Rendiconti Sem. Mat. Univ. Pol. Torino
Vol. 77, 1 (2019), 13 – 23

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ON EXCHANGE π-JU UNITAL RINGS

Abstract. We define and study 2-*JU* and π-*JU* rings and certain specifications of them, as especially we explore their intersection with clean and exchange rings. Specifically, we establish a necessary and sufficient condition for an arbitrary unitary ring to be 2-JU exchange. Our results parallel to those established by the present author in Toyama Math. J. (2016), Internat. J. Algebra (2017) and Missouri J. Math. Sci. (2017).

Key words: exchange rings, clean rings, JU rings, π-JU rings, bounded unit groups

1. Introduction and Background

Everywhere in the text of the current paper, all our rings R are assumed to be associative, containing the identity element 1 which, in general, differs from the zero element 0. Our terminology and notations are mainly in agreement with [17]. For instance, as usual, $U(R)$ stands for the unit group, $Nil(R)$ for the set of nilpotents and $J(R)$ for the Jacobson radical of a ring R – thus both the inclusions $\pm 1 + Nil(R) \subseteq U(R)$ and $\pm 1 + J(R) \subseteq U(R)$ hold. Also, $Id(R)$ denotes the set of all idempotents in R and $Inv(R)$ denotes the set of all involutions (that are, units of order not exceeding 2) in R – thus $1 \in Inv(R)$. Following standard notation, $C(R)$ designates the center of the ring R.

Some classical classes of rings are of importance for the successful presentation, so we will recall them explicitly as follows: Referring to [19], a ring R is *clean* if $R = U(R) + Id(R)$. A few specific sorts of clean rings are these: A ring R is *J-clean* (also named *semi-boolean*) provided $R = J(R) + Id(R)$ and *weakly J-clean* (also named *weakly semi-boolean*) provided $R = J(R) \pm Id(R)$ (see, for more details, cf. [20] and [8]).

The next two concepts are our crucial tools.

Definition 1.1. Suppose $n \in \mathbb{N}$ is fixed. A ring R is called *n-JU* if, for every $u \in U(R)$, the relation $u^n \in 1 + J(R)$ is fulfilled, that is, $U^n(R) \subseteq 1 + J(R)$.

Indeed, the ring epimorphism $R \to R/J(R)$ restricts to the group epimorphism $U(R) \to U(R/J(R))$ with kernel $1 + J(R)$ and, resultantly, the isomorphism $U(R)/[1 + J(R)] \cong U(R/J(R))$ is valid. Thus, the inclusion $U^n(R) \subseteq 1 + J(R)$ amounts to the equality $U^n(R/J(R)) = \{\bar{1}\}$, i.e., R is an n-JU ring exactly when $U(R/J(R))$ is an n-bounded group.

It is obviously true that 1-JU rings coincide with the already known JU rings from [6]. Also, rings R for which $u^{n+1} - u \in J(R)$ holds for some fixed $n \in \mathbb{N}$ are n-JU, and vice versa.

Definition 1.2. A ring R is called π-*JU* if, for each $u \in U(R)$, there exists $i \in \mathbb{N}$ depending on u such that the relation $u^i \in 1 + J(R)$ is valid.

A question which immediately arises is of whether or not R is a π-JU ring precisely when $U^{\omega}(R) = \cap_{i<\omega} U^i(R) \subseteq 1 + J(R)$. If not, this connection will, eventually, form a new proper subclass of rings.

The aim of this article is to promote a comprehensive investigation of the defined above ring classes and more specially what is their transversal with the classical clean and exchange rings. Precisely, we shall state and prove two explicit criteria, namely Theorems 2.7 and 2.8 quoted below, for an arbitrary ring to be clean 2-JU and exchange 2-JU, respectively. We shall also investigate some variations of 2-JU rings such as UJI rings. Before doing that, we shall recollect some principally well-known notions and establishments.

In [6] a ring R was termed *JU*, provided $U(R) = 1 + J(R)$. This was considerably extended in [11] to the so-termed *WJU* rings, that are rings for which $U(R) = \pm 1 + J(R)$. Among the non-trivial examples given there, one sees incidentally that the ring of integers $\mathbb{Z}$ is WJU, too. In fact, it is well known that $J(\mathbb{Z}) = \{0\}$, so that the two equalities $U(\mathbb{Z}) = \{\pm 1\} = \pm 1 + J(\mathbb{Z})$ are fulfilled, as needed.

This suggests us to try proving that WJU rings are Dedekind finite (or, in other words, directly finite abbreviated as DF), that is, for any two elements a, b of a ring the equality $ab = 1$ implies that $ba = 1$. This can be successfully realized in the next statement. For convenience, $\overline{a} := a + J(R)$ whenever $a \in R$.

Proposition 1.3. *WJU rings are DF.*

Proof. Since $U(R) = \pm 1 + J(R)$, it follows immediately that $U(R/J(R)) = \{\pm\overline{1}\}$. We assert now that $Nil(R/J(R)) = \{\overline{0}\}$. In fact, as $\overline{1} + Nil(R/J(R)) \subseteq U(R/J(R))$, it follows at once that $\overline{1} + Nil(R/J(R)) \subseteq \{\pm\overline{1}\}$. Let us now $\overline{2} \in Nil(R/J(R))$. However, $R/J(R)$ does not contain non-trivial central nilpotent elements leading to $\overline{2} = J(R)$ and thus to $2 \in J(R)$. That is why, $U(R) = 1 + J(R)$ whence $U(R/J(R)) = \{\overline{1}\}$. This, finally, means that $R/J(R)$ is a reduced factor-ring, indeed. Since reduced rings are known to be DF (see, e.g., [17]), we infer that $R/J(R)$ is DF, so that it can be easily checked that the same follows of R, as claimed. In fact, given $ab = 1$ for any two elements $a, b \in R$, we deduce that $ba \in Id(R)$ and $\overline{a}\overline{b} = \overline{1}$. Hence $\overline{b}\overline{a} = \overline{1}$, because $R/J(R)$ is DF. Therefore, $\overline{a}, \overline{b} \in U(R/J(R))$ forcing that $\overline{a} = \overline{b} = \overline{1}$ which directly implies that $ba \in 1 + J(R) = U(R)$. Finally, $ba \in U(R) \cap Id(R) = \{1\}$, i.e., $ba = 1$, as required. □

Accordingly, the following implications are true:

$$\text{J-clean rings} \Rightarrow \text{JU rings} \Rightarrow \text{WJU rings} \Rightarrow \text{DF rings}.$$

We thus can do the following helpful observation.

Remark 1.4. In view of the presented above facts, the result by Calci et al. in [1] that J-clean rings are DF is now trivial. In that aspect, we remark that in [16, Proposition 2.3] it was established that JU rings are DF. Thereby, our Proposition 1.3 supersedes this claim as well.

However, although WJU rings are themselves 2-JU, we strongly suspect that a 2-JU ring may in general not be DF. In fact, let we just notice that there exists a ring P

which is *not* DF such that $U^i(P) = \{1\}$ for some $i \geqslant 4$ and $J(P) = \{0\}$. To show this, let F be a field and let we set $P = F\langle x,y\rangle/(xy-1)$ with $xy \neq yx$. A simple check shows now that $U(P) \cong F^*$ and so, if we additionally take F to be of cardinality greater than or equal to 5, it can be shown that P is primitive (semiprimitive) and *not* a DF ring, as pursued.

We also note that the limitation on F to be with $|F| \geqslant 5$ is essential (for instance, it could be taken $F = \mathbb{Z}_5$), because if $|F| = 2$ or $|F| = 3$, then $F^* = \{1\}$ or respectively $F^* = \{\pm 1\}$ and so we will have that $U(P) = \{1\} = 1 + J(P)$ or that $U(P) = \{\pm 1\} = \pm 1 + J(P)$ which substantiates that P is a JU ring or a WJU ring, thus manifestly contradicting Proposition 1.3. Moreover, $(F^*)^2 = \{1\}$ will also easily imply that $F \cong \mathbb{Z}_2$ or $F \cong \mathbb{Z}_3$, respectively, so that by what we have already shown the inequality for the exponent i cannot be decreased.

After all the information detected so far, we now naturally come to the following common extension of JU and WJU rings by slightly modifying Problem 2 from [11].

Definition 1.5. A ring R is said to be *UJI*, provided that the equality $U(R) = Inv(R) + J(R)$ holds.

The stated equality provides us with one more equivalent characterization like this:

(i) $U(R) = Inv(R)[1 + J(R)] = [1 + J(R)]Inv(R)$, which is obviously left-right symmetric.

On the other side, in view of our comments after Definition 2.3, point (i) implies the equality

(ii) $U(R/J(R)) = Inv(R/J(R))$, i.e., $U^2(R/J(R)) = \{\bar{1}\}$.

In other words, UJI rings are 2-JU rings. Unfortunately, the converse is manifestly not true as the specific construction of a ring R with nil $J(R) \neq \{0\}$, arising from Theorems 2.7 and 2.8 both quoted below, shows (compare with [7] as well).

Similarly to above, we thus obtain the sequence of implications:

$$\text{JU rings} \Rightarrow \text{WJU rings} \Rightarrow \text{UJI rings} \Rightarrow \text{2-JU rings}.$$

Nevertheless, one can deduce the following: *If* $2 \in J(R)$ *and* R *is a* 2*-JU ring, then* $C(R)$ *is a JU ring*. Indeed, for any element $z \in U(C(R))$, we have that $z^2 - 1 \in J(R)$, i.e., $(z-1)(z+1) \in J(R)$. Therefore, $(z-1)(z-1+2) = (z-1)^2 + 2(z-1) \in J(R)$ and hence $(z-1)^2 \in J(R)$. This ensures that $z - 1 \in J(R)$, that is, $z \in 1 + J(R)$. Finally, $U(C(R)) \subseteq 1 + J(R)$, as required.

The relationship between nilpotent elements and elements from the Jacobson radical in UJI rings is elucidated by the following assertion.

Proposition 1.6. *Let* R *be a UJI ring. Then, for any* $q \in Nil(R)$*, there exists a natural number* $n \geqslant 2$ *which depends on* q *such that* $2^{n-1}q \in J(R)$ *is true. In particular, if* R *has odd characteristic, then* $Nil(R) \subseteq J(R)$.

Proof. For an arbitrary non-zero $q \in Nil(R)$ with $q^n = 0$ for some $n \in \mathbb{N}$, one writes that $-1+q = j+i$ for some $j \in J(R)$ and $i \in Inv(R)$. Thus $q-j = 1+i$ and, since by ordinary induction it follows that $(1+i)^n = 2^{n-1}(1+i)$, we obtain that $(q-j)^n = 2^{n-1}(1+i) = 2^{n-1}(q-j) = 2^{n-1}q - 2^{n-1}j$. But it is easily checked that $(q-j)^n \in J(R)$, so that $2^{n-1}q \in J(R)$, as asserted. As for the remaining claim, since $(2^{n-1}, char(R)) = 1$, we have immediately that $q \in J(R)$, as promised. □

We will close this section with a special type of UJI rings, that kind of rings generalizes J-clean and weakly J-clean rings (see, for more account, cf. [10, Definition 1.1] and [8] as well).

Definition 1.7. A ring R is said to be *feebly J-clean* if, for every $x \in R$, there exist $z \in J(R)$ and $e, f \in Id(R)$ with $ef = fe$ such that $x = z+e-f$.

Mimicking [19], we recall that idempotents of a ring R *lift* modulo $J(R)$ if, for any $x \in R$, given the truthfulness of the condition $x^2 - x \in J(R)$ implies the existence of $e \in Id(R)$ such that $e - x \in J(R)$.

An element r of an arbitrary ring is said to be *tripotent*, provided $r^3 = r$. If each element of a ring is a tripotent, we shall say that this ring is *tripotent* as well.

So, we now have all the information necessary to establish the following expansion of [10, Proposition 2.2].

Proposition 1.8. *Feebly J-clean rings are UJI clean rings.*

Proof. Firstly, we shall show that a feebly J-clean ring R is UJI. To this purpose, given $u \in U(R)$, we write that $u = z+e-f$ for $z \in J(R)$ and two commuting $e, f \in Id(R)$. Since $u-z = e-f \in U(R)$ and $(e-f)^3 = e-f$, we obtain that $(u-z)^3 = u-z$, that is, $(u-z)^2 = 1$. This shows that $u-z \in Inv(R)$, i.e., $U(R) = Inv(R)+J(R)$, as claimed.

Secondly, we shall show that a feebly J-clean ring R is clean. To that goal, we firstly intend to demonstrate that all idempotents of R can be lifted modulo $J(R)$. In fact, writing $x = z+e-f$ for each $x \in R$, where $z \in J(R)$, $e, f \in Id(R)$ with $ef = fe$, we may assume with no loss of generality that $ef = fe = 0$ by replacing $e-f = e(1-f) - f(1-e)$ as $e(1-f), f(1-e) \in Id(R)$. Thus $x^2 = z' + e + f$ for some $z' \in J(R)$, whence given the validity of the condition $x^2 - x = z'' + 2f \in J(R)$ for some $z'' \in J(R)$ implies that $2f \in J(R)$. We claim that there is $g \in Id(R)$ with $g - x \in J(R)$. Indeed, setting $g = e+f$, we obtain that $g \in Id(R)$ because e and f are orthogonal idempotents, as well as that $g - x = e+f-(z+e-f) = 2f - z \in J(R)$. Therefore, idempotents really lift modulo $J(R)$. On the other hand, $R/J(R) = \{\overline{e} - \overline{f}\}$ is clearly verified to be a tripotent factor-ring as $(e-f)^3 = e-f$, and thus it is a clean ring being a (commutative) unit regular ring (see the main theorem in [2]). Our statement about the cleanness of R then follows employing directly a proof rather similar to that in [19, Proposition 1.5]. □

2. Main Results

We start our considerations with a few simple but useful technicalities.

Lemma 2.1. *Let R be a ring and $e \in R$ an idempotent. Then the following two statements are true:*

(a) If R is an n-JU ring, then eRe is an n-JU ring.

(b) If R is a π-JU ring, then eRe is a π-JU ring.

Proof. (a) Given $u \in U(eRe)$ with inverse $v \in U(eRe)$, that is, $uv = vu = e$, and knowing that $u + (1-e) \in U(R)$ with inverse $v + (1-e) \in U(R)$ as $u(1-e) = (1-e)u = 0 = v(1-e) = (1-e)v$, we are able to write that $[u + (1-e)]^n = u^n + (1-e) = 1 + j$ for some $j \in J(R)$. Thus $u^n - e = j \in eRe \cap J(R) = J(eRe)$ and hence $u^n = e + j \in 1_{eRe} + J(eRe)$, as required.

(b) To proof goes on similar arguments as these in point (a). $\square$

The next example shows that the property of being 2-JU is not inherited by the full matrix ring.

Example 2.2. For each ring $R \neq J(R)$ (whence $R \neq \{0\}$), the matrix ring $\mathbb{M}_n(R)$ is not a 2-JU ring for all $n \in \mathbb{N}$.

Proof. We know that $J(\mathbb{M}_n(R)) = \mathbb{M}_n(J(R))$ (see, e.g., [17]). In view of Lemma 2.1, we just need to consider the case $n = 2$ only. Also, $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in U(\mathbb{M}_2(R))$ with the inverse $\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$ and so write in a way of contradiction that

$$(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix})^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where the latter matrix in the sum belongs to $J(\mathbb{M}_2(R))$. Therefore, it follows that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in \mathbb{M}_2(J(R))$ which leads to $1 \in J(R)$ and, consequently, to the obvious impossibility $R = J(R)$. $\square$

We recall that, imitating [14], a ring R is called *UU*, provided the equality $U(R) = 1 + Nil(R)$ is true. Likewise, mimicking [15], a ring R is called *nil-clean*, provided the equality $R = Nil(R) + Id(R)$ is true.

We are now ready to prove the following comprehensive assertion.

Proposition 2.3. *The following statements are equivalent:*

(a) R is a nil-clean UJI ring.

(b) R is a nil-clean UU ring.

(c) $J(R)$ is nil and $R/J(R)$ is a Boolean ring.

Proof. (a) $\Rightarrow$ (b). It is known by [15, Proposition 3.16] that $J(R) \subseteq Nil(R)$, whence $Nil(R) + J(R) = Nil(R)$. Moreover, for each $v \in Inv(R)$ one has that $v^2 = 1$ whence $(v-1)^2 = 2(1-v) \in Nil(R)$, because $2 \in Nil(R)$ again in virtue of [15, Proposition 3.14].

Hence $v-1 \in Nil(R)$ and so $Inv(R) \subseteq 1+Nil(R)$. Consequently, $U(R) = Inv(R) + J(R) \subseteq 1+Nil(R)+J(R) = 1+Nil(R)$, so that R is definitely a UU ring.

(c) $\Rightarrow$ (a). It is immediate, since $U(R)/[1+J(R)] \cong U(R/J(R)) \cong \{\bar{1}\}$ and thus even $U(R) = 1+J(R)$.

The equivalence (b) $\iff$ (c) is just a direct consequence from [14, Theorem 4.3]. □

It is worth to notice that the same type of result, hopefully slightly modified, could be considered via the usage of corresponding results from [5] and [9] for the class of weakly nil-clean rings by accomplishing with the so-called WUU rings R, that are rings for which $U(R) = \pm 1 + Nil(R)$ (see also [9]). We leave that problem to the interested readers, however.

In such an environment, having the necessary machinery, we are now in position to proceed by proving the following description of the ring structure in accomplishment with the corresponding group structure. To this purpose, let us recall that an element r of a ring R is said to be J-clean (resp., weakly J-clean, and feebly J-clean), provided the existence of an element $z \in J(R)$ and of two (orthogonal) commuting idempotents e, f such that $r = z+e$ (resp., $r = z \pm e$, and $r = z+e-f$).

Theorem 2.4. *Suppose that R is a ring. The next three items hold:*

(i) R is J-clean $\iff$ *R is clean and all units of R are J-clean.*

(ii) Let $6 = 0$*. Then R is weakly J-clean* $\iff$ *R is clean and all units of R are weakly J-clean.*

(iii) Let $6 = 0$ *and let R be commutative. Then R is feebly J-clean* $\iff$ *R is clean and all units of R are feebly J-clean.*

Proof. (i) "$\Rightarrow$". It is self-evident.

"$\Leftarrow$". For an arbitrary element $u \in U(R)$, we write that $u = z+e$ for some $z \in J(R)$ and $e \in Id(R)$. Since $U(R)+J(R) = U(R)$, we deduce that $u-z = e \in U(R) \cap Id(R) = \{1\}$ and hence $u = 1+z$. Hereafter, we may apply [6] to get our wanted claim. However, we may process directly like this: For every $r \in R$, write $r = w-f$, where $w \in U(R)$ and $f \in Id(R)$. Hence $r = (z+1)-f = z+(1-f) \in J(R)+Id(R)$, as required.

(ii) "$\Rightarrow$". It is trivial.

"$\Leftarrow$". For an arbitrary element $x \in R$, we write that $x = u+h$ for some $u \in U(R)$ and $h \in Id(R)$. Since $u = z+e$ or $u = z-e$ for some $z \in J(R)$ and $e \in Id(R)$, it must be as above that $e = 1$, so that $u = z+1$ or $u = z-1$. In the latter case, it follows immediately that $x = z-1+h = z-(1-h) \in J(R)-Id(R)$, as required. In the former case, we have $x = z+1+h$. But a direct check shows that $(1-2h)^2 = 1$ and thus $1-2h \in U(R)$ assures that $2h \in J(R)$ or $2(1-h) \in J(R)$. Therefore, $x = (z+2h)+(1-h) \in J(R)+Id(R)$ when $2h \in J(R)$ or $x = (z+2(1-h))-(1-3h) \in J(R)-Id(R)$, because $(1-3h)^2 = 1+3h = 1-3h$, taking into account that $6 = 0$.

(iii) "$\Rightarrow$". It is elementary.

"$\Leftarrow$". For an arbitrary element $x \in R$, we write that $x = u + h$, where $u \in U(R)$ and $h \in Id(R)$. Since $u = z + e - f$ such that $z \in J(R)$ and $e, f \in Id(R)$ with $ef = fe = 0$, we have $w = u - z = e - f \in U(R)$ and thereby $w^3 = w$. Thus $w^2 = 1 = (e - f)^2 = e + f$ whence $u = z + (1 - 2f) \in J(R) + Inv(R)$. We, consequently, can derive that $x = z + 1 - 2f + h = z - (1 - h) + 2(1 - f) = z - (1 - h) - 4(1 - f) \in J(R) - Id(R) - Id(R)$, because $[4(1 - f)]^2 = 16(1 - f) = 4(1 - f)$ as $6 = 0$ and so $12 = 0$. Since x is an arbitrary element, replacing x by $x - 1$, we conclude that $x = z + [1 - (1 - h)] - 4(1 - f) = z + h - 4(1 - f) \in J(R) + Id(R) - Id(R)$, as required. □

We proceed our work with one more definition (cf. [10] as well).

Definition 2.5. We shall say that tripotents *strongly lift* modulo $J(R) \iff \forall\, x \in R$ given the truthfulness of the condition $x^3 - x \in J(R)$ implies that there exist two idempotents $e, f \in Id(R)$ with $e.f = f.e = 0$ such that $e - f - x \in J(R)$.

It is worth to notice that the element $e - f$ is obviously tripotent, that is, $(e - f)^3 = e - f$.

We have now all of the necessary ingredients in order to establish our next basic result, which is just stated in terms of Definition 2.5 (see [10, Theorem 2.3] too).

Proposition 2.6. *A ring R is feebly J-clean if, and only if, $R/J(R)$ is a tripotent ring and all tripotents strongly lift modulo $J(R)$.*

Proof. "$\Rightarrow$". For every $x \in R$, we write $x = z + e - f$ for some $z \in J(R)$ and two commuting $e, f \in Id(R)$. So, as noticed above, $x^3 = z' + (e - f)^3 = z' + e - f$ for some $z' \in J(R)$, and hence the inclusion $x^3 - x = z' - z \in J(R)$ is always fulfilled. Also, $e - f - x = -z \in J(R)$ is also always fulfilled. On the other side, the quotient $R/J(R) = \{\overline{e} - \overline{f}\}$ being a difference of two commuting idempotents is clearly tripotent too, as expected.

"$\Leftarrow$". Since for any $x \in R$ we have that $[x + J(R)]^3 = x + J(R)$, it follows that $x^3 - x \in J(R)$. Hence there exist two commuting $e, f \in Id(R)$ with $e - f - x \in J(R)$, i.e., $x \in J(R) + Id(R) - Id(R)$, as needed. □

We continue with two more structural results of some importance, the first of which somewhat improves on the listed above Proposition 2.3 in the commutative case. They are our chief theorems in the article.

Theorem 2.7. *Suppose R is a commutative ring whose $J(R)$ is nil. Then the next two conditions are equivalent:*

(1) R is clean UJI.

(2) R is decomposed as $R = R_1 \times R_2$, where $R_1/J(R_1) \subseteq \prod_\lambda \mathbb{Z}_2$ and $R_2/J(R_2) \subseteq \prod_\mu \mathbb{Z}_3$ for some ordinals λ, μ.

Proof. "(1) $\Rightarrow$ (2)". One observes that $R = U(R) + Id(R) = J(R) + Inv(R) + Id(R)$, because $U(R) = J(R) + Inv(R)$. Write $3 = j + i + e$, where $j \in J(R)$, $i \in Inv(R)$ and $e \in Id(R)$. Since $1 - e = -2 + j + i \in Id(R)$, we have $(-2 + j + i)^2 = -2 + j + i$. This forces that there is $z \in J(R)$ with $7 + z = 5i$. Squaring the last equality, one drives that $2^3.3 = 24 \in J(R)$. So, $2^3.3^3 = 6^3 \in J(R)$ giving that $6 \in J(R)$. But, taking into account that $J(R) \subseteq Nil(R)$, one decomposes with the Chinese Remainder Theorem at hand that $R \cong R_1 \times R_2$, where $2 \in Nil(R_1)$ and $3 \in Nil(R_2)$ and, using elementary manipulations, both R_1, R_2 continue to be UJI rings.

In the first case, we may copy the idea from Proposition 2.3 to get that $R_1/J(R_1)$ is a Boolean ring, and so it is a subdirect product of a family of a single copy or isomorphic copies of the field $\mathbb{Z}_2$.

Concerning the second direct summand R_2, we can process like this: In view of [19], we derive that $R_2/J(R_2)$ is a clean ring of characteristic 3 such that $U(R_2/J(R_2)) = Inv(R_2/J(R_2))$, whence $R_2/J(R_2)$ is a clean 2-UU ring. This enables us with the aid of [7, Theorem 2.5] (see cf. [9], [12] and [13], respectively, too) that $R_2/J(R_2)$ is really a subdirect product of a family of a single copy or isomorphic copies of the field $\mathbb{Z}_3$, as needed.

"(2) $\Rightarrow$ (1)". Since it is obvious that both R_1 and R_2 are clean rings, it is immediately true that so does R.

Moreover, $U(R_1/J(R_1)) = \{\overline{1}\}$ implying that $U(R_1) = 1 + J(R_1) = Inv(R_1) + J(R_1)$, as wanted.

Besides, $U(R_2/J(R_2)) = Inv(R_2/J(R_2))$ meaning that $U(R_2) = Inv(R_2) + J(R_2)$. In fact, letting $u_2 \in U(R_2)$, it has to be that $(u_2 + J(R_2))^2 = 1 + J(R_2)$, i.e., $u_2^2 - 1 \in J(R_2)$. Hence $(-u_2 - 1)^2 - (-u_2 - 1) \in J(R_2)$, because $3 \in J(R_2)$. Since $J(R_2)$ is nil, there is $e_2 \in Id(R_2)$ with the property $e_2 - (-u_2 - 1) = u_2 + (1 + e_2) \in J(R_2)$. But certainly $(2e_2 - 1)^2 = 1$ means that $2e_2 - 1 \in Inv(R_2)$ and, since $u_2 - (2e_2 - 1) + 3e_2 \in J(R_2)$, we finally infer that $u_2 \in Inv(R_2) + J(R_2)$, as required.

Thus, bearing in mind that R_1 and R_2 are already both UJI rings, one plainly concludes that the same follows of R, as stated. □

In particular, we detect that the quotient $R/J(R)$ is a subdirect product of a family of a single copy or isomorphic copies of the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$.

Let us recall that a ring R is said to be *exchange* if, for any element $a \in R$, there exists an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$ (this is actually a criterion due to Goodearl and Nicholson – see, for instance, [19]). It is well known that both von Neumann regular rings and clean rings are exchange, which implications are both not reversible. Notice that some rings very close to von Neumann regular rings and having specific properties were examined in [18].

So, we somewhat can now slightly extend the previous theorem to the following one:

Theorem 2.8. *For a ring R the next three items are equivalent:*

(1) R is an exchange 2-JU ring.

(2) R is a clean 2*-JU ring.*

(3) $R/J(R)$ is a subdirect product of a family of a single copy or isomorphic copies of the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$, and all idempotents of R lift modulo $J(R)$.

Proof. "(2) ⇒ (1)". This implication is well-known since clean rings are ever exchange (see, e.g., [19]).

"(1) ⇒ (3)". That $R/J(R)$ remains an exchange ring and all idempotents of R are lifting modulo $J(R)$ follows directly again from [19]. But the inclusion $U^2(R) \subseteq 1+J(R)$, being equivalent to $U^2(R/J(R)) = \{\overline{1}\}$, allows us to deduce again with the aid of [7, Theorem 2.5] (see also [12, 13]) that $R/J(R)$ is a subdirect product of a family of a single copy or isomorphic copies of the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$, as stated.

"(3) ⇒ (2)". Since the quotient $R/J(R)$ is tripotent, whence it is a clean factor-ring owing to the main theorem from [2], we just employ once again [19] to get that R is clean, indeed. Moreover, it is not too hard to check that $U^2(R/J(R)) = \{\overline{1}\}$, that is, $U^2(R) \subseteq 1+J(R)$, showing that R is a 2-JU ring, as expected. □

As an immediate observation, one sees that (semiprimitive) exchange 2-JU rings are always DF (as being commutative) – compare with Remark 1.4 listed above. In that way, UU rings from [14], and their common generalization WUU rings from [5], are DF rings.

In closing our work, we state one more comment.

Remark 2.9. In [3, Lemma 2.2] the requirement "feebly clean" is absolutely redundant according to [7, Proposition 2.1] (compare also with [4]). Moreover, our theorems established above somewhat shed a new impact on the major results from [3] and [4].

We end our article with the following three questions of interest:

Problem 2.10. Does it follow that exchange π-JU rings R with nil $J(R)$ are π-regular?

Problem 2.11. Characterize up to an isomorphism arbitrary exchange UJI rings.

Problem 2.12. Describe the isomorphic structure of UJI rings R for which $R = Id(R) + Id(R)$.

In that connection, let $R = Id(R) + Id(R)$ be a JU ring. Then the canonical map $R \to R/J(R)$ and its restriction $U(R) \to U(R/J(R))$ lead us to $R/J(R) = Id(R/J(R)) + Id(R/J(R))$ and $U(R/J(R)) = \{\overline{1}\}$. Hence $R/J(R)$ has to be reduced and thus abelian. This means that $R/J(R)$ is commutative and so a Boolean ring, because $\overline{2} = \overline{0}$. However, what can be said of $J(R)$ is unknown yet.

Similarly, using the machinery from [5] and [11], respectively, we can describe the isomorphic class of WJU rings R in the case when $R = Id(R) + Id(R)$ or even when $R = Id(R) \pm Id(R)$.

Acknowledgement. The author owes his sincere thanks to the expert referee for his/her helpful structural comments and suggestions, refining substantially the shape of the paper.

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AMS Subject Classification: Primary 16U99; 16E50; 13B99

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Lavoro pervenuto in redazione il 08.02.2018.