|t_i^\perp - t_i'^\perp|, |t_i - t_i'|) \leq 1/2$ and therefore $\bar{K}_\infty^Y(t - t') = +\infty$. □

Let now B be the set defined by

$$B := \left\{ (t_i'^\perp, t_i^\perp) \in [0, L)^{d-1} \times [0, L)^{d-1} : r^i(u, t_i^\perp) \neq r^i(u, t_i'^\perp),\ |t_i'^\perp - t_i^\perp| \leq 1/2 \right\}.$$

Then, by Lemma 2,

$$\int_{[0,L)^{d-1}} \int_{[0,L)^{d-1}} \not\Vdash_B^\infty (t_i^\perp, t_i'^\perp)\, \mathrm{d}t_i \perp\, \mathrm{d}t_i'^\perp \leq \int_0^L \int_0^L \mathrm{Int}(t_i^\perp, t_i'^\perp)\, \mathrm{d}t_i^\perp\, \mathrm{d}t_i'^\perp \lesssim C < +\infty. \tag{25}$$

Hence, $r^i(u, t_i^\perp) = r^i(u, t_i'^\perp)$ whenever $|t_i^\perp - t_i'^\perp| \leq 1/2$ and therefore $r^i(u, \cdot)$ is constant for every u.

3.4. Stability argument

This step (together with 5.) concludes the proof of Theorem 7.

Recall ((7)) that we started from

$$\mathcal{F}_{M,L}^Y(E) \geq \frac{M^2}{L^d} \Big(\sum_{i=1}^d \mathcal{G}_{M,L}^{i,Y}(E) + \sum_{i=1}^d I_{M,L}^{i,Y}(E) \Big). \tag{26}$$

Since $I^{i,Y}_{M,L}(E)=0$ if and only if E is a union of stripes, one has that the l.h.s.and r.h.s.of the above are equal whenever E is a union of stripes. Thus, it is sufficient to show that optimal stripes are minimizers of the r.h.s.of the above.

For the r.h.s., in order to show optimality of stripes we will initially use Theorem 10 in order to reduce ourselves to a situation in which the minimizer of $\mathcal{F}_{M,L}$ are close to optimal stripes S in L^1. This holds for $M>\bar M$, where $\bar M$ depends on L.

The idea at this point is to show that oscillations of the characteristic function of the set E in directions which are orthogonal to the direction of S increase necessarily the r.h.s.of (26) (this is done in Lemma 3). Thus the r.h.s.of (26) can be further bounded from below by

$$\frac{1}{L^d}\,\mathcal{G}^{i,Y}_{M,L}(E), \tag{27}$$

where e_i is the orientation of the stripes. Finally, optimal periodic stripes minimize (27), since it corresponds to the one-dimensional problem (5.) already studied in the literature with the help of reflection positivity (see e.g. [13, 14, 15, 16]).

The main result is the following:

LEMMA 3 (Stability). *Let $E\subset\mathbb{R}^d$ be a $[0,L)^d$-periodic set of locally finite perimeter, and S be a set which is a union of periodic stripes, i.e. (up to exchange of coordinates and translations) there exists $\hat E\subset\mathbb{R}$ such that $S=\hat E\times\mathbb{R}^{d-1}$ and*

$$\hat E=\bigcup_{k\in\mathbb{Z}}[2kh,(2k+1)h), \tag{28}$$

for a suitable h. Then, there exist $\bar\varepsilon,\bar M>0$ such that if $\|\chi_E-\chi_S\|_{L^1}<\bar\varepsilon$ and $M>\bar M$, one has that, for $i\in\{2,\dots,d\}$,

$$\mathcal{G}^{i,Y}_{M,L}(E)+I^{i,Y}_{M,L}(E)\geq 0. \tag{29}$$

Moreover, in (29) equality holds if and only if E is a union of stripes in direction e_1.

Proof: Let s^-,s,s^+ be three consecutive points for $\partial E_{t_i^\perp}$. By Lemma 1, for all $0<\eta_0<1$, there exists $M_0>0$ such that if $M>M_0$

$$\min(|s-s^-|,|s^+-s|)<\eta_0 \quad\text{then}\quad r_{i,M}(E,t_i^\perp,s)>0.$$

Thus without loss of generality we may assume that $\min(|s-s^-|,|s^+-s|)\geq\eta_0$.

Thus, given that, for every s, $r_{i,M}(E,t_i^\perp,s)>-e^{-cM}$ for some $c>0$ (see Lemma 1), one has that

$$\begin{aligned} r_{i,M}(E,t_i^\perp,s)&+\frac{1}{2d}\int_{s^-}^{s^+}\int_{\mathbb{R}^d}f_E(t_i^\perp,u,\zeta_i^\perp,\zeta_i)\bar K^Y_M(\zeta)\,\mathrm{d}\zeta\,\mathrm{d}u\\ &\geq -e^{-cM}+\frac{1}{2d}\int_{s^-}^{s^+}\int_{\mathbb{R}^d}f_E(t_i^\perp,u,\zeta_i^\perp,\zeta_i)\bar K^Y_M(\zeta)\,\mathrm{d}\zeta\,\mathrm{d}u.\end{aligned} \tag{30}$$

Let now $0<\varepsilon<\eta_0$. By assumption, for some $t_i\in\partial E_{t_i^\perp}$ one of the following holds:

(i) $(t_i-\varepsilon,t_i)\subset E_{t_i^\perp}$ and $(t_i,t_i+\varepsilon)\subset E^c_{t_i^\perp}$

(ii) $(t_i-\varepsilon,t_i)\subset E^c_{t_i^\perp}$ and $(t_i,t_i+\varepsilon)\subset E_{t_i^\perp}$.

W.l.o.g., we may assume that (i) above holds and that $i=d$.

Setting $Q^\perp_\varepsilon(t^\perp_d)=\{z^\perp_d\in[0,L)^{d-1}:|z^\perp_d-t^\perp_d|_\infty\leq\varepsilon\}$, one has that

$$\max\Big(\frac{|Q^\perp_\varepsilon(t^\perp_d)\times(t_d-\varepsilon,t_d)\cap E^c|}{|Q^\perp_\varepsilon(t^\perp_d)\times(t_d-\varepsilon,t_d)|},\frac{|Q^\perp_\varepsilon(t^\perp_d)\times(t_d,t_d+\varepsilon)\cap E|}{|Q^\perp_\varepsilon(t^\perp_d)\times(t_d-\varepsilon,t_d)|}\Big)\geq\frac{7}{16}. \tag{31}$$

Thus, we can further assume that

$$(t_d-\varepsilon,t_d)\subset E_{t^\perp_d}\quad\text{and}\quad\frac{|Q^\perp_\varepsilon(t^\perp_d)\times(t_d-\varepsilon,t_d)\cap E^c|}{|Q^\perp_\varepsilon(t^\perp_d)\times(t_d-\varepsilon,t_d)|}\geq\frac{7}{16}. \tag{32}$$

For every $s\in(t_d-\varepsilon,t_d)$, $(\zeta^\perp_d,s)\notin E$ and $\zeta_d+s\in(t_d,t_d+\varepsilon)$ we have that $f_E(t^\perp_d,s,\zeta^\perp_d,\zeta_d)=1$. Thus by integrating initially in ζ_d and using (11), we have that

$$\begin{aligned}\int_{t_d-\varepsilon}^{t_d+\varepsilon}\int_{t_d-s}^{t_d+\varepsilon-s}\int_{Q^\perp_\varepsilon(t^\perp_d)}&f_E(t^\perp_d,s,\zeta^\perp_d,\zeta_d)\bar K^Y_M(\zeta)\,\mathrm{d}\zeta^\perp_d\,\mathrm{d}\zeta_d\,\mathrm{d}s\geq\\&\geq\frac{e^{M(\gamma_M-\varepsilon)}}{\varepsilon^{d-2}}\varepsilon\int_{Q^\perp_\varepsilon(t^\perp_d)}\int_{t_d-\varepsilon}^{t_d}|\chi_{E_{t^\perp_d}}(s)-\chi_{E_{t^\perp_d+\zeta^\perp_d}}(s)|\,\mathrm{d}s\,\mathrm{d}\zeta^\perp_d\\&\geq\frac{e^{M(\gamma_M-\varepsilon)}}{\varepsilon^{d-2}}\varepsilon\int_{Q^\perp_\varepsilon(t^\perp_d)}\int_{t_d-\varepsilon}^{t_d}|1-\chi_{E_{t^\perp_d+\zeta^\perp_d}}(s)|\,\mathrm{d}s\,\mathrm{d}\zeta^\perp_d\\&\geq\frac{e^{M(\gamma_M-\varepsilon)}}{\varepsilon^{d-2}}\varepsilon|Q^\perp_\varepsilon(t^\perp_d)\times(t_d-\varepsilon,t_d)\cap E^c|\geq\frac{7e^{M(\gamma_M-\varepsilon)}\varepsilon^{d+1}}{16\varepsilon^{d-2}},\end{aligned}$$

which tends to $+\infty$ as $M\to+\infty$.

Therefore, for $\tilde M$ sufficiently large depending on ε the r.h.s. of (30) is positive. Up to a permutation of coordinates, this naturally holds also for $i=2,\dots,d-1$. Therefore the lemma is proved.

□

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