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RANKS AND GENERAL LINEAR PROJECTIONS

Abstract. Let $X \subset \mathbb{P}^N$ be an integral and non-degenerate variety. We study the X-ranks and the cactus X-ranks of the points of $\mathbb{P}^N$ when X is obtained as a general linear projection from a sufficiently positive linear embedding of X in an axiomatic set-up for the embedding $X \hookrightarrow \mathbb{P}^N$ (*ideal embedding*). In particular we give an upper bound for the X-cactus rank of a general $q \in \mathbb{P}^N$.

1. Introduction

Let $X \subset \mathbb{P}^N$ be an integral and non-degenerate n-dimensional variety defined over an algebraically closed field $\mathbb{K}$. For each $q \in \mathbb{P}^N$ the *X-rank* $r_X(q)$ of q is the minimal cardinality of a finite set $S \subset X$ such that $q \in \langle S \rangle$, where $\langle \ \rangle$ denote the linear span ([6, 11, 13, 18, 19]). The notion of X-rank unifies several important notions:

1. the tensor rank (taking as X the Segre embedding of a multiprojective space);
2. the additive decomposition of a homogeneous polynomial as a sum of powers of linear forms (taking as X a Veronese embedding of a projective space);
3. the rank decomposition of a partially symmetric tensor (taking the Segre-Veronese embeddings of a multiprojective space).

See [19] for several real-life applications of these X-ranks.

For any positive integer k let $\sigma_k(X) \subseteq \mathbb{P}^N$ denote the k-secant variety of X, i.e. the closure in $\mathbb{P}^N$ the union of all linear spaces $\langle S \rangle$, where S is a finite subset of X with cardinality k. Each secant variety $\sigma_k(X)$ is irreducible and $\dim \sigma_k(X) \leqslant \min\{N, (n+1)k-1\}$. X is said to be *secant defective* if there is k such that $\sigma_k(X) < \{N, (n+1)k-1\}$. For any $q \in \mathbb{P}^N$ the *border rank* $b_X(q)$ of q is the minimal integer k such that $q \in \sigma_k(X)$. Set $r_g := \lceil (N+1)/(n+1) \rceil$. X is not secant-defective if and only if $\sigma_{r_g}(X) = \mathbb{P}^N$ and $\dim \sigma_k(X) = (n+1)k-1$ for all $k < r_g$. Let ρ_X (resp. ρ'_X, resp. ρ''_X) denote the maximal positive integer t such that all zero-dimensional (resp. zero-dimensional and smoothable, resp. finite set) $Z \subset X$ with $\deg(Z) \leqslant t$ are linearly independent, i.e. $\dim\langle Z \rangle = \deg(Z) - 1$. The maximum exists because any $N+2$ points of X are linearly dependent. We have $\rho_X \leqslant \rho'_X \leqslant \rho''_X$. A degree x zero-dimensional scheme $Z \subset X$ is said to be *smoothable in* X if it is a flat limit of a family of subschemes of X; a zero-dimensional scheme A is said to be *smoothable* if there is an embedding $j : A \to B$ with B a smooth quasi-projective variety and $j(A)$ is smoothable in B ([10, §2]). If X is smooth, a zero-dimensional scheme $Z \subset X$ is smoothable if and only if it is smoothable in X ([10, Proposition 2.1]). Smoothable zero-dimensional schemes

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occur in any use of secant varieties for the following reasons. Take a zero-dimensional scheme $Z \subset X \subset \mathbb{P}^N$ and set $a := \deg(Z)$. Fix any $p \in \langle Z \rangle$. Taking a flat family of sets with cardinality a with Z as its limits we get $p \in \sigma_a(X)$ ([8, page 294], [10, Proposition 2.5]). Thus if $k \leqslant \rho'(X)$ for each $q \in \sigma_k(X)$ there is a smoothable zero-dimensional scheme $Z \subset X$ such that $\deg(Z) \leqslant k$ and $q \in \langle Z \rangle$.

DEFINITION 1. *We say that X is* ideal *or that the embedding $X \hookrightarrow \mathbb{P}^N$ is ideal if the following conditions are satisfied:*

1. *X is not secant-defective;*

2. *$\rho'_X \geqslant r_g - 1$.*

The second condition is the most restrictive one. It is easy check that it implies that $\dim \sigma_k(X) = (n+1)k - 1$ for all $k < r_g$ ([11, Lemma 2.3], [10, Proposition 2.5]), so that if we have condition (2), then to have condition (1) it is sufficient to check that $\sigma_{r_g}(X) = \mathbb{P}^N$. If X is the order d Veronese embedding of $\mathbb{P}^n$, then $\rho_X = \rho'_X = \rho''_X = d+1$, and $N = -1 + \binom{n+d}{n}$ and so $r_g = \lceil \binom{n+d}{n}/(n+1) \rceil$. Thus for $n \geqslant 2$ and $d \geqslant 5$ the Veronese embedding of $\mathbb{P}^n$ is not ideal, although it is not secant-defective by the Alexander-Hirschowitz theorem ([2, 3, 4, 5, 9, 14, 21]).

All embedded varieties in the list we gave are linearly normal embeddings of a homogeneous space. However, in this paper we almost always only consider general projections of some very positive embedding of X in another very large projective space and so we do not claim that the notion of *ideal embedding* may be applied to any X in that list, except the rational normal curve case, a case for which the ranks and border ranks are known by Sylvester's theorem ([15, 18, 19]). It may be applied to general linear projections of certain high degree Veronese and Segre-Veronese embeddings with a precise description of the positivity we need.

We prove the following result.

THEOREM 1. *Let T be an integral projective variety and $\mathcal{L}$ an ample line bundle on T. Set $n := \dim(T)$. Fix an integer $N \geqslant \max\{2n+1, n+x\}$, where x is the maximal dimension of a Zariski tangent space of T. There is an integer t_0 such that for all integers $t \geqslant t_0$ the following holds:*

(a) the image $X \subset \mathbb{P}^N$ of a general projection in $\mathbb{P}^N$ of the image of T by the very ample linear system $|\mathcal{L}^{\otimes t}|$ is ideal;

(b) $\rho'_X \leqslant r_g$ and equality holds if and only if $N+1 \equiv 0 \pmod{n+1}$.

Let $Y \subset \mathbb{P}^M$, $M \gg 0$, be an integral and non-degenerate variety; in the set-up of Theorem 1 one can take as Y the image of T by the complete linear system $|\mathcal{L}^{\otimes t}|$ for some large t. A natural question is if there is a linear subspace $W \subset \mathbb{P}^M$ such that $\dim W = M - N - 1$, $W \cap Y = \varnothing$, the linear projection $\ell_W : \mathbb{P}^M \to \mathbb{P}^N$ induces a morphism birational to its image and with either $\rho'_X = r_g$ or $\rho''_X = r_g$, where $X := \ell_W(Y)$ when $(N+1)/(n+1) \notin \mathbb{N}$. The answer is negative for ρ'_X if we require that $\ell_{W|Y}$ is an embedding (Remark 6), but we do not know if either $\rho''_X = r_g$ or $\rho'_g = r_g$ in some other cases if we allow the case that $\ell_{W|Y}$ is only an injective morphism . For positive, but not very positive, embeddings of smooth curves, this question (with injective maps, but not embeddings) has sometimes a positive answer ([20]).

Take any integral and non-degenerate $X \subset \mathbb{P}^N$. For any $q \in \mathbb{P}^N$ the *cactus X-rank* (resp. the *smoothable X-rank*) $c_X(q)$ (resp. $r_{X,\mathrm{sm}}(q)$) of q is the minimal integer t such that there is a zero-dimensional (resp. zero-dimensional and smoothable) scheme $Z \subset X$ such that $q \in \langle Z \rangle$ and $\deg(Z) = t$ ([6, 7, 10, 12, 11, 16, 22]). In [18] the cactus X-rank is called scheme-rank. The notion of cactus X-rank is quite important and well-studied, because when $\dim X$ is large, the integer r_g may be much larger that the maximum of all cactus X-ranks ([8, Theorems 3 and 4] are striking results for homogeneous polynomials). Easy examples show that in general a zero-dimensional (even if smoothable) scheme computing the cactus X-rank of a point may not be linearly independent. For each positive integer t set $c(X, \leqslant t) := \{q \in \mathbb{P}^N \mid c_X(q) \leqslant t\}$. Note that in the definition of $c(X, \leqslant t)$ we do not take a closure in $\mathbb{P}^N$, so this is not a cactus variety of X in the sense of [10, 11]. The set $c(X, \leqslant t)$ is constructible by Chevalley's theorem ([17, Ex. II.3.19]) and the existence of the Hilbert scheme of X. In particular it makes sense to ask what is the maximum among all cactus X-ranks $c_X(q)$, $q \in X$, and what is the *generic cactus X-rank* $c(X, \mathrm{gen})$, i.e. the only integer t such that there is a non-empty open subset $U \subset \mathbb{P}^N$ with $c_X(q) = t$ for all $q \in U$. For each positive integer b such that $\sigma_b(X) \subsetneq \mathbb{P}^N$ we may ask what is the *generic cactus X-rank of* $\sigma_b(X)$, i.e. the only integer t such that there is a non-empty open subset U of the variety $\sigma_b(X)$ with $c_X(q) = t$ for all $q \in U$.

THEOREM 2. *Fix an integer $N > n$ and let $Y \subset \mathbb{P}^r$, $r > N$, be an integral and non-degenerate n-dimensional variety. Let k be the minimal integer such that* $\dim c(Y, \leqslant k) \geqslant N$. *Fix a general $W \in G(r-N, r+1)$ and let $\ell_W : \mathbb{P}^r \setminus W \to \mathbb{P}^N$ denote the linear projection from W. We have $W \cap Y = \varnothing$ and we set $X := \ell_W(Y)$. Then* $c(X, \mathrm{gen}) \leqslant k$.

2. The proofs

REMARK 1. Let $X \subset \mathbb{P}^N$ be an ideal embedding. Fix an integer $d \geqslant 2$ and let $\nu_d : \mathbb{P}^N \to \mathbb{P}^r$, $r := \binom{n+d}{n} - 1$, be the order d Veronese embedding of $\mathbb{P}^N$. Let M be the linear span of $\nu_d(X)$ in $\mathbb{P}^r$. It is easy to check that $\nu_d(X)$ is ideal in M.

REMARK 2. Every zero-dimensional scheme $Z \subset \mathbb{P}^N$ with $\deg(Z) = 2$ spans a line, i.e. it is linearly independent. Thus $\rho_X \geqslant 2$ for any X. Hence X is ideal if either $\sigma_2(X) = \mathbb{P}^N$ or X is not secant-defective and $\sigma_3(X) = \mathbb{P}^N$.

EXAMPLE 1. Let $X \subset \mathbb{P}^N$, $N = d - g$, be a linearly normal embedding of a smooth curve of genus $g \geqslant 3$. Assume $d := \deg(X) > 2g - 2$ and so $h^1(\mathcal{O}_X(1)) = 0$. By Riemann-Roch we have $\rho_X \geqslant d - 2g + 1$. Hence X is ideal if $d \geqslant 3g - 3$.

REMARK 3. Fix an integer $k > 0$. Let $\mathbb{S} \subset G(k, r+1)$ be a constructible family of linear subspaces with $\dim \mathbb{S} \leqslant d_k$. Consider the incidence correspondence

$\mathbb{I} \subset G(k, r+1) \times \mathbb{P}^r$ and let $\pi_1 : \mathbb{I} \to G(k, r+1)$ and $\pi_2 : \mathbb{I} \to \mathbb{P}^r$ be the maps induced by projections. Let $\mathbb{S}$ be the closure of $\pi_2(\pi_1^{-1}(\mathbb{S}))$ in $\mathbb{P}^r$. We have $\dim \mathbb{S} \leqslant d_k + k - 1$. Thus if $d_k + k - 1 \leqslant N - 1$ we have $W \cap \mathbb{S} = \emptyset$ for a general $W \in G(r-N, r+1)$. Thus for a general $W \in G(r-N, r+1)$ the linear projection $\ell_W : \mathbb{P}^r \backslash W \to \mathbb{P}^N$ sends each $(k-1)$-dimensional space $M \in \mathbb{S}$ isomorphically onto a $(k-1)$-dimensional linear subspace of $\mathbb{P}^N$. Hence for any $M \in \mathbb{S}$ and any degree k scheme $Z \subset M$ spanning M the scheme $\ell_W(Z)$ has degree k and, since it spans the $(k-1)$-dimensional linear space $\ell_W(M)$, it is linearly independent.

REMARK 4. Let $Y \subset \mathbb{P}^r$ be an integral and non-degenerate n-dimensional variety. In the set-up of Remark 3 we take as $\mathbb{S}$ the family of all degree k smoothable zero-dimensional schemes $Z \subset Y$ (or of all finite sets of Y with cardinality $\leqslant k$). Then we may take $d_k = kn$. Moreover if $\rho_Y \geqslant 2k+1$, we cannot take as d_k any integer $< nk$ ([11, Theorem 1.18]).

REMARK 5. Take $q \in \mathbb{P}^N$ such that $b_X(q) \leqslant \rho'_X$. Then $r_{X,\mathrm{sm}}(q) \leqslant b_X(q)$ ([10, Proposition 2.5], [11, Lemma 2.6]).

Proof of Theorem 1: Let k_1 be the minimal positive integer t such that $\mathcal{L}^{\otimes t}$ is very ample and $h^0(\mathcal{L}^{\otimes t}) \geqslant N+1$. Fix an integer $t \geqslant k_1$ and set $r := h^0(\mathcal{L}^{\otimes t}) - 1$. Let $Y \subset \mathbb{P}^r$ be the image of T by the complete linear system $|\mathcal{L}^{\otimes t}|$. We have $Y \cong T$. For any $(r-N-1)$-dimensional linear subspace W of $\mathbb{P}^r$ let $\ell_W : \mathbb{P}^r \backslash W \to \mathbb{P}^N$ denote the linear projection from W. Since $N \geqslant \{2n+1, n+x\}$, for a general $W \in G(r-N, r+1)$ we have $W \cap Y = \emptyset$ and $\ell_{W|Y}$ is an isomorphism onto its image. We fix a general $W \in G(r-N, r+1)$ and set $X := \ell_W(Y) \subset \mathbb{P}^N$. We need to check under what assumptions on t the embedded variety X is ideal. Fix $S \subset T_{\mathrm{reg}}$ with $|S| = r_g = \lceil (N+1)/(n+1) \rceil$. For each $a \in S$ let $2a$ the closed subscheme of T with $(I_a)^2$ as its ideal sheaf. The scheme $2a$ is zero-dimensional and $\deg(2a) = n+1$. Set $Z := \cup_{a \in S} 2a$. There is an integer $k_2 \geqslant 0$ such that $h^1(I_Z \otimes \mathcal{L}^{\otimes t}) = 0$ for all $t \geqslant k_2$. Taking $t \geqslant \max\{k_1, k_2\}$ we see by Terracini's lemma (in arbitrary characteristic) that $\dim \sigma_{r_g}(Y) = (n+1)r_g - 1$ ([1, part (1) of Corollary 1.11]). In particular we have $r \geqslant (n+1)r_g - 1$. We take $k_0 \geqslant \max\{k_1, k_2\}$ and take $t \geqslant k_0$. Since $t \geqslant k_2$ we have $r - N \geqslant (n+1)(r_g - 1) - 1$, where $r := -1 + h^0(\mathcal{L}^{\otimes t})$. By Remarks 3 and 4 we have $\rho'_X \geqslant r_g - 1$, i.e. X satisfies the second condition of Definition 1. By [11, Lemma 2.3], or [10, Proposition 2.5] to check the first condition it is sufficient to prove that $\sigma_{r_g}(X) = \mathbb{P}^N$ (for a suitable k). By the definition of the integer k_2 we have $\dim \sigma_{r_g}(Y) = (n+1)r_g - 1$ and for a general finite set $S \subset Y_{\mathrm{reg}}$ with cardinality r_g the linear span M of the union of the Zariski tangent spaces T_aY, $a \in S$, has dimension $(n+1)r_g - 1$. For a general $W \in G(r-N, r+1)$ we have $\dim W \cap M = (n+1)r_g - N - 1$. Thus the linear space $\ell_W(M \backslash M \cap W) \subseteq \mathbb{P}^N$ has dimension N, i.e. it is $\mathbb{P}^N$. By the characteristic free part of Terracini's lemma ([1, part (1) of Corollary 1.11]) applied to the finite set $\ell_W(S)$ we have $\sigma_{r_g}(X) = \mathbb{P}^N$. Hence X is not secant-defective, concluding the proof of part (a).

Now we prove part (b). Let $\mathcal{Z}$ be the family of all degree r_g smoothable zero-dimensional subschemes of Y. Recall that the union $\cup_{Z \in \mathcal{Z}} \langle Z \rangle$ is a constructible set $\mathbb{T} \subset \mathbb{P}^r$ of dimension $nr_g + r_g - 1$ (Remark 4). We have $nr_g + r_g - 1 \geqslant N$ with equality if and only if $N+1 \equiv 0 \pmod{n+1}$. If $nr_g + r_g - 1 = N$ a general W does not intersect $\mathbb{T}$ and hence $\rho'_X \geqslant r_g$. If $nr_g + r_g - 1 > N$, then a general W meets $\mathbb{T}$ and hence $\rho'_X < r_g$. If $nr_g + r_g - 1 = N$ call $\mathcal{Z}_1$ the set of all degree $r_g + 1$ zero-dimensional smoothable schemes and set $\mathbb{T}_1 := \cup_{Z \in \mathcal{Z}_1} \langle Z \rangle$. Since $\dim \mathbb{T}_1 > N$ and W is general, we get $\rho'_X < r_g + 1$ and so $\rho'_X = r_g$. □

REMARK 6. Take r_g, T and Y as in the statement of Theorem 1 with $(N+1)/(n+1) \notin \mathbb{Z}$, i.e. $(n+1)r_g - 1 > N$. The set $\mathcal{Z}$ of all smoothable zero-dimensional schemes of Y with degree r_g is a projective scheme, because by its definition it is closed in the Hilbert scheme of all degree r_g subschemes of t. For large t, say $t \geqslant 2kr_g$ with $k > 0$ with $\mathcal{L}^{\otimes k}$ very ample, we have $\rho_Y \geqslant 2r_g - 1$ and hence if $Z, Z' \in \mathcal{Z}$, then $\langle Z \rangle \cap \langle Z' \rangle = \emptyset$. Thus the set of $\langle Z \rangle$, $Z \in \mathcal{Z}$, is a closed irreducible subset $\mathbb{S}$ of $G(r_g, r+1)$. Hence the set $c(Y, \leqslant r_g)$ is a closed irreducible subvariety of $\mathbb{P}^r$ with dimension $nr_g + r - 1 > N$. Thus $W \cap c(Y, \leqslant r_g) \neq \emptyset$ for all $W \in G(r-N, r+1)$.

Proof of Theorem 2: Fix an irreducible component Γ of the constructible subset $c(Y, \leqslant k)$ of $\mathbb{P}^r$. There is an irreducible component Ψ of the Hilbert scheme of Y such that the linear span of a general element of Ψ contains a general element of Γ. Fix a general $q \in \Gamma$. Take $z \in \Psi$ such that $q \in \langle Z \rangle$. Since $\dim W = r - N - 1$, $\dim \langle Z \rangle \leqslant k - 1$ and W is general, we may assume that $W \cap \langle Z \rangle = \emptyset$. Thus ℓ_W is defined at each point of $\langle Z \rangle$ and ℓ_W send isomorphically $\langle Z \rangle$ onto a linear subspace of $\mathbb{P}^N$ with the same dimension. The scheme $\ell_W(Z)$ shows that $c_X(\ell_W(q)) \leqslant k$. Now we fix W. We move Z to a general element Z' of Ψ and call q' a general element of $\langle Z' \rangle$. Since Z and Z' are general in Ψ, we have $\deg(Z) = \deg(Z')$ and $\dim \langle Z \rangle = \dim \langle Z' \rangle$. Since $W \cap \langle Z \rangle = \emptyset$, for a general Z' we have $W \cap \langle Z' \rangle = \emptyset$. Thus $\ell_W(q')$ is well-defined and $c_X(\ell_W(q')) \leqslant \deg(Z') \leqslant k$. Since Z' is general in Ψ and q' is general in $\langle Z' \rangle$, q' is general in Γ. To conclude the proof it is sufficient to prove that $\ell_W(q')$ is a general point of $\mathbb{P}^N$, i.e. it is sufficient to prove that $\ell_W(\Gamma \backslash \Gamma \cap W)$ contains a non-empty open subset of $\mathbb{P}^N$. By the definition of k we have $\dim \Gamma \geqslant N$. Take $\Gamma_1 \subseteq \Gamma$ with $\dim \Gamma_1 = N$. Since W is general, we have $W \cap \Gamma_1 = \emptyset$. Thus $\dim \ell_W(\Gamma_1) = N$. □

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