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J. M. Almira*, A. Romero

A NEW PROOF OF A CLASSICAL RESULT ON THE TOPOLOGY OF ORIENTABLE CONNECTED AND COMPACT SURFACES BY MEANS OF THE BOCHNER TECHNIQUE

Abstract. As an application of the Bochner formula, we prove that if a 2-dimensional Riemannian manifold admits a non-trivial smooth tangent vector field X then its Gauss curvature is the divergence of a tangent vector field, constructed from X, defined on the open subset out the zeroes of X. Thanks to the Whitney embedding theorem and a standard approximation procedure, as a consequence, we give a new proof of the following well-known fact: if on an orientable, connected and compact 2-dimensional smooth manifold there exists a continuous tangent vector field with no zeroes, then the manifold is diffeomorphic (or equivalently homeomorphic) to a torus.

1. Introduction

One of the most famous results in topology is the so-called "Hairy Ball Theorem" which states that every continuous tangent vector field defined on an even-dimensional sphere vanishes somewhere, a fact that can be rephrased in several forms and has some interesting consequences. For example, the result can be restated by claiming that, if $n = 2N$ is an even natural number, then there are no continuous unitary tangent vector fields on $\mathbb{S}^n$, and can be used to demonstrate that every continuous map $f : \mathbb{S}^{2N} \to \mathbb{S}^{2N}$ has either a fixed point or an antipodal point (a point x such that $f(x) = -x$) [21].

The best-known proof of the Hairy Ball Theorem is due to Brouwer [4] and is based on the homology of the spheres. Other proofs use degree theory [7] and other algebraic topology arguments [5, 6], but there are also analytic proofs [13], combinatorial proofs [10], proofs based on the vector analysis of differential forms [3], on differential topology [12, 18], etc. The proof we present in this paper is based on Riemannian geometry arguments, is restricted to the case $n = 2$ and, as far as we know, it is the first one of this type that exists.

The case $n = 2$ has special relevance and was first proved by Poincaré [18] using his index theorem, which characterizes Euler characteristic as the sum of the indices of a tangent vector field whose singularities are isolated. In particular, if a nowhere vanishing tangent vector field exists on a connected compact surface M, then $\chi(M) = 0$, which represents an obstruction to the existence of such tangent vector fields on the sphere, since $\chi(\mathbb{S}^2) = 2$. Note that Euler's characteristic can be used to reformulate the result as a characterization of the topological torus. Indeed, if M is

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an orientable connected compact 2-dimensional smooth manifold, then $\chi(M) = 2 - 2g$, where g is the genus of M. Thus, M is a torus if and only if $\chi(M) = 0$, since $g = 1$ is the unique solution of $2 - 2g = 0$. On the other hand, it is not difficult to construct nowhere vanishing tangent vector fields on the torus (a fact that, surprisingly, has relevant applications in fusion theory [15, Chapter 4]). Thus, it is natural to state Hairy Ball Theorem as follows:

THEOREM 1. *Assume M is a 2-dimensional orientable, connected and compact smooth manifold. Then the following are equivalent statements:*

(a) *M admits a continuous tangent vector field with no zeroes.*

(b) *M is diffeomorphic (or equivalently homeomorphic) to a torus.*

In this note, we present a new proof of Theorem 1. The main ingredients of the proof are, after endowing M with a Riemannian metric g, a combination of the Bochner technique, Gauss-Bonnet theorem, and a standard approximation argument. We only demonstrate the implication $(a) \Rightarrow (b)$, since the other implication is an easy exercise that consists of the explicit definition of a nowhere vanishing tangent vector field on the torus.

2. Proof of the main result

One of the most important techniques in Riemannian geometry is the Bochner technique [16, Chapter 9], [22]. Roughly, it consists of relating the existence of certain non-trivial tangent vector fields on a Riemannian manifold with properties of the curvature of such a manifold. In the compact case, it leads to the vanishing of certain geometrically relevant tangent vector fields (Killing or conformal) under the assumption of positive or negative curvature everywhere [22].

Our argument for the proof of $(a) \Rightarrow (b)$ in Theorem 1 will split into three steps. In the first one, given a 2-dimensional Riemannian manifold (M, g), if it admits a smooth tangent vector field with no zeroes, we use the well known Bochner formula (see, e.g., [22, 19]) to construct a tangent vector field on M whose divergence, with respect to g, is equal to the Gauss curvature of g. In the second step, we will start from a compact and connected 2-dimensional smooth manifold M which admits a nowhere zero continuous tangent vector field. We will endow M with a Riemannian metric g. Then we will prove, by a classical approximation argument, that existence of the nowhere zero continuous tangent vector field leads to the existence of a smooth one also with no zeroes. Finally, in the last step, we will make use of the global Gauss-Bonnet theorem [1, 8] to get that M must be a torus.

Step 1. An application of the Bochner formula

Consider a compact, orientable and connected 2-dimensional manifold M. The classical Whitney embedding theorem [2, Theorem 5.4.8] guarantees that we can get a smooth embedding of M into $\mathbb{R}^n$, for n large enough. Then we can consider on M the metric g induced by the usual one of $\mathbb{R}^n$ and this is precisely what we do.

The Bochner formula states that

$$X(\operatorname{div}(X)) = -\operatorname{Ric}(X,X) + \operatorname{div}(\nabla_X X) - \operatorname{trace}(A_X^2),$$

for all smooth tangent vector field X, where Ric is the Ricci tensor of (M,g), div denotes the divergence on (M,g) and A_X is the linear operator field defined by $A_X(v) = -\nabla_v X$, being ∇ the Levi-Civita connection of g and $v \in T_xM$, $x \in M$ [22, 19].

Now observe that

$$\operatorname{div}(X) = -\operatorname{trace}(A_X),$$

for any X and any dimension of M.

On the other hand, when $\dim M = 2$, we have

$$\operatorname{Ric}(X,Y) = K\, g(X,Y),$$

for all X, Y, where K denotes the Gauss curvature of g.

We want to apply Bochner formula to the unitary tangent vector field $\mathcal{T} := T/g(T,T)^{\frac{1}{2}}$ in the case $\dim M = 2$. Taking into account $g(\mathcal{T},\mathcal{T}) = 1$, we have that the matrix of $A_{\mathcal{T}}$ respect to an orthonormal basis $(\mathcal{T}, E)$ is

$$\begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}.$$

Therefore, we have

$$\operatorname{trace}(A_{\mathcal{T}}^2) = (\operatorname{trace}(A_{\mathcal{T}}))^2 = (\operatorname{div}(\mathcal{T}))^2.$$

Hence, we arrive to the following formula,

$$\mathcal{T}(\operatorname{div}(\mathcal{T})) = -K + \operatorname{div}(\nabla_{\mathcal{T}}\mathcal{T}) - (\operatorname{div}(\mathcal{T}))^2. \tag{2.1}$$

On the other hand, using the well known formula

$$\operatorname{div}(fX) = X(f) + f\operatorname{div}(X),$$

which holds true for any tangent vector field X and any smooth function f on M, we get

$$(\operatorname{div}(\mathcal{T}))^2 = \operatorname{div}\big(\operatorname{div}(\mathcal{T})\,\mathcal{T}\big) - \mathcal{T}(\operatorname{div}(\mathcal{T})), \tag{2.2}$$

and substituting (2.2) in (2.1), we get

$$\begin{aligned} K &= \operatorname{div}(\nabla_{\mathcal{T}}\mathcal{T}) - \operatorname{div}(\operatorname{div}(\mathcal{T})\,\mathcal{T}) \\ &= \operatorname{div}\Big(\nabla_{\mathcal{T}}\mathcal{T} - \operatorname{div}(\mathcal{T})\,\mathcal{T}\Big). \end{aligned}$$

Thus, taking $Y = \nabla_{\mathcal{T}}\mathcal{T} - \mathrm{div}(\mathcal{T})\mathcal{T}$ we have the announced formula

$$K = \mathrm{div}(Y)\,. \tag{2.3}$$

Step 2. The approximation argument

Every continuous tangent vector field X on M may be identified with a continuous map

$$X = (X_1, \cdots, X_n) : M \longrightarrow \mathbb{R}^n,$$

satisfying that $X(x)$ lies in the tangent space $T_x(M)$ (contemplated into $\mathbb{R}^n$ via the embedding), for all $x \in M$. Now, assume a continuous tangent vector field X that vanishes nowhere and consider

$$Z := \frac{X}{\|X\|} = (Z_1, \cdots, Z_n),$$

where $\|\cdot\|$ denotes the Euclidean norm of $\mathbb{R}^n$. Note that every function $Z_i : M \to \mathbb{R}$, $1 \leqslant i \leqslant n$, is continuous.

On the other hand, compactness of M implies that

$$M \subset \overline{\mathbf{B}}_0(r) = \{x \in \mathbb{R}^n : \|x\| \leqslant r\},$$

for some $r > 0$. Thus, Tietze's extension theorem [11, 14] implies that there exist functions $F_i \in \mathbf{C}(\overline{\mathbf{B}}_0(r), \mathbb{R})$ such that $F_{i|M} = Z_i$ for $i = 1, \cdots, n$.

Apply now the Stone-Weierstrass approximation theorem [17, Theorem 4.1] (see also [20]) to the space of continuous functions $\mathbf{C}(\overline{\mathbf{B}}_0(r), \mathbb{R})$ and its subalgebra $\mathbb{R}[x_1, \cdots, x_n]$ to conclude that there are polynomials $P_i \in \mathbb{R}[x_1, \cdots, x_n]$, $1 \leqslant 1 \leqslant n$, such that

$$\sup_{\|\mathbf{x}\| \leqslant r} |F_i(\mathbf{x}) - P_i(\mathbf{x})| < \frac{1}{2\sqrt{n}}, \; i = 1, \cdots, n,$$

which implies that

$$\begin{aligned}
\sup_{\mathbf{x}\in M}\left(\sum_{i=1}^{n}|Z_i(\mathbf{x}) - P_i(\mathbf{x})|^2\right)^{\frac{1}{2}} &\leqslant \sup_{\|\mathbf{x}\|\leqslant r}\left(\sum_{i=1}^{n}|F_i(\mathbf{x}) - P_i(\mathbf{x})|^2\right)^{\frac{1}{2}} \\
&\leqslant \left(n\big(\max_{1\leqslant i\leqslant n}\sup_{\|\mathbf{x}\|\leqslant r}|F_i(\mathbf{x}) - P_i(\mathbf{x})|\big)^2\right)^{\frac{1}{2}} \\
&< \left(n\frac{1}{4n}\right)^{\frac{1}{2}} = \sqrt{\frac{1}{4}} = \frac{1}{2}\,.
\end{aligned}$$

In particular,

$$P = (P_1, \cdots, P_n) : M \longrightarrow \mathbb{R}^n,$$

defines a smooth map. Let us now consider the tangential component T of P. In other words, we consider, for each $\mathbf{x} \in M$, the decomposition of $T_{\mathbf{x}}\mathbb{R}^n$ as a direct sum $T_{\mathbf{x}}\mathbb{R}^n = T_{\mathbf{x}}M \oplus T_{\mathbf{x}}^{\perp}M$ and decompose

$$P = T + U,$$

along the embedding, with $T(\mathbf{x}) \in T_{\mathbf{x}}M$ and $U(\mathbf{x}) \in T_{\mathbf{x}}^{\perp}M$, for each $\mathbf{x} \in M$. It follows that $T(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in M$ since $T(\mathbf{x}_0) = 0$ for some $\mathbf{x_0} \in M$ implies that

$$\frac{1}{4} > \|P(\mathbf{x}_0) - Z(\mathbf{x}_0)\|^2 = \|U(\mathbf{x}_0) - Z(\mathbf{x}_0)\|^2 = \|U(\mathbf{x}_0)\|^2 + 1 > 1,$$

which is impossible (note that in the formula above we have used that $U(\mathbf{x}_0) \perp Z(\mathbf{x}_0)$). Hence T is a smooth tangent vector field with no zeroes on M.

Remark. Obviously, the same conclusion can be obtained for M of any dimension.

Step 3. The Gauss-Bonnet argument

If we apply the classical divergence theorem taking in mind (2.3), we have that

$$\int_M K\,dA = 0,$$

(compare with [8, Theorem p. 68]). However, from the Gauss-Bonnet theorem we know

$$\int_M K\,dA = 2\pi\chi(M),$$

where $\chi(M)$ is the Euler number of M. Therefore, $\chi(M) = 0$ and consequently M is a torus [9, Theorem 9.3.11].

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J. M. ALMIRA,
Departamento de Ingeniería y Tecnología de Computadores, Universidad de Murcia.
30100 Murcia, SPAIN
e-mail: `jmalmira@um.es`

A. ROMERO,
Departamento de Geometría y Topología, Universidad de Granada
18071 Granada, SPAIN.
e-mail: `aromero@ugr.es`