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SOME MYERS TYPE THEOREMS AND HITCHIN–THORPE INEQUALITIES FOR SHRINKING RICCI SOLITONS, II

Abstract. This expository paper is a continuation of the previous expository paper on some compactness criteria and the validity of the Hitchin–Thorpe inequality for shrinking Ricci solitons (H. Tadano, Rend. Semin. Mat. Univ. Politec. Torino **73** (2015), 183–199). The Myers type theorems and the validity of the Hitchin–Thorpe inequality for shrinking Ricci solitons presented in the previous expository paper shall be improved. We also introduce diameter bounds and a gap theorem for compact self–shrinkers of the mean curvature flow.

1. Ricci Solitons

In this expository paper, we shall continue the survey in [52] on recent progress of compactness criteria and the validity of the Hitchin–Thorpe inequality for shrinking Ricci solitons. Ricci solitons were introduced by R. Hamilton [23] and are natural generalizations of Einstein manifolds. They correspond to self-similar solutions to the Ricci flow and often arise as singularity models of the flow [6, 24]. The importance of Ricci solitons was demonstrated in a series of three papers by G. Perelman [42, 43, 44], where Ricci solitons played crucial roles in his affirmative solution of the Poincaré conjecture.

DEFINITION 1. A complete Riemannian manifold (M,g) is called a *Ricci soliton* [23] if there exists a vector field $V \in \mathfrak{X}(M)$ satisfying the equation

$$\mathrm{Ric}_g + \frac{1}{2}\mathcal{L}_V g = \lambda g \tag{1.1}$$

for some constant $\lambda \in \mathbb{R}$, where Ric_g denotes the Ricci curvature of (M,g) and $\mathcal{L}_V$ is the Lie derivative in the direction of V. We refer to V as a *potential vector field*. We say that the soliton (M,g) is *shrinking*, *steady*, and *expanding* described as $\lambda > 0, \lambda = 0$, and $\lambda < 0$, respectively. A typical example of Ricci solitons is an Einstein manifold, where V is given by a Killing vector field. In such a case, we say that the soliton is *trivial*. When V is replaced with the gradient vector field ∇f for some smooth function $f : M \to \mathbb{R}$ as $V = \nabla f$, the soliton (M,g) is called a *gradient Ricci soliton*. We refer to f as a *potential function*. Then (1.1) becomes the equation

$$\mathrm{Ric}_g + \mathrm{Hess}\, f = \lambda g, \tag{1.2}$$

where $\mathrm{Hess}\, f$ denotes the Hessian of the potential function f.

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EXAMPLE 1. A typical example of gradient Ricci solitons is the *Gaussian soliton* $(\mathbb{R}^n, g_0)$ which satisfies

$$\mathrm{Ric}_{g_0} + \mathrm{Hess}\, f = \pm\frac{1}{2} g_0,$$

where g_0 is the canonical flat metric on $\mathbb{R}^n$ and the potential function is given by $f(x) = \pm\frac{1}{4}|x|^2$. This is a complete non-compact shrinking or expanding Ricci soliton, respectively.

G. Perelman [42] proved that any potential vector field on compact Ricci solitons must be the sum of a gradient vector field and a Killing vector field. It is well-known now that compact steady and expanding Ricci solitons must be trivial, as well as compact shrinking Ricci solitons in dimensions two and three [6]. Examples of compact non-trivial shrinking Kähler–Ricci solitons have been discovered by N. Koiso [26], H.-D. Cao [5], X.-J. Wang and X. Zhu [60], and F. Podestà and A. Spiro [45]. Examples of non-compact non-trivial Kähler–Ricci solitons have been constructed by M. Feldman, T. Ilmanen, and D. Knopf [15], and A. S. Dancer and M. Y. Wang [13].

2. Some Compactness Theorems

In this section, we shall introduce some improvements on the compactness theorems for complete Riemannian manifolds presented in [52]. Throughout this paper, we assume that (M, g) is an n-dimensional smooth connected Riemannian manifold without boundary. Let $X, Y, Z \in \mathfrak{X}(M)$ be three smooth vector fields on M. For any smooth function $f : M \to \mathbb{R}$, a *gradient vector field* and a *Hessian* of f are defined, respectively, by

$$g(\nabla f, X) = df(X) \quad \text{and} \quad \mathrm{Hess}\, f(X,Y) = g(\nabla_X \nabla f, Y).$$

The *Riemannian curvature*, the *Ricci curvature*, and the *scalar curvature* are defined, respectively, by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

$$\mathrm{Ric}_g(X,Y) = \sum_{i=1}^{n} g(R(e_i, X)Y, e_i), \quad \text{and} \quad R = \sum_{i=1}^{n} \mathrm{Ric}_g(e_i, e_i),$$

where $\{e_i\}_{i=1}^n$ is an orthonormal frame of (M, g).

One of the most important problems in Riemannian geometry is to investigate the relation between topology and geometric structure on Riemannian manifolds. J. Lohkamp [35] proved that in dimensions at least three, any manifold admits a complete Riemannian metric of negative Ricci curvature.

THEOREM 1 (J. Lohkamp [35]). *Any n-dimensional manifold M, $n \geqslant 3$, admits a complete Riemannian metric whose Ricci curvature is everywhere negative.*

Hence, in dimensions at least three, there are no topological obstructions to the existence of a complete Riemannian metric of negative Ricci curvature. However, the situation of Riemannian metrics of positive Ricci curvature is quite different from that of Riemannian metrics of negative Ricci curvature. The celebrated theorem of S. B. Myers [39] implies the compactness of a complete Riemannian manifold under some positive lower bound on the Ricci curvature.

THEOREM 2 (S. B. Myers [39]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some positive constant $\lambda > 0$ such that the Ricci curvature satisfies $\mathrm{Ric}_g \geqslant \lambda g$. Then (M,g) must be compact with finite fundamental group. Moreover, the diameter of (M,g) has the upper bound*

$$\mathrm{diam}(M,g) \leqslant \pi\sqrt{\frac{n-1}{\lambda}}.$$

Theorem 2 may be considered as a topological obstruction to the existence of a complete Riemannian metric whose Ricci curvature is bounded from below by a positive constant. To give a nice compactness criterion for complete Riemannian manifolds is one of the most interesting problems in Riemannian geometry. Theorem 2 has been widely generalized in various directions by many authors, see [1, 4, 9, 21, 22, 37, 62] for example. The first generalization was established by W. Ambrose [1], where the positive lower bound on the Ricci curvature as in Theorem 2 was replaced with some integral condition on the Ricci curvature along some geodesics.

THEOREM 3 (W. Ambrose [1]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some point $p \in M$ for which every geodesic $\gamma : [0,+\infty) \to M$ emanating from p satisfies*

$$\int_0^{+\infty} \mathrm{Ric}_g(\dot{\gamma}(s),\dot{\gamma}(s))ds = +\infty.$$

Then (M,g) must be compact.

On the other hand, motivated by relativistic cosmology, G. J. Galloway [21] proved another generalization by perturbing the positive lower bound on the Ricci curvature as in Theorem 2 by the derivative in the radial direction of some bounded function.

THEOREM 4 (G. J. Galloway [21]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exist some constants $\lambda > 0$ and $L \geqslant 0$ such that for every pair of points in M and minimal geodesic γ joining those points, it holds*

$$\mathrm{Ric}_g(\dot{\gamma},\dot{\gamma})|_{\gamma(s)} \geqslant \lambda + \frac{d\phi}{ds}(s),$$

where ϕ is some smooth function of the arc length satisfying $|\phi| \leqslant L$ along γ. Then (M,g) must be compact. Moreover, the diameter of (M,g) has the upper bound

$$\mathrm{diam}(M,g) \leqslant \frac{\pi}{\lambda}\left(L + \sqrt{L^2 + (n-1)\lambda}\right).$$

Note that by taking $L = 0$, Theorem 4 is reduced to Theorem 2. One of the most important features of Theorem 3 and 4 is that the Ricci curvature is not required to be everywhere non-negative. On the other hand, J. Cheeger, M. Gromov, and M. Taylor [9] established a further generalization of Theorem 2 assuming some asymptotic condition on the Ricci curvature outside some domain of a complete Riemannian manifold.

THEOREM 5 (J. Cheeger, M. Gromov, and M. Taylor [9]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exist some point $p \in M$ and positive constants $r_0 > 0$ and $\nu > 0$ such that*

$$\mathrm{Ric}_g(x) \geqslant (n-1)\frac{\left(\frac{1}{4}+\nu^2\right)}{r^2(x)}g(x) \tag{2.1}$$

for all $x \in M$ satisfying $r(x) \geqslant r_0$, where $r(x)$ denotes the distance between p and x. Then (M,g) must be compact. Moreover, the diameter of (M,g) from p has the upper bound

$$\mathrm{diam}_p(M,g) \leqslant r_0 \exp\left(\frac{\pi}{\nu}\right).$$

Note that Theorem 5 is not true if $\nu = 0$ in (2.1). In fact, J. Cheeger, M. Gromov, and M. Taylor [9] pointed out that a metric on the n-dimensional real space $\mathbb{R}^n$ which is of the form $dr^2 + rg(\theta)$ outside some domain satisfies the asymptotic condition (2.1) with $\nu = 0$, where $g(\theta)$ is the standard metric on the sphere $\mathbb{S}^{n-1}$.

In the previous expository paper [52], we have introduced some generalizations of Theorem 3 and 4. In this section, we shall introduce further generalizations of Theorem 3 and 4, as well as Theorem 5 to the case of smooth metric measure spaces via (m–) Bakry–Émery and (m–) modified Ricci curvatures. We first recall definition of a smooth metric measure space.

DEFINITION 2. A *smooth metric measure space* is a complete Riemannian manifold (M,g) with the weighted volume form $d\mu := e^{-f} d\mathrm{vol}_g$, where $f : M \to \mathbb{R}$ is a smooth function on M and $d\mathrm{vol}_g$ is the Riemannian density with respect to the metric g. For a smooth metric measure space (M,g) and $m \in \mathbb{R} \cup \{\pm\infty\} \setminus \{0\}$, we put

$$\mathrm{Ric}_f := \mathrm{Ric}_g + \mathrm{Hess}\, f \quad \text{and} \quad \mathrm{Ric}_f^m := \mathrm{Ric}_g + \mathrm{Hess}\, f - \frac{1}{m} df \otimes df \tag{2.2}$$

and call them a *Bakry–Émery Ricci curvature* and an *m–Bakry–Émery Ricci curvature*, respectively. We refer to f as a *potential function*. More generally, for a smooth vector field $V \in \mathfrak{X}(M)$ and $m \in \mathbb{R} \cup \{\pm\infty\} \setminus \{0\}$, we put

$$\mathrm{Ric}_V := \mathrm{Ric}_g + \frac{1}{2}\mathcal{L}_V g \quad \text{and} \quad \mathrm{Ric}_V^m := \mathrm{Ric}_g + \frac{1}{2}\mathcal{L}_V g - \frac{1}{m} V^* \otimes V^*,$$

where V^* is the metric dual of V with respect to g. We call them a *modified Ricci curvature* and an *m–modified Ricci curvature*, respectively. We refer to V as a *potential vector field*. Note that if the potential vector field V is replaced with the gradient vector field ∇f for some smooth function $f : M \to \mathbb{R}$ as $V = \nabla f$, then the modified and m–modified Ricci curvatures are reduced to the Bakry–Émery and m–Bakry–Émery Ricci curvatures, respectively. We also put

$$\Delta_f := \Delta_g - \nabla f \cdot \nabla \quad \text{and} \quad \Delta_V := \Delta_g - V \cdot \nabla \tag{2.3}$$

and call them a *Witten–Laplacian* and a *V–Laplacian*, respectively. Here, Δ_g is the Laplacian with respect to g.

Note that if $f : M \to \mathbb{R}$ is a constant function in (2.2) and (2.3), then the (m–) Bakry–Émery Ricci curvature and Witten–Laplacian are reduced to the Ricci curvature and Laplacian, respectively. As in the classical case, for all smooth functions u, v on M with compact support, we have

$$\int_M g(\nabla u, \nabla v) d\mu = -\int_M (\Delta_f u) v d\mu = -\int_M u(\Delta_f v) d\mu.$$

Moreover, D. Bakry and M. Émery [3] proved that for any smooth function u on M, we obtain

$$\frac{1}{2}\Delta_f |\nabla u|^2 = |\operatorname{Hess} u|^2 + \operatorname{Ric}_f(\nabla u, \nabla u) + g(\nabla \Delta_f u, \nabla u), \tag{2.4}$$

which may be regarded as a natural extension of the Bochner–Weitzenböck formula

$$\frac{1}{2}\Delta_g |\nabla u|^2 = |\operatorname{Hess} u|^2 + \operatorname{Ric}_g(\nabla u, \nabla u) + g(\nabla \Delta_g u, \nabla u). \tag{2.5}$$

The (m–) Bakry–Émery and (m–) modified Ricci curvatures have recently received much attention in various areas of mathematics as they naturally extend a lot of interesting theorems via Ricci curvature to the case of smooth metric measure spaces such as eigenvalue estimates [19], Li–Yau Harnack inequalities [30], and comparison theorems [61]. Moreover, they have recently been important tools in the optimal transport theory [58]. Since the (m–) Bakry–Émery and (m–) modified Ricci curvatures are generalizations of the Ricci curvature, a natural question to ask is whether Theorem 2, 3, 4, and 5 remain valid in the case of the (m–) Bakry–Émery and (m–) modified Ricci curvatures.

In this section, we shall introduce some compactness theorems via (m–) Bakry–Émery and (m–) modified Ricci curvatures. Although some of these compactness theorems were already introduced in the previous expository paper [52], we here recall them for the reader's convenience.

2.1. Compactness theorems via Bakry–Émery and modified Ricci curvatures

First, we shall introduce some compactness theorems via Bakry–Émery and modified Ricci curvatures. A natural question to ask is whether we may replace the positive lower bound on the Ricci curvature as in Theorem 2 with a positive lower bound on the

Bakry–Émery and modified Ricci curvatures. However, a positive lower bound on the modified Ricci curvature does not imply the compactness of a complete Riemannian manifold. In fact, the shrinking Gaussian soliton $(\mathbb{R}^n, g_0)$ is non-compact although it has a positive lower bound on the Bakry–Émery Ricci curvature.

G. Wei and W. Wylie [61] established a Myers type theorem via Bakry–Émery Ricci curvature assuming that the potential function is bounded.

THEOREM 6 (G. Wei and W. Wylie [61]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some positive constant $\lambda > 0$ such that the Bakry–Émery Ricci curvature satisfies $\mathrm{Ric}_f \geqslant \lambda g$. If the potential function satisfies $|f| \leqslant H$ for some non-negative constant $H \geqslant 0$, then (M,g) must be compact. Moreover, the diameter of (M,g) has the upper bound*

$$\mathrm{diam}(M,g) \leqslant \pi\sqrt{\frac{n-1}{\lambda}} + \frac{4H}{\sqrt{(n-1)\lambda}}. \tag{2.6}$$

M. Limoncu [34] proved another Myers type theorem by showing

$$\mathrm{diam}(M,g) \leqslant \frac{\pi}{\sqrt{\lambda}}\sqrt{n-1+2\sqrt{2}H} \tag{2.7}$$

under the same assumption as in Theorem 6. Note that the diameter estimate (2.7) is sharper than (2.6) if $H > \frac{(n-1)\pi}{8}(\sqrt{2}\pi - 4)$. On the other hand, the author [49] established a new diameter estimate which is sharper than (2.6) and (2.7) without any assumption.

THEOREM 7 ([49]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some positive constant $\lambda > 0$ such that $\mathrm{Ric}_f \geqslant \lambda g$. If the potential function satisfies $|f| \leqslant H$ for some non-negative constant $H \geqslant 0$, then (M,g) must be compact. Moreover, the diameter of (M,g) has the upper bound*

$$\mathrm{diam}(M,g) \leqslant \frac{\pi}{\sqrt{\lambda}}\sqrt{n-1+\frac{8H}{\pi}}.$$

M. Limoncu [34] also proved a Myers type theorem via Bakry–Émery Ricci curvature assuming some decay condition on the norm of the gradient vector field of the potential function.

THEOREM 8 (M. Limoncu [34]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some positive constant $\lambda > 0$ such that $\mathrm{Ric}_f \geqslant \lambda g$. If there exist some point $p \in M$ and non-negative constant $k \geqslant 0$ such that the potential function satisfies*

$$|\nabla f|(x) \leqslant \frac{k}{r(x)}$$

for all $x \in M \backslash \{p\}$, *where* $r(x)$ *denotes the distance between* p *and* x, *then* (M,g) *must be compact. Moreover, the diameter of* (M,g) *from* p *has the upper bound*

$$\mathrm{diam}_p(M,g) \leqslant \frac{\pi}{\sqrt{\lambda}}\sqrt{n-1+4k}.$$

M. Fernández-López and E. García-Río [16] established a Myers type theorem via modified Ricci curvature assuming that the norm of the potential vector field is bounded.

THEOREM 9 (M. Fernández-López and E. García-Río [16]). *Let* (M,g) *be an* n*-dimensional complete Riemannian manifold. Suppose that there exists some positive constant* $\lambda > 0$ *such that the modified Ricci curvature satisfies* $\mathrm{Ric}_V \geqslant \lambda g$. *Then* (M,g) *is compact if and only if the norm of the potential vector field* $|V|$ *is bounded on* (M,g).

However, no upper diameter estimate was given in Theorem 9. M. Limoncu [33] established an upper diameter estimate assuming some upper bound on the norm of the potential vector field.

THEOREM 10 (M. Limoncu [33]). *Let* (M,g) *be an* n*-dimensional complete Riemannian manifold. Suppose that there exists some positive constant* $\lambda > 0$ *such that* $\mathrm{Ric}_V \geqslant \lambda g$. *If the potential vector field satisfies* $|V| \leqslant K$ *for some non-negative constant* $K \geqslant 0$, *then* (M,g) *must be compact. Moreover, the diameter of* (M,g) *has the upper bound*

$$\mathrm{diam}(M,g) \leqslant \frac{\pi}{\lambda}\left(\frac{K}{\sqrt{2}} + \sqrt{\frac{K^2}{2} + (n-1)\lambda}\right). \tag{2.8}$$

The author [50] improved the diameter estimate (2.8) by showing

$$\mathrm{diam}(M,g) \leqslant \frac{1}{\lambda}\left(2K + \sqrt{4K^2 + (n-1)\lambda\pi^2}\right) \tag{2.9}$$

under the same assumption as in Theorem 10. Recently, by applying the generalized mean curvature comparison theorem to the excess function, J.-Y. Wu [63] established a new diameter estimate which is sharper than (2.8) and (2.9).

THEOREM 11 (J.-Y. Wu [63]). *Let* (M,g) *be an* n*-dimensional complete Riemannian manifold. Suppose that there exists some positive constant* $\lambda > 0$ *such that* $\mathrm{Ric}_V \geqslant \lambda g$. *If the potential vector field satisfies* $|V| \leqslant K$ *for some non-negative constant* $K \geqslant 0$, *then* (M,g) *must be compact. Moreover, the diameter of* (M,g) *has the upper bound*

$$\mathrm{diam}(M,g) \leqslant \frac{1}{\lambda}\left(2K + \pi\sqrt{(n-1)\lambda}\right).$$

An interesting problem on smooth metric measure spaces is to establish Ambrose and Galloway type theorems via (m–) Bakry–Émery and (m–) modified Ricci curvatures. S. Zhang [65] established an Ambrose type theorem via Bakry–Émery Ricci curvature assuming that the potential function has at most linear growth in the distance function.

THEOREM 12 (S. Zhang [65]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some point $p \in M$ for which every geodesic $\gamma : [0,+\infty) \to M$ emanating from p satisfies*

$$\int_0^{+\infty} \mathrm{Ric}_f(\dot\gamma(s),\dot\gamma(s))ds = +\infty.$$

If there exist some constants α and β such that the potential function satisfies $f(x) \leqslant \alpha r(x) + \beta$ for all $x \in M$, where $r(x)$ denotes the distance between p and x, then (M,g) must be compact.

More generally, we may prove an Ambrose type theorem via modified Ricci curvature which may be considered as a generalization of Theorem 9.

THEOREM 13 ([51]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some point $p \in M$ for which every geodesic $\gamma : [0,+\infty) \to M$ emanating from p satisfies*

$$\int_0^{+\infty} \mathrm{Ric}_V(\dot\gamma(s),\dot\gamma(s))ds = +\infty.$$

If the potential vector field satisfies $|V| \leqslant K$ for some non-negative constant $K \geqslant 0$, then (M,g) must be compact.

M. P. Cavalcante, J. Q. Oliveira, and M. S. Santos [7] established a Galloway type theorem via Bakry–Émery Ricci curvature assuming that the potential function is bounded.

THEOREM 14 (M. P. Cavalcante, J. Q. Oliveira, and M. S. Santos [7]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exist some constants $\lambda > 0$ and $L \geqslant 0$ such that for every pair of points in M and minimal geodesic γ joining those points, it holds*

$$\mathrm{Ric}_f(\dot\gamma,\dot\gamma)|_{\gamma(s)} \geqslant \lambda + \frac{d\phi}{ds}(s),$$

where ϕ is some smooth function of the arc length satisfying $|\phi| \leqslant L$ along γ. If the potential function satisfies $|f| \leqslant H$ for some non-negative constant $H \geqslant 0$, then (M,g) must be compact. Moreover, the diameter of (M,g) has the upper bound

$$\mathrm{diam}(M,g) \leqslant \frac{\pi}{\lambda}\left(L + \sqrt{L^2 + \{(n-1) + 2\sqrt{2}H\}\lambda}\right). \tag{2.10}$$

The author [51] improved the diameter estimate (2.10) by showing

$$\mathrm{diam}(M,g) \leqslant \frac{1}{\lambda}\left(2L + \sqrt{4L^2 + \{(n-1)\pi + 8H\}\lambda\pi}\right) \tag{2.11}$$

under the same assumption as in Theorem 14. Note that by taking $L = 0$, Theorem 14 with (2.11) is reduced to Theorem 7. On the other hand, the author [51] proved a Galloway type theorem via modified Ricci curvature assuming some upper bound on the norm of the potential vector field.

THEOREM 15 ([51]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exist some constants $\lambda > 0$ and $L \geqslant 0$ such that for every pair of points in M and minimal geodesic γ joining those points, it holds*

$$\mathrm{Ric}_V(\dot\gamma,\dot\gamma)|_{\gamma(s)} \geqslant \lambda + \frac{d\phi}{ds}(s),$$

where ϕ is some smooth function of the arc length satisfying $|\phi| \leqslant L$ along γ. If the potential vector field satisfies $|V| \leqslant K$ for some non-negative constant $K \geqslant 0$, then (M,g) must be compact. Moreover, the diameter of (M,g) has the upper bound

$$\mathrm{diam}(M,g) \leqslant \frac{1}{\lambda}\left(2(L+K) + \sqrt{4(L+K)^2 + (n-1)\lambda\pi^2}\right).$$

Note that by taking $L = 0$, Theorem 15 is reduced to Theorem 10 with (2.9).

Y. Soylu [48] established a Cheeger–Gromov–Taylor type theorem via Bakry–Émery Ricci curvature assuming that the potential function is bounded.

THEOREM 16 (Y. Soylu [48]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exist some point $p \in M$ and positive constants $r_0 > 0$ and $\nu > 0$ such that*

$$\mathrm{Ric}_f(x) \geqslant (n-1)\frac{\left(\frac{1}{4}+\nu^2\right)}{r^2(x)}g(x)$$

for all $x \in M$ satisfying $r(x) \geqslant r_0$, where $r(x)$ denotes the distance between p and x. If the potential function satisfies $|f| \leqslant (n-1)H$ for some non-negative constant $H \geqslant 0$, then (M,g) must be compact. Moreover, the diameter of (M,g) from p has the upper bound

$$\mathrm{diam}_p(M,g) \leqslant r_0 \exp\left(\frac{1}{\nu^2}\sqrt{\pi^2\nu^2 + 8H^2 + 4H\sqrt{\pi^2\nu^2(1+4\nu^2)+4H^2}}\right).$$

On the other hand, the author [54] proved a Cheeger–Gromov–Taylor type theorem via modified Ricci curvature assuming some decay condition on the norm of the potential vector field.

THEOREM 17 ([54]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exist some point $p \in M$ and positive constants $r_0 > 0$ and $\nu > 0$ such that*

$$\mathrm{Ric}_V(x) \geqslant (n-1)\frac{\left(\frac{1}{4}+\nu^2\right)}{r^2(x)}g(x)$$

for all $x \in M$ *satisfying* $r(x) \geqslant r_0$*, where* $r(x)$ *denotes the distance between* p *and* x*. If there exists some non-negative constant* $k \geqslant 0$ *such that the potential vector field satisfies*

$$|V|(x) \leqslant \frac{(n-1)k}{r(x)} \quad \text{and} \quad k < \nu^2$$

for all $x \in M$ *satisfying* $r(x) \geqslant r_0$*, then* (M,g) *must be compact. Moreover, the diameter of* (M,g) *from* p *has the upper bound*

$$\operatorname{diam}_p(M,g) \leqslant r_0 \exp\left(\frac{2k + \sqrt{4k^2 + (\nu^2 - k)\pi^2}}{\nu^2 - k}\right).$$

An interesting problem is to improve the growth conditions on the norm of the potential vector field as in Theorem 9, 10, 11, and 15. M. Fernández-López and E. García-Río [16] also proved the compactness of a complete Riemannian manifold assuming some growth condition on the norm of the potential vector field in terms of the distance function.

THEOREM 18 (M. Fernández-López and E. García-Río [16]). *Let* (M,g) *be an* n*-dimensional complete Riemannian manifold. Suppose that there exists some positive constant* $\lambda > 0$ *such that* $\operatorname{Ric}_V \geqslant \lambda g$*. If there exist some point* $p \in M$*, constants* α*,* β*, and* $\theta \in (0,1)$ *such that the potential vector field satisfies*

$$|V|(x) \leqslant \alpha r(x)^\theta + \beta$$

for all $x \in M$*, where* $r(x)$ *denotes the distance between* p *and* x*, then* (M,g) *must be compact.*

Theorem 18 was improved by the author [51] by showing that the compactness of a complete Riemannian manifold may still be obtained assuming some linear growth condition on the norm of the potential vector field in terms of the distance function, as well as assuming that the modified Ricci curvature is not required to be everywhere non-negative.

THEOREM 19 ([51]). *Let* (M,g) *be an* n*-dimensional complete Riemannian manifold. Suppose that there exist some constants* $\lambda > 0$ *and* $L \geqslant 0$ *such that for every pair of points in* M *and minimal geodesic* γ *joining those points, it holds*

$$\operatorname{Ric}_V(\dot\gamma,\dot\gamma)|_{\gamma(s)} \geqslant \lambda + \frac{d\phi}{ds}(s),$$

where ϕ *is some smooth function of the arc length satisfying* $\phi \geqslant -L$ *along* γ*. If there exist some point* $p \in M$ *and constants* $\alpha < \lambda$ *and* β *such that the potential vector field satisfies*

$$|V|(x) \leqslant \alpha r(x) + \beta \tag{2.12}$$

for all $x \in M$*, where* $r(x)$ *denotes the distance between* p *and* x*, then* (M,g) *must be compact.*

By taking $L = 0$ and $V = \nabla f$ for some smooth function $f : M \to \mathbb{R}$ in Theorem 19, we may recover a Myers type theorem via Bakry–Émery Ricci curvature due to S. Zhang [65].

THEOREM 20 (S. Zhang [65]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some positive constant $\lambda > 0$ such that $\mathrm{Ric}_f \geqslant \lambda g$. If there exist some point $p \in M$ and constants $\alpha < \frac{\lambda}{2}$ and β such that the potential function satisfies*

$$f(x) \leqslant \alpha(r(x)+\beta)^2 \tag{2.13}$$

for all $x \in M$, where $r(x)$ denotes the distance between p and x, then (M,g) must be compact.

The shrinking Gaussian soliton $(\mathbb{R}^n, g_0)$ satisfies

$$\mathrm{Ric}_{g_0} + \mathrm{Hess}\, f = \frac{1}{2} g_0,$$

where g_0 is the canonical flat metric on $\mathbb{R}^n$ and the potential function is given by $f(x) = \frac{1}{4}|x|^2$. The shrinking Gaussian soliton is an example to show that Theorem 20 is not true if $\alpha = \frac{\lambda}{2}$, since the shrinking Gaussian soliton is non-compact. Hence, the growth condition (2.13) with $\alpha < \frac{\lambda}{2}$ is sharp in obtaining the compactness of a complete Riemannian manifold under a positive lower bound on the Bakry–Émery Ricci curvature. The same observation is also true for (2.12) with $\alpha < \lambda$, since the potential function of the shrinking Gaussian soliton satisfies $|\nabla f|(x) = \frac{1}{2}|x|$.

2.2. Compactness theorems via *m*–Bakry–Émery and *m*–modified Ricci curvatures with positive *m*

Next, we shall introduce some compactness theorems via m–Bakry–Émery and m–modified Ricci curvatures with positive m. It is known that the m–Bakry–Émery and m–modified Ricci curvatures with positive m are more natural generalizations of the Ricci curvature than the Bakry–Émery and modified Ricci curvatures. Z. Qian [46] established a Myers type theorem via m–Bakry–Émery Ricci curvature with positive m without making any assumption on the potential function.

THEOREM 21 (Z. Qian [46]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some positive constant $\lambda > 0$ such that the m–Bakry–Émery Ricci curvature satisfies $\mathrm{Ric}_f^m \geqslant \lambda g$, where $m \in (0,+\infty)$. Then (M,g) must be compact. Moreover, the diameter of (M,g) has the upper bound*

$$\mathrm{diam}(M,g) \leqslant \pi\sqrt{\frac{n+m-1}{\lambda}}.$$

M. Limoncu [33] generalized Theorem 21 to the case of the m–modified Ricci curvature with positive m. Note that no condition on the potential vector field was assumed.

THEOREM 22 (M. Limoncu [33]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some positive constant $\lambda > 0$ such that the m–modified Ricci curvature satisfies $\mathrm{Ric}_V^m \geqslant \lambda g$, where $m \in (0,+\infty)$. Then (M,g) must be compact. Moreover, the diameter of (M,g) has the upper bound*

$$\mathrm{diam}(M,g) \leqslant \pi\sqrt{\frac{n+m-1}{\lambda}}.$$

On the other hand, M. P. Cavalcante, J. Q. Oliveira, and M. S. Santos [7] established an Ambrose type theorem via m–Bakry–Émery Ricci curvature with positive m.

THEOREM 23 (M. P. Cavalcante, J. Q. Oliveira, and M. S. Santos [7]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some point $p \in M$ for which every geodesic $\gamma : [0,+\infty) \to M$ emanating from p satisfies*

$$\int_0^{+\infty} \mathrm{Ric}_f^m(\dot\gamma(s),\dot\gamma(s))ds = +\infty,$$

where $m \in (0,+\infty)$. Then (M,g) must be compact.

The key ingredient in proving Theorem 23 is the Riccati inequality

$$\mathrm{Ric}_f^m(\partial_r,\partial_r) \leqslant -\dot{\mathfrak{m}}_f - \frac{(\mathfrak{m}_f)^2}{n+m-1},$$

which may be derived by applying the Bochner–Weitzenböck formula (2.4) to the distance function $r(x) := d(p,x)$. Here $\mathfrak{m}_f := \Delta_f r$. Recently, Y. Li [31] proved that for any smooth function u on M, we have

$$\frac{1}{2}\Delta_V|\nabla u|^2 = |\,\mathrm{Hess}\,u|^2 + \mathrm{Ric}_V(\nabla u,\nabla u) + g(\nabla\Delta_V u,\nabla u). \tag{2.14}$$

Note that if the potential vector field V is replaced with the gradient vector field ∇f for some smooth function $f : M \to \mathbb{R}$ as $V = \nabla f$, then (2.14) is reduced to the Bochner–Weitzenböck formula (2.4). By applying the Bochner–Weitzenböck formula (2.14) to the distance function $r(x) := d(p,x)$, we may obtain the Riccati inequality

$$\mathrm{Ric}_V^m(\partial_r,\partial_r) \leqslant -\dot{\mathfrak{m}}_V - \frac{(\mathfrak{m}_V)^2}{n+m-1},$$

where $\mathfrak{m}_V := \Delta_V r$. By using this Riccati inequality, we may prove an Ambrose type theorem via m–modified Ricci curvature with positive m quite similarly to Theorem 23.

THEOREM 24 ([51]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some point $p \in M$ for which every geodesic $\gamma : [0,+\infty) \to M$ emanating from p satisfies*

$$\int_0^{+\infty} \mathrm{Ric}_V^m(\dot\gamma(s),\dot\gamma(s))ds = +\infty,$$

where $m \in (0,+\infty)$. Then (M,g) must be compact.

M. Rimoldi [47] established a Galloway type theorem via m–Bakry–Émery Ricci curvature with positive m.

THEOREM 25 (M. Rimoldi [47]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exist some constants $\lambda > 0$ and $L \geqslant 0$ such that for every pair of points in M and minimal geodesic γ joining those points, it holds*

$$\mathrm{Ric}_f^m(\dot\gamma,\dot\gamma)|_{\gamma(s)} \geqslant \lambda + \frac{d\phi}{ds}(s),$$

where ϕ is some smooth function of the arc length satisfying $|\phi| \leqslant L$ along γ and $m \in (0,+\infty)$. Then (M,g) must be compact. Moreover, the diameter of (M,g) has the upper bound

$$\mathrm{diam}(M,g) \leqslant \frac{1}{\lambda}\left(2L + \sqrt{4L^2 + (n+m-1)\lambda\pi^2}\right).$$

On the other hand, M. P. Cavalcante, J. Q. Oliveira, and M. S. Santos [7] proved a Galloway type theorem via m–modified Ricci curvature with positive m.

THEOREM 26 (M. P. Cavalcante, J. Q. Oliveira, and M. S. Santos [7]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exist some constants $\lambda > 0$ and $L \geqslant 0$ such that for every pair of points in M and minimal geodesic γ joining those points, it holds*

$$\mathrm{Ric}_V^m(\dot\gamma,\dot\gamma)|_{\gamma(s)} \geqslant \lambda + \frac{d\phi}{ds}(s),$$

where ϕ is some smooth function of the arc length satisfying $|\phi| \leqslant L$ along γ and $m \in (0,+\infty)$. Then (M,g) must be compact. Moreover, the diameter of (M,g) has the upper bound

$$\mathrm{diam}(M,g) \leqslant \frac{\pi}{\lambda}\left(L + \sqrt{L^2 + (n+m-1)\lambda}\right). \tag{2.15}$$

The author [51] improved the diameter estimate (2.15) by showing

$$\mathrm{diam}(M,g) \leqslant \frac{1}{\lambda}\left(2L + \sqrt{4L^2 + (n+m-1)\lambda\pi^2}\right)$$

under the same assumption as in Theorem 26. Note that by taking $L = 0$, Theorem 26 is reduced to Theorem 22.

L. F. Wang [59] established a Cheeger–Gromov–Taylor type theorem via m–Bakry–Émery Ricci curvature with positive m.

THEOREM 27 (L. F. Wang [59]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exist some point $p \in M$ and positive constants $r_0 > 0$ and $\nu > 0$ such that*

$$\mathrm{Ric}_f^m(x) \geqslant (n+m-1)\frac{\left(\frac{1}{4}+\nu^2\right)}{(1+r(x))^2}g(x)$$

for all $x \in M$*, where* $r(x)$ *denotes the distance between* p *and* x*, and* $m \in (0, +\infty)$*. Then* (M,g) *must be compact. Moreover, the diameter of* (M,g) *from* p *has the upper bound*

$$\mathrm{diam}_p(M,g) < \exp\left(\frac{2\pi}{\nu}\right) - 1.$$

The author [54] proved a Cheeger–Gromov–Taylor type theorem via m–modified Ricci curvature with positive m.

THEOREM 28 ([54]). *Let* (M,g) *be an* n*-dimensional complete Riemannian manifold. Suppose that there exist some point* $p \in M$ *and positive constants* $r_0 > 0$ *and* $\nu > 0$ *such that*

$$\mathrm{Ric}_V^m(x) \geqslant (n+m-1)\frac{\left(\frac{1}{4}+\nu^2\right)}{r^2(x)}g(x)$$

for all $x \in M$ *satisfying* $r(x) \geqslant r_0$*, where* $r(x)$ *denotes the distance between* p *and* x*, and* $m \in (0, +\infty)$*. Then* (M,g) *must be compact. Moreover, the diameter of* (M,g) *from* p *has the upper bound*

$$\mathrm{diam}_p(M,g) \leqslant r_0 \exp\left(\frac{\pi}{\nu}\right).$$

2.3. Compactness theorems via m–Bakry–Émery and m–modified Ricci curvatures with negative m

Traditionally, the m–Bakry–Émery and m–modified Ricci curvatures have been investigated when m is a positive constant or infinity. However, the m–Bakry–Émery and m–modified Ricci curvatures with negative m have recently been investigated [27, 38, 40, 41, 64]. W. Wylie [64] established a Myers type theorem via m–Bakry–Émery Ricci curvature with negative m, which may be compared with Theorem 6 and 7.

THEOREM 29 (W. Wylie [64]). *Let* (M,g) *be an* n*-dimensional complete Riemannian manifold. Suppose that there exists some positive constant* $\lambda > 0$ *such that*

$$\mathrm{Ric}_f^{-(n-1)} := \mathrm{Ric}_g + \mathrm{Hess}\, f + \frac{1}{n-1} df \otimes df \geqslant (n-1)\lambda g.$$

If the potential function f *is bounded, then* (M,g) *must be compact. Moreover, the diameter of* (M,g) *has the upper bound*

$$\mathrm{diam}(M,g) \leqslant \frac{\pi}{\sqrt{\lambda}}\left(\frac{u_{\max}}{u_{\min}}\right)^{\frac{1}{n-1}},$$

where $u := \exp(f)$*, and* $u_{\max}$ *and* $u_{\min}$ *denote, respectively, the maximum and minimum values of* u*.*

The key ingredient in proving Theorem 29 is the Riccati type inequality

$$v^2 \operatorname{Ric}_V^{-(n-1)}(\partial_r, \partial_r) \leqslant -\partial_r \left(v^2 \Delta_V r\right) - v^2 \frac{(\Delta_V r)^2}{n-1} \tag{2.16}$$

due to W. Wylie [64], which may be derived by applying the Bochner–Weitzenböck formula (2.14) to the distance function $r(x) := d(p,x)$, where

$$v := \exp\left(\frac{f_\gamma}{n-1}\right), \quad f_\gamma(t) := \int_0^t g_{\gamma(s)}(V(\gamma(s)), \dot{\gamma}(s))ds,$$

and $\gamma = \gamma(s)$ is a fixed geodesic. By using this Riccati type inequality, we may prove a Myers type theorem via m–modified Ricci curvature with negative m, assuming some decay condition on the norm of the potential vector field.

THEOREM 30 ([55]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some positive constant $\lambda > 0$ such that*

$$\operatorname{Ric}_V^{-(n-1)} := \operatorname{Ric}_g + \frac{1}{2}\mathcal{L}_V g + \frac{1}{n-1} V^* \otimes V^* \geqslant (n-1)\lambda g. \tag{2.17}$$

If there exist some point $p \in M$ and constants $k \geqslant 0$ and $l > 0$ such that the potential vector field satisfies $|V|(x) \leqslant k\exp(-lr(x))$ for all $x \in M$, where $r(x)$ denotes the distance between p and x, then (M,g) must be compact. Moreover, the diameter of (M,g) has the upper bound

$$\operatorname{diam}(M,g) \leqslant \frac{\pi}{\sqrt{\lambda}} \left(\exp\left(\frac{k}{l}\right)\right)^{\frac{2}{n-1}}.$$

A natural problem is to generalize Theorem 29 and 30 by establishing Ambrose, Galloway, and Cheeger–Gromov–Taylor type theorems via m–Bakry–Émery and m–modified Ricci curvatures with negative m. The key ingredient in proving Theorem 3 is the Riccati inequality

$$\operatorname{Ric}_g(\partial_r, \partial_r) \leqslant -\dot{\mathfrak{m}} - \frac{\mathfrak{m}^2}{n-1},$$

which may be derived by applying the Bochner–Weitzenböck formula (2.5) to the distance function $r(x) := d(p,x)$. Here $\mathfrak{m} := \Delta_g r$ is the mean curvature. See [8] for details. By using the Riccati type inequality (2.16), the author [55] proved an Ambrose type theorem via m–Bakry–Émery Ricci curvature with negative m, which may be compared with Theorem 29.

THEOREM 31 ([55]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some point $p \in M$ for which every geodesic $\gamma : [0,+\infty) \to M$ emanating from p satisfies*

$$\int_0^{+\infty} \operatorname{Ric}_f^{-(n-1)}(\dot{\gamma}(s), \dot{\gamma}(s))ds = +\infty.$$

If the potential function f is bounded and $\operatorname{Ric}_f^{-(n-1)} > 0$, then (M,g) must be compact.

We may prove an Ambrose type theorem via m–modified Ricci curvature with negative m assuming some decay condition on the norm of the potential vector field.

THEOREM 32 ([55]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some point $p\in M$ for which every geodesic $\gamma:[0,+\infty)\to M$ emanating from p satisfies*

$$\int_0^{+\infty}\mathrm{Ric}_V^{-(n-1)}(\dot\gamma(s),\dot\gamma(s))ds=+\infty.$$

If there exist some constants $k\geqslant 0$ and $l>0$ such that the potential vector field satisfies $|V|(x)\leqslant k\exp(-lr(x))$ for all $x\in M$, where $r(x)$ denotes the distance between p and x, and $\mathrm{Ric}_V^{-(n-1)}>0$, then (M,g) must be compact.

On the other hand, we may prove a Galloway type theorem via m–modified Ricci curvature with negative m.

THEOREM 33 ([55]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exist some constants $\lambda>0$ and $L\geqslant 0$ such that for every pair of points in M and minimal geodesic γ joining those points, it holds*

$$\mathrm{Ric}_V^m(\dot\gamma,\dot\gamma)|_{\gamma(s)}\geqslant\lambda+\frac{d\phi}{ds}(s),$$

where ϕ is some smooth function of the arc length satisfying $|\phi|\leqslant L$ along γ and $m\in(-\infty,0)$. If the potential vector field satisfies $|V|\leqslant K$ for some non-negative constant $K\geqslant 0$ such that $0\leqslant K<\sqrt{-\lambda m}$, then (M,g) must be compact. Moreover, the diameter of (M,g) has the upper bound

$$\mathrm{diam}(M,g)\leqslant\frac{2m(L+K)-\sqrt{4m^2(L+K)^2+(n-1)(m^2\lambda+mK^2)\pi^2}}{m\lambda+K^2}.$$

By taking $m=-(n-1)$ and $V=\nabla f$ for some smooth function $f:M\to\mathbb{R}$ in Theorem 33, we may obtain a Galloway type theorem via m–Bakry–Émery Ricci curvature with negative m, which may be compared with Theorem 29.

COROLLARY 1 ([55]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exist some constants $\lambda>0$ and $L\geqslant 0$ such that for every pair of points in M and minimal geodesic γ joining those points, it holds*

$$\mathrm{Ric}_f^{-(n-1)}(\dot\gamma,\dot\gamma)|_{\gamma(s)}\geqslant\lambda+\frac{d\phi}{ds}(s),$$

where ϕ is some smooth function of the arc length satisfying $|\phi|\leqslant L$ along γ. If the potential function satisfies $|\nabla f|\leqslant K$ for some non-negative constant $K\geqslant 0$ such that $0\leqslant K<\sqrt{\lambda(n-1)}$, then (M,g) must be compact. Moreover, the diameter of (M,g) has the upper bound

$$\mathrm{diam}(M,g)\leqslant\frac{2(n-1)(L+K)+(n-1)\sqrt{4(L+K)^2+((n-1)\lambda-K^2)\pi^2}}{(n-1)\lambda-K^2}.$$

We may prove a Cheeger–Gromov–Taylor type theorem via m–modified Ricci curvature with negative m assuming some decay condition on the norm of the potential vector field.

THEOREM 34 ([55]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exist some point $p \in M$ and positive constants $r_0 > 0$ and $\nu > 0$ such that*

$$\mathrm{Ric}_V^m(x) \geqslant (n-1)\frac{\left(\frac{1}{4}+\nu^2\right)}{r^2(x)} g(x)$$

for all $x \in M$ satisfying $r(x) \geqslant r_0$, where $r(x)$ denotes the distance between p and x, and $m \in (-\infty, 0)$. If there exists some non-negative constant $k \geqslant 0$ such that the potential vector field satisfies

$$|V|(x) \leqslant \frac{(n-1)k}{r(x)} \quad \textit{and} \quad k < \frac{m+\sqrt{m^2-4(n-1)m\nu^2}}{2(n-1)}$$

for all $x \in M$ satisfying $r(x) \geqslant r_0$, then (M,g) must be compact. Moreover, the diameter of (M,g) from p has the upper bound

$$\mathrm{diam}_p(M,g) \leqslant r_0 \exp\left(\frac{2mk-\sqrt{4m^2k^2+(m^2\nu^2-m^2k+(n-1)mk^2)\pi^2}}{m\nu^2-mk+(n-1)k^2}\right).$$

By taking $m = -(n-1)$ and $V = \nabla f$ for some smooth function $f : M \to \mathbb{R}$ in Theorem 34, we may obtain a Cheeger–Gromov–Tylor type theorem via m–Bakry–Émery Ricci curvature with negative m, which may be compared with Theorem 29.

COROLLARY 2 ([55]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exist some point $p \in M$ and positive constants $r_0 > 0$ and $\nu > 0$ such that*

$$\mathrm{Ric}_f^{-(n-1)}(x) \geqslant (n-1)\frac{\left(\frac{1}{4}+\nu^2\right)}{r^2(x)} g(x)$$

for all $x \in M$ satisfying $r(x) \geqslant r_0$, where $r(x)$ denotes the distance between p and x. If there exists some non-negative constant $k \geqslant 0$ such that the potential function satisfies

$$|\nabla f|(x) \leqslant \frac{(n-1)k}{r(x)} \quad \textit{and} \quad k < \frac{-1+\sqrt{1+4\nu^2}}{2}$$

for all $x \in M$ satisfying $r(x) \geqslant r_0$, then (M,g) must be compact. Moreover, the diameter of (M,g) from p has the upper bound

$$\mathrm{diam}_p(M,g) \leqslant r_0 \exp\left(\frac{2k+\sqrt{4k^2+(\nu^2-k-k^2)\pi^2}}{\nu^2-k-k^2}\right).$$

We here summarize Myers, Ambrose, Galloway, and Cheeger–Gromov–Taylor type theorems introduced in Section 2.

Curvature \ Type	Myers	Ambrose	Galloway	Cheeger–Gromov–Taylor
Ric_g	Theorem 2	Theorem 3	Theorem 4	Theorem 5
Ric_f	Theorem 6, 7	Theorem 12	Theorem 14	Theorem 16
Ric_V	Theorem 9, 10, 11	Theorem 13	Theorem 15	Theorem 17
Ric_f^m $(m>0)$	Theorem 21	Theorem 23	Theorem 25	Theorem 27
Ric_V^m $(m>0)$	Theorem 22	Theorem 24	Theorem 26	Theorem 28
Ric_f^m $(m<0)$	Theorem 29	Theorem 31	Corollary 1	Corollary 2
Ric_V^m $(m<0)$	Theorem 30	Theorem 32	Theorem 33	Theorem 34

3. Diameter Bounds for Compact Ricci Solitons

In this section, we shall introduce some lower and upper diameter bounds for compact shrinking Ricci solitons. Although these diameter bounds were already introduced in the previous expository paper [52], we here recall them to compare with a lower diameter bound for compact self–shrinkers of the mean curvature flow which shall be introduced in Section 5.

We recall some useful facts on gradient Ricci solitons. Since any compact Ricci soliton must be a gradient Ricci soliton, we here assume that (M,g) is a gradient Ricci soliton satisfying (1.2). First, by taking the trace of (1.2), we obtain

$$R+\Delta_g f = n\lambda. \tag{3.1}$$

By combining (3.1) and the second Bianchi identity, we may obtain a nice formula between the potential function and the scalar curvature.

PROPOSITION 1 (R. Hamilton [24]). *Let (M,g) be an n-dimensional gradient Ricci soliton satisfying* (1.2). *Then the potential function satisfies*

$$R+|\nabla f|^2-2\lambda f = C \tag{3.2}$$

for some real constant C.

We denote by $f_{\max}$ and $f_{\min}$ the maximum and minimum values of the potential function on a compact gradient Ricci soliton, respectively. We also denote by $R_{\max}$ and $R_{\min}$ the maximum and minimum values of the scalar curvature on a compact Ricci soliton, respectively.

PROPOSITION 2 (M. Fernández-López and E. García-Río [17]). *Let (M,g) be an n-dimensional compact shrinking Ricci soliton satisfying* (1.2)*. Then the potential function satisfies*

$$2\lambda f_{\max} - 2\lambda f_{\min} \geqslant R_{\max} - n\lambda. \tag{3.3}$$

Moreover, if the soliton has positive Ricci curvature, then

$$2\lambda f_{\max} - 2\lambda f_{\min} = R_{\max} - R_{\min}. \tag{3.4}$$

By combining the maximum principle for the Laplacian and (3.2), we may obtain a gradient estimate of the potential function on a compact shrinking Ricci soliton.

PROPOSITION 3 (M. Fernández-López and E. García-Río [18]). *Let (M,g) be an n-dimensional compact shrinking Ricci soliton satisfying* (1.2)*. Then the potential function satisfies*

$$|\nabla f|^2 \leqslant R_{\max} - R. \tag{3.5}$$

3.1. Lower diameter bounds

From now on, we assume that the dimension of a Ricci soliton is bigger than or equal to four. Diameter bounds for compact shrinking Ricci solitons have recently been investigated by many authors [2, 12, 17, 19, 20, 49, 50, 53]. In particular, a lower diameter bound for compact non-trivial shrinking Ricci solitons was first investigated by M. Fernández-López and E. García-Río [17] in terms of the Ricci curvature and the range of the potential function.

THEOREM 35 (M. Fernández-López and E. García-Río [17]). *Let (M,g) be an n-dimensional compact non-trivial shrinking Ricci soliton satisfying* (1.2)*. Then the diameter of (M,g) has the lower bound*

$$\operatorname{diam}(M,g) \geqslant \max\left\{\sqrt{\frac{2(f_{\max}-f_{\min})}{C-\lambda}}, \sqrt{\frac{2(f_{\max}-f_{\min})}{\lambda-c}}, 2\sqrt{\frac{2(f_{\max}-f_{\min})}{C-c}}\right\}. \tag{3.6}$$

In Theorem 35 and throughout this paper, the numbers

$$C := \max_{v\in TM}\{\mathrm{Ric}_g(v,v) : |v| = 1\} \quad \text{and} \quad c := \min_{v\in TM}\{\mathrm{Ric}_g(v,v) : |v| = 1\}$$

denote, respectively, the maximum and minimum values of the Ricci curvature on the unit sphere bundle over (M,g). Note that $cg \leqslant \mathrm{Ric}_g \leqslant Cg$.

When the soliton has positive Ricci curvature, the diameter estimate (3.6) may be written in terms of the range of the scalar curvature.

COROLLARY 3 (M. Fernández-López and E. García-Río [17]). *Let (M,g) be an n-dimensional compact non-trivial shrinking Ricci soliton satisfying* (1.2)*. Suppose that* *the soliton has positive Ricci curvature. Then the diameter of* (M,g) *has the lower bound*

$$\operatorname{diam}(M,g) \geqslant \max\left\{\sqrt{\frac{R_{\max}-R_{\min}}{\lambda(C-\lambda)}},\sqrt{\frac{R_{\max}-R_{\min}}{\lambda(\lambda-c)}},2\sqrt{\frac{R_{\max}-R_{\min}}{\lambda(C-c)}}\right\}. \tag{3.7}$$

Based on the lower diameter bound (3.7), M. Fernández-López and E. García-Río [18] proved that if a compact Ricci soliton is sufficiently close to an Einstein manifold, then the soliton must be Einstein.

COROLLARY 4 (M. Fernández-López and E. García-Río [18]). *Let* (M,g) *be an n-dimansional compact shrinking Ricci soliton satisfying* (1.2). *Then* (M,g) *is trivial if and only if one of the following conditions holds:*

(1) $\operatorname{Ric}_g \geqslant \left(1-\dfrac{R_{\max}-R_{\min}}{(n-1)\lambda\pi^2+R_{\max}-R_{\min}}\right)\lambda g,$

(2) $cg \leqslant \operatorname{Ric}_g \leqslant \left(\lambda+\dfrac{c\,(R_{\max}-R_{\min})}{(n-1)\lambda\pi^2}\right) g$ *for some* $c>0$,

(3) $cg \leqslant \operatorname{Ric}_g \leqslant \left(1+\dfrac{4\,(R_{\max}-R_{\min})}{(n-1)\lambda\pi^2}\right) cg$ *for some* $c>0$.

Hence, Corollary 4 gives us a gap phenomenon between Einstein manifolds and non-trivial Ricci solitons. M. Fernández-López and E. García-Río [18] also proved a similar gap theorem in terms of the L^2-norm of the gradient vector field of the potential function.

THEOREM 36 (M. Fernández-López and E. García-Río [18]). *Let* (M,g) *be an n-dimensional compact shrinking Ricci soliton satisfying* (1.2). *Then* (M,g) *is trivial if and only if*

$$|\operatorname{Ric}_g-\lambda g| \leqslant \frac{-\Lambda+\sqrt{\Lambda^2+8(n-1)\lambda\Lambda}}{4(n-1)},$$

where $\Lambda := \frac{1}{\operatorname{vol}(M,g)}\int_M |\nabla f|^2$ *denotes the average of the* L^2*-norm of* ∇f.

Note that H. Li [29] proved a similar gap theorem to Theorem 36 for compact Kähler–Ricci solitons. We may also prove a gap theorem for compact Ricci solitons in terms of the scalar curvature. If a compact Ricci soliton is trivial, then the scalar curvature satisfies $R=n\lambda$, and therefore we have $R_{\max}=n\lambda$. We may characterize a compact Ricci soliton to be trivial by giving some upper bound on $R_{\max}-n\lambda$ in terms of the L^2-norm of the gradient vector field of the potential function.

THEOREM 37 (M. Fernández-López and E. García-Río [18]). *Let* (M,g) *be an n-dimensional compact shrinking Ricci soliton satisfying* (1.2). *Then* (M,g) *is trivial if and only if*

$$R_{\max}-n\lambda \leqslant \left(1+\frac{2}{n}\right)\frac{1}{\operatorname{vol}(M,g)}\int_M |\nabla f|^2.$$

On the other hand, a universal lower diameter bound for compact non-trivial shrinking Ricci solitons was first established by A. Futaki and Y. Sano [20] in the relation to the study of the first non-zero eigenvalue of the Witten–Laplacian.

THEOREM 38 (A. Futaki and Y. Sano [20]). *Let (M,g) be an n-dimensional compact non-trivial shrinking Ricci soliton satisfying (1.2). Then the diameter of (M,g) has the lower bound*

$$\mathrm{diam}(M,g) \geqslant \frac{10\pi}{13\sqrt{\lambda}}. \tag{3.8}$$

The lower diameter estimate (3.8) has been improved by many authors [2, 12, 19]. In particular, a sharper lower diameter estimate was given independently by Y. Chu and Z. Hu [12] and by A. Futaki, H. Li, and X.-D. Li [19].

THEOREM 39 (Y. Chu and Z. Hu [12] and A. Futaki, H. Li, and X.-D. Li [19]). *Let (M,g) be an n-dimensional compact non-trivial shrinking Ricci soliton satisfying (1.2). Then the diameter of (M,g) has the lower bound*

$$\mathrm{diam}(M,g) \geqslant \frac{2(\sqrt{2}-1)\pi}{\sqrt{\lambda}}. \tag{3.9}$$

Theorem 39 says that if the diameter of a compact shrinking Ricci soliton (M,g) satisfies the inequality

$$\mathrm{diam}(M,g) < \frac{2(\sqrt{2}-1)\pi}{\sqrt{\lambda}},$$

then the soliton (M,g) must be trivial. Hence, Theorem 39 gives us a gap phenomenon between Einstein manifolds and non-trivial Ricci solitons. An interesting problem is to improve the diameter estimate (3.9). Base on the gradient estimate (3.5), we may prove a lower diameter bound for compact shrinking Ricci solitons in terms of the scalar curvature.

THEOREM 40 ([53]). *Let (M,g) be an n-dimensional compact non-trivial shrinking Ricci soliton satisfying (1.2). Then the diameter of (M,g) has the lower bound*

$$\mathrm{diam}(M,g) \geqslant \frac{R_{\max}-n\lambda}{2\lambda\sqrt{R_{\max}-R_{\min}}}. \tag{3.10}$$

Note that if the maximum value of the scalar curvature is sufficiently large, then the diameter estimate (3.10) is sharper than (3.9).

3.2. Upper diameter bounds

Theorem 10 and 11 are closely related to an upper diameter bound for compact shrinking Ricci solitons. Theorem 35 and Corollary 3 give a lower diameter bound for compact non-trivial shrinking Ricci solitons in terms of the range of the potential function, as well as in terms of the range of the scalar curvature. Stimulated by these lower diameter bounds, M. Fernández-López and E. García-Río [17] conjectured that an upper diameter bound for compact shrinking Ricci solitons would also be given in terms of the range of the potential function, as well as in terms of the range of the scalar curvature. By combining the diameter estimate (2.9) and the gradient estimate (3.5), we may give an upper diameter bound for compact shrinking Ricci solitons in terms of the range of the scalar curvature.

THEOREM 41 ([50]). *Let (M,g) be an n-dimensional compact shrinking Ricci soliton satisfying (1.2). Then the diameter of (M,g) has the upper bound*

$$\operatorname{diam}(M,g) \leqslant \frac{1}{\lambda}\left(2\sqrt{R_{\max}-R_{\min}}+\sqrt{4(R_{\max}-R_{\min})+(n-1)\lambda\pi^2}\right).$$

By combining Theorem 11 and the gradient estimate (3.5), J.-Y. Wu [63] gave a sharper diameter estimate for compact shrinking Ricci solitons.

THEOREM 42 (J.-Y. Wu [63]). *Let (M,g) be an n-dimensional compact shrinking Ricci soliton satisfying (1.2). Then the diameter of (M,g) has the upper bound*

$$\operatorname{diam}(M,g) \leqslant \frac{2}{\lambda}\sqrt{R_{\max}-R_{\min}}+\pi\sqrt{\frac{n-1}{\lambda}}. \tag{3.11}$$

Thanks to the relation (3.4), when the soliton has positive Ricci curvature, the diameter estimate (3.11) may be written in terms of the range of the potential function.

COROLLARY 5. *Let (M,g) be an n-dimensional compact shrinking Ricci soliton satisfying (1.2). Suppose that the soliton has positive Ricci curvature. Then the diameter of (M,g) has the upper bound*

$$\operatorname{diam}(M,g) \leqslant 2\sqrt{\frac{2(f_{\max}-f_{\min})}{\lambda}}+\pi\sqrt{\frac{n-1}{\lambda}}.$$

4. The Hitchin–Thorpe Inequality for Compact Ricci Solitons

In this section, we shall introduce some validities of the Hitchin–Thorpe inequality for four-dimensional compact shrinking Ricci solitons. J. A. Thorpe [57] and N. Hitchin [25] proved independently that if a four-dimensional compact manifold M admits an Einstein metric, then the Euler number $\chi(M)$ and signature $\tau(M)$ must satisfy the inequality

$$2\chi(M) \geqslant 3|\tau(M)|.$$

This inequality is known as the *Hitchin–Thorpe inequality* and has various geometric implications. For instance, if a four-dimensional compact manifold M does not satisfy the inequality, then M never admit any Einstein metric. Hence, the Hitchin–Thorpe inequality may be considered as a topological obstruction to the existence of an Einstein metric on a four-dimensional compact manifold. On the other hand, C. LeBrun

[28] proved that there are infinitely many four-dimensional compact simply connected Riemannian manifolds which do not admit any Einstein metric, but nevertheless satisfy the *strict Hitchin–Thorpe inequality*

$$2\chi(M) > 3|\tau(M)|.$$

Since a Ricci soliton is a natural generalization of an Einstein manifold, we may expect some topological obstruction to the existence of a four-dimensional compact Ricci soliton. The validity of the Hitchin–Thorpe inequality for four-dimensional compact shrinking Ricci solitons was first investigated by L. Ma [36] assuming some upper bound on the L^2-norm of the scalar curvature.

THEOREM 43 (L. Ma [36]). *Let (M,g) be a four-dimensional compact shrinking Ricci soliton satisfying (1.2). If the scalar curvature satisfies*

$$\int_M R^2 \leqslant 24\lambda^2 \mathrm{vol}(M,g),$$

then the soliton must satisfy the Hitchin–Thorpe inequality.

M. Fernández-López and E. García-Río [17] proved the validity of the Hitchin–Thorpe inequality assuming some upper diameter bound in terms of the Ricci curvature.

THEOREM 44 (M. Fernández-López and E. García-Río [17]). *Let (M,g) be a four-dimensional compact non-trivial shrinking Ricci soliton satisfying (1.2). If the diameter of (M,g) has the upper bound*

$$\mathrm{diam}(M,g) \leqslant \max\left\{\sqrt{\frac{2}{C-\lambda}}, \sqrt{\frac{2}{\lambda-c}}, 2\sqrt{\frac{2}{C-c}}\right\},$$

then the soliton must satisfy the Hitchin–Thorpe inequality.

We may also prove the validity of the Hitchin–Thorpe inequality assuming some upper diameter bound in terms of the scalar curvature.

THEOREM 45 ([53]). *Let (M,g) be a four-dimensional compact non-trivial shrinking Ricci soliton satisfying (1.2). If the diameter of (M,g) has the upper bound*

$$\mathrm{diam}(M,g) \leqslant \frac{2\sqrt{2}(\sqrt{2}-1)\pi}{\sqrt{R_{\max}-R_{\min}}},$$

then the soliton must satisfy the Hitchin–Thorpe inequality.

On the other hand, the validity of the Hitchin–Thorpe inequality for four-dimensional compact shrinking Ricci solitons may also be obtained assuming some lower diameter bound in terms of the scalar curvature.

THEOREM 46 ([50]). *Let (M,g) be a four-dimensional compact shrinking Ricci soliton satisfying (1.2). If the diameter of (M,g) has the lower bound*

$$\left(2+\sqrt{4+\frac{3}{2}\pi^2}\right)\frac{\sqrt{R_{\max}-R_{\min}}}{\lambda}\leqslant \operatorname{diam}(M,g), \tag{4.1}$$

then the soliton must satisfy the Hitchin–Thorpe inequality.

Theorem 46 follows from Theorem 41 and 43. By combining Theorem 42 and 43, we may prove another validity of the Hitchin–Thorpe inequality for four-dimensional compact shrinking Ricci solitons.

THEOREM 47 ([56]). *Let (M,g) be a four-dimensional compact shrinking Ricci soliton satisfying (1.2). If the diameter of (M,g) has the lower bound*

$$\left(2+\frac{\sqrt{6}}{2}\pi\right)\frac{\sqrt{R_{\max}-R_{\min}}}{\lambda}\leqslant \operatorname{diam}(M,g), \tag{4.2}$$

then the soliton must satisfy the Hitchin–Thorpe inequality.

Note that since

$$2+\frac{\sqrt{6}}{2}\pi<2+\sqrt{4+\frac{3}{2}\pi^2},$$

the condition (4.2) is sharper than (4.1). In Theorem 47, if the soliton has constant scalar curvature, then the soliton appears as an Einstein manifold and the assumption (4.2) is trivially satisfied. Hence, Theorem 47 may be regarded as a certain generalization of the Hitchin–Thorpe inequality [25, 57] for four-dimensional compact Einstein manifolds of positive Ricci curvature.

5. Self–Shrinkers of the Mean Curvature Flow

In this section, we shall introduce some diameter bounds and a gap theorem for compact self–shrinkers of the mean curvature flow. Let $x : M \to \mathbb{R}^{n+p}$ be an n-dimensional submanifold in the $(n+p)$-dimensional Euclidean space. When we evolve the position vector field x in the direction of the mean curvature $\vec{H}$, it gives rise to a solution to the *mean curvature flow*

$$x : M\times[0,T)\to\mathbb{R}^{n+p}, \quad \frac{\partial x}{\partial t}=\vec{H}.$$

DEFINITION 3. The immersed manifold M is called a *self–shrinker* if there exists a positive constant $\lambda > 0$ such that

$$\vec{H}=-\lambda x^{\perp}, \tag{5.1}$$

where $\perp$ denotes the projection onto the normal bundle of M.

It is well-known that [32] any self–shrinker satisfies

$$|\vec{H}|^2+\frac{\lambda}{2}\Delta|x|^2=n\lambda \quad \text{and} \quad \nabla|x|^2=2x^T, \tag{5.2}$$

where Δ and ∇ denote the Laplacian and Levi–Civita connection with respect to the induced metric on M, respectively. Note that the first equation in (5.2) corresponds to (3.1). Define a *potential function* on M by

$$\phi:=\frac{\lambda}{2}|x|^2-\frac{n}{2}.$$

Define a *Witten–Laplacian* by

$$\Delta_\phi:=\Delta-\nabla\phi\cdot\nabla.$$

Then we have a similar formula to (3.2) for a self–shrinker of the mean curvature flow.

PROPOSITION 4 ([53]). *Let $x:M\to\mathbb{R}^{n+p}$ be an n-dimensional self–shrinker satisfying* (5.1). *Then the potential function satisfies*

$$|\vec{H}|^2+|\nabla\phi|^2-2\lambda\phi=n\lambda. \tag{5.3}$$

We denote by $\phi_{\max}$ and $\phi_{\min}$ the maximum and minimum values of the potential function on a compact self–shrinker, respectively. Recall that the gradient estimate (3.5) follows from (3.2). Thanks to (5.3), we may prove a similar gradient estimate of the potential function on a compact self–shrinker.

PROPOSITION 5 ([53]). *Let $x:M\to\mathbb{R}^{n+p}$ be an n-dimensional compact self–shrinker satisfying* (5.1). *Then the potential function satisfies*

$$2\lambda\phi_{\max}-2\lambda\phi_{\min}\geqslant|\vec{H}|^2_{\max}-n\lambda \tag{5.4}$$

and

$$|\nabla\phi|^2\leqslant|\vec{H}|^2_{\max}-|\vec{H}|^2, \tag{5.5}$$

where $|\vec{H}|_{\max}$ and $|\vec{H}|_{\min}$ denote, respectively, the maximum and minimum values of the norm of the mean curvature.

Recently, many similarities between gradient shrinking Ricci solitons and self–shrinkers of the mean curvature flow have been pointed out by many authors [10, 11, 14, 19, 32]. In particular, A. Futaki, H. Li, and X.-D. Li [19] proved a lower diameter bound for compact self–shrinkers of the mean curvature flow in the relation to the study of the first non-zero eigenvalue of the Witten–Laplacian.

THEOREM 48 (A. Futaki, H. Li, and X.-D. Li [19]). *Let $x:M\to\mathbb{R}^{n+p}$ be an n-dimensional compact self–shrinker satisfying* (5.1) *such that $x(M)$ is not a minimal* *submanifold in* $\mathbb{S}^{n+p-1}(\sqrt{\frac{n}{\lambda}})$ *and* h^{α}_{ij} *be the components of the second fundamental form of M. Then the diameter of* $(M, x^* g_0)$ *has the lower bound*

$$\operatorname{diam}(M, x^* g_0) \geqslant \frac{\pi}{\sqrt{\frac{3}{2}\lambda + \frac{1}{2}K_0}},$$

where g_0 *is the canonical flat metric on the Euclidean space* $\mathbb{R}^{n+p}$ *and*

$$K_0 := \max_{1 \leqslant i \leqslant n} \left(\sum_{\alpha,k} h^{\alpha}_{ik} h^{\alpha}_{ki} \right).$$

Theorem 40 essentially follows from the formula (3.3) and the gradient estimate (3.5). By using the corresponding formula (5.4) and the corresponding gradient estimate (5.5), we may prove a lower diameter bound for compact self–shrinkers of the mean curvature flow in terms of the mean curvature.

THEOREM 49 ([53]). *Let* $x : M \to \mathbb{R}^{n+p}$ *be an n-dimensional compact self–shrinker satisfying* (5.1) *such that* $x(M)$ *is not a minimal submanifold in* $\mathbb{S}^{n+p-1}(\sqrt{\frac{n}{\lambda}})$. *Then the diameter of* $(M, x^* g_0)$ *has the lower bound*

$$\operatorname{diam}(M, x^* g_0) \geqslant \frac{|\vec{H}|^2_{\max} - n\lambda}{2\lambda\sqrt{|\vec{H}|^2_{\max} - |\vec{H}|^2_{\min}}}.$$

We may prove a gap theorem for compact self–shrinkers in terms of the mean curvature. If a compact self–shrinker of the mean curvature flow has constant norm of the mean curvature, then the norm satisfies $|\vec{H}|^2 = n\lambda$, and therefore we have $|\vec{H}|^2_{\max} = n\lambda$. We may characterize a compact self–shrinker to have constant norm of the mean curvature by giving some upper bound on $|\vec{H}|^2_{\max} - n\lambda$ in terms of the L^2-norm of the gradient vector field of the potential function.

THEOREM 50 ([53]). *Let* $x : M \to \mathbb{R}^{n+p}$ *be an n-dimensional compact self–shrinker satisfying* (5.1). *Then the self–shrinker has constant norm of the mean curvature if and only if*

$$|\vec{H}|^2_{\max} - n\lambda \leqslant \left(1 + \frac{2}{n}\right) \frac{1}{\operatorname{vol}(M, x^* g_0)} \int_M |\nabla \phi|^2.$$

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