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PROPERTIES OF A BARYCENTRIC RATIONAL INTERPOLANT*

Abstract. We study two properties of a rational interpolant which has been introduced thirty years ago, is linear in the interpolated function and converges exponentially rapidly when the nodes are conformal images of Chebyshev points. About its conditioning, we show that it is remarkably well–conditioned, as its Lebesgue constant grows about as slowly as that of polynomial interpolation between Chebyshev nodes, and this with very little dependence on the interpolation points. About its convergence for arbitrary matrices of nodes, we prove that, at least under some conditions about the relative location of the nodes, the interpolation error decays as the square of the maximal mesh distance.

1. Introduction

In its simplest version, polynomial interpolation consists in associating to a vector $\mathrm{x} = [x_0, \dots, x_n]^T$ of, say, $n+1$ distinct abscissae, and a vector $\mathrm{f} = [f_0, \dots, f_n]^T$ of corresponding ordinates, which may be function values of a function f or not, the unique polynomial $p_n[\mathrm{f}]$ of degree at most n that takes on the value f_j at x_j, $i = 0, \dots, n$. The corresponding application has several nice properties. It is linear: the polynomial interpolating a linear combination of f–vectors is the linear combination of the corresponding polynomials, i.e., $p_n[\alpha\mathrm{f} + \beta\mathrm{g}] = \alpha p_n[\mathrm{f}] + \beta p_n[\mathrm{g}]$, $\forall\ \alpha, \beta \in \mathbb{R}$; moreover, the uniqueness makes it a projection, a property that permits the study of the conditioning by means of the Lebesgue constant.

But it has its drawbacks, the main one being the fact that it is useful for large n only if the nodes are located in particular ways on the interpolation interval $[a,b] \equiv [\min x_j, \max x_j]$ [15, chap. 5]. Long before the advent of electronic computing, for instance, Runge has shown that, for the most important case of equidistant nodes, the interpolant of a meromorphic function converges only if the poles of the latter lie outside a certain ellipse containing $[a,b]$ [10]. And even for functions, such as e^x, for which the interpolant converges for equidistant points, it is extremely ill–conditioned for n large [11, Section 5.5].

It therefore is legitimate to search for alternative infinitely smooth interpolants which work well for general sets of nodes. In [3], the author took advantage of the observation by Werner [18] that, by modifying the so–called *weights* λ_j in the barycentric formula

$$p_n[\mathrm{f}](x) = \sum_{j=0}^{n} \frac{\lambda_j}{x - x_j} f_j \Bigg/ \sum_{j=0}^{n} \frac{\lambda_j}{x - x_j}, \qquad \lambda_j := 1 \Big/ \prod_{k \neq j} (x_j - x_k), \tag{1.1}$$

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for p_n, one obtains a rational function of degree $\leqslant n$, i.e., one with both numerator and denominator degrees at most n; he suggested choosing weights β_j alternative to the λ_j to obtain a good denominator for the given set of nodes, instead of the unique denominator 1 for all sets of nodes in polynomial interpolation. That way the two properties of linearity and projection are preserved [4].

A good choice of the β_j requires the ordering of the nodes according to

$$a = x_0 < x_1 < \cdots < x_n = b\,; \tag{1.2}$$

indeed, the weights ought to alternate in sign, for otherwise the interpolant will not reproduce all linear functions [12]. In view of this, the simplest possible choice of the weights, as proposed in [3], is

$$\beta_j = (-1)^j, \tag{1.3}$$

independently of the value of the nodes. It was also shown in [3] that the corresponding interpolant, say R_0, does not possess any real poles; moreover, its $O(h)$–convergence and well–conditioning have been conjectured. The first has been proved in [9] under a local mesh ratio condition. For an approximation operator which is a projection, the second is best addressed through its Lebesgue constant Λ_n. For R_0, the latter has been shown to increase merely logarithmically for the most frequently used sets of nodes [7], a behaviour as nice as that of polynomial interpolation with the most favourable nodes (see also [17]).

However, by considering the even rational trigonometric interpolant obtained by transferring the polynomial one to the circle with the transform $x = \cos\phi$ [3], one comes to the conclusion that the first and last terms of the sums in both the numerator and the denominator of the interpolant should appear only once and the other terms twice. After simplifying by a factor of two, one sees that the weights [3]

$$\beta_j = (-1)^j \delta_j, \qquad \delta_j := \begin{cases} 1/2, & j = 0 \text{ or } j = n, \\ 1, & \text{otherwise}, \end{cases} \tag{1.4}$$

should be better than (1.3). (To be precise, this formula for the weights holds in the present case, where $x_0 = a$ and $x_n = b$, i.e., where the interval $[x_0, x_n]$ covers the whole interval of interpolation: when the interpolant is to be used beyond one or both extremities, slightly more complicated formulae involving trigonometric functions should be used, see [3].) *We shall denote the corresponding interpolant by* R_1. When the nodes are the Chebyshev points of the second kind, the weights (1.4) coincide with the λ_j [11, p. 252], $R_1 = p_n[\mathrm{f}]$, and exponential convergence is achieved [16, p. 57]. It has been shown in [3] that R_1 does not have any poles in $[a,b]$ for any set of nodes, and conjectured in the same work that it is well–conditioned. Besides, it has been shown in [1] that it retains the exponential convergence when the x_j are conformally shifted Chebyshev nodes, and conjectured that the convergence is $O(h^2)$ in general.

In 2007, Floater and Hormann [9] have presented a family of interpolants with a higher order of convergence. Let the latter be given as $d+1$. To every d and every

set of nodes x, these authors give a linear rational interpolant $r_n[\mathrm{f}]$, depending on d, such that $\|r_n[\mathrm{f}]-f\| = O(h^{d+1})$. After constructing it as a blend of the $n-d$ interpolating polynomials corresponding to the subvectors of $d+1$ consecutive nodes, they provide the weights of its barycentric representation (which always exist, see [5]). For equidistant nodes, their $r_n[\mathrm{f}]$ to $d=0$ and $d=1$ correspond to R_0 and R_1, respectively. However, this is not the case with other sets of nodes.

In the present paper, we give conditions on the set of nodes under which the $O(h^2)$ convergence of R_1 is achieved; moreover we study its Lebesgue constant for equidistant and well–spaced nodes such as conformally shifted Chebyshev points of the second kind.

2. The conditioning of the interpolant

Given real abscissae (1.2) and a function $f:[a,b]\to\mathbb{R}$, we thus consider the barycentric rational interpolant

$$R_1(x) = \sum_{j=0}^{n}{}'' \frac{(-1)^j f_j}{x-x_j} \Bigg/ \sum_{i=0}^{n}{}'' \frac{(-1)^j}{x-x_j}, \tag{2.5}$$

where the double prime means that the first and last terms of the sum are multiplied by $1/2$. Let

$$h := \max_{i=0,\dots,n-1}(x_{j+1}-x_j).$$

The conditioning of a problem is the sensitivity of its solution to perturbations of its data. In many contexts, it is called stability, but in numerical analysis the latter should be reserved for the stability of an algorithm [11, p. 33]. Here the data are the function values f and the solution is the interpolant R_1.

The fact that R_1 is a linear projection operator into a linear space of functions with a Lagrange basis, i.e., a basis $\{b_j(x)\}$ with the Lagrange property $b_j(x_i)=\delta_{ij}$, induces that the norm of that operator, called its Lebesgue constant and denoted Λ_n, provides a good characterization of the conditioning.

DEFINITION 1. *We shall say that* Λ_n grows at most logarithmically with *n, when*

$$\Lambda_n \leqslant a + b\ln n \tag{2.6}$$

for some positive constants a and b and every $n > n_0$, $n_0 \in \mathbb{N}$.

3.1. Equidistant points

THEOREM 1. *For uniformly spaced nodes, the Lebesgue constant of the interpolant R_1 grows at most logarithmically with n.*

This result is already known as the case $d = 1$ of [6]. However, our proof of the bound for D does not make use of the original formula for the Floater–Hormann interpolant, but merely of the barycentric formula.

3.2. Well–spaced nodes

The article [7] introduces the following property of interpolation points.

DEFINITION 2. *For each $n \in \mathbb{N}$, let X_n be a set of interpolation nodes satisfying property* (1.2) *and let*

$$h_k := x_{k+1} - x_k, \qquad k = 0, \dots, n-1.$$

A matrix $X = (X_n)_{n\in\mathbb{N}}$ of such sets is called well–spaced *if there exist constants K and R, $R \geqslant 1$, both independent of n, such that the three conditions*

$$\frac{x_k - x_{k-1}}{x_k - x_j} \leqslant \frac{K}{k-j}, \qquad k = 1, \dots, n, \quad j = 0, \dots, k-1, \tag{2.7}$$

$$\frac{x_{k+1} - x_k}{x_j - x_k} \leqslant \frac{K}{j-k}, \qquad k = 0, \dots, n-1, \quad j = k+1, \dots, n, \tag{2.8}$$

$$\frac{h_k}{h_{k-1}} \leqslant R, \quad \frac{h_{k-1}}{h_k} \leqslant R, \qquad k = 1, \dots, n-1, \tag{2.9}$$

hold for every set of nodes X_n.

Note that the property neither depends on a shift of the nodes nor on a multiplication of all of them with the same constant.

The authors of [7] have discovered that every such matrix leads to a logarithmic growth of the Lebesgue constant of R_0.

THEOREM 2. *If $X = (X_n)_{n\in\mathbb{N}}$ is a matrix of well–spaced nodes, then the Lebesgue constant of the interpolant R_0 grows at most logarithmically with n.*

We show the same for R_1.

THEOREM 3. *If $X = (X_n)_{n\in\mathbb{N}}$ is a matrix of well–spaced nodes, then the Lebesgue constant of the interpolant R_1 grows at most logarithmically with n.*

3.3. Conformally shifted nodes

Bos and al. [7] give the following definition of a (regular) distribution function of nodes.

DEFINITION 3. *A function $F \in C[0,1]$ is a* distribution function *if it is a strictly increasing bijection of the interval $[0,1]$. F is* regular*, if $F \in C^1[0,1]$ and F' has a finite number of zeros in $[0,1]$ with finite multiplicities.*

Given a distribution function F and some $n \in \mathbb{N}$, the authors define the associated interpolation nodes $X_n = X_n(F)$ as the set of x_k with

$$x_k := F(k/n), \qquad k = 0, 1, \ldots, n. \tag{2.10}$$

They then prove the following result.

THEOREM 4. *If F is a regular distribution function and X_n are the associated interpolation nodes from (2.10) for every $n \in \mathbb{N}$, then the matrix of nodes $X = (X_n)_{n\in\mathbb{N}}$ is well–spaced.*

Since the derivative of a conformal map g cannot have any zero, Theorems 2 and 3 automatically yield the following results.

COROLLARY 1. *The Lebesgue constants of R_0 as well as of R_1 with conformally shifted* equidistant *points grow at most logarithmically with n.*

Classical examples of conformal shifts are the Kosloff–Tal Ezer and the Bayliss-Turkel maps; see [13] for a recent list of such maps and a new one introduced by the authors.

For R_1, the same can be proven about conformally shifted Chebyshev points. Indeed, the chain rule immediately implies the following property.

LEMMA 1. *A composition of regular distribution functions is itself a regular distribution function.*

If a distribution function is conformal, then it is regular, as its derivative cannot vanish. The corresponding matrix of nodes then is automatically well–spaced, according to Theorem 4.

COROLLARY 2. *The Lebesgue constant of R_1 with conformally shifted* Chebyshev *points of the second kind grows at most logarithmically with n.*

This guarantees the good conditioning of the exponentially convergent methods based on R_1 such as that introduced in [2] and later used in [14] and [8].

3. Convergence of R_1

THEOREM 5. *Suppose $f \in C^3[a,b]$. If there is a constant C such that*

$$\sum_{k=0}^{i}(-1)^k(x_{k+1}-x_k) \leqslant Ch, \tag{3.11}$$

for all $i = 0,1,\ldots,n-1$, and if

$$\begin{aligned} &(x_{i+1}-x_i)(x_{i+2}-x_0) \geqslant (x_{i+2}-x_{i+1})(x_i-x_0), && i = 1,\ldots,n-2,\\ &(x_i-x_{i-1})(x_n-x_{i-2}) \geqslant (x_{i-1}-x_{i-2})(x_n-x_i), && i = 2,\ldots,n-1, \end{aligned} \tag{3.12}$$

then

$$\|r-f\| = O(h^2) \quad \text{as } h \to 0.$$

The numerical experiments to be presented in the complete version of this work confirm the above results.

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