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CONVERGENCE AND STABILITY OF SPLIT-STEP-THETA METHODS FOR STOCHASTIC DIFFERENTIAL EQUATIONS WITH JUMPS UNDER NON-GLOBAL LIPSCHITZ DRIFT COEFFICIENT

Abstract. In this paper we present some of our ongoing researches where we investigate the convergence and stability of the split step theta method (SSTM) and its compensated form for stochastic differential equations with jumps (SDEwJs) under non-global Lipschitz condition of the drift term. The methods converge strongly to the exact solution in the root mean square with order $1/2$. Stability analysis reveals that the compensated split-step-theta method (CSSTM) holds the A-stability property for $\theta \in [1/2, 1]$ for both linear and nonlinear cases. For a linear test equation with a negative drift and positive jump coefficients, there exists $\theta \leqslant 1/2$ for which the SSTM is A-stable. This overcome the barrier of θ by D. J. Higham & P. E. Kloeden (2006) and X. Wang & S. Gan (2010). In the nonlinear case the SSTM holds the B-stability property. We give some numerical experiments to illustrate our theoretical results.

1. Introduction

In this work, we consider the jump-diffusion Itô's stochastic differential equations in the following form

$$(1.1) \quad dX(t) = f(X(t^-))dt + g(X(t^-))dW(t) + h(X(t^-))dN(t), \quad X(0) = X_0,$$

where $T > 0$ is a fixed final time, $t \in [0, T]$, $W(t)$ is a m-dimensional Brownian motion, the function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $n \in \mathbb{N}$ satisfies the one-sided Lipschitz condition, the local Lipschitz condition and the polynomial growth condition. The functions $g : \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times m}$ and $h : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfy the global Lipschitz condition, $N(t)$ is a one dimensional Poisson process with intensity λ. Extension to vector-valued jumps with independent entries is straightforward. Equations of type (1.1) arise in a range of scientific, engineering and financial applications [1, 12]. Since most SDEwJs are rarely solved explicitly, numerical analysis is an important tool in studying stochastic models. To characterise a numerical scheme, it is necessary to examine its convergence and its stability. Strong convergence, weak convergence and stability of numerical methods are thoroughly investigated for SDEs and SDEwJs with global Lipschitz coefficients, see e.g., [5, 7, 9, 13, 17] and references therein. In many models, the drift coefficient does not satisfy the global Lipschitz condition. A typical example is a stochastic Ginzburg-Landau equations [2] with a cubic nonlinear drift term $x^3 - x$, $x \in \mathbb{R}$. Recently it has been proved [11] that the Euler-Maruyama method often fails to converge strongly to the exact solution of nonlinear SDEs of the form (1.1) without jump term when at least one of the functions f or g grows superlinearly. To overcome this drawback of the Euler-Maruyama method, convergence of tamed explicit numerical approximations was proposed in [8, 9, 11, 15] and references therein. The stability properties of tamed schemes are quite limited due to the fact that such schemes are explicit [15]. Due to their stability behaviour, implicit methods with non global Lipschitz condition were proposed in [5]. Many implicit theta methods proposed to solve SDEwJs require the global Lipschitz condition, see e.g., [4, 16, 17] and references therein. However, for SDEs without jump, with non-global Lipschitz condition, some theta methods (e.g., [18, 19]) converge strongly for $1/2 \leqslant \theta \leqslant 1$. To sum up convergence and stability of SDEwJs with non-global Lipschitz condition are still not well understood. Here we study the convergence and stability of the SSTM and CSSTM for SDEwJs under non-global Lipschitz condition of the drift f. Note that the CSSTM for SDEwJs was introduced in [7, Section 5] where only the exponential mean square stability was studied under restrictive assumptions than the current assumptions (see [19, Section 1]). To the best of our knowledge, there is no numerical method for SDEwJs in the non compensated form which holds the A-stability property. In this work we propose results of recent researches whose proofs are detailed in [14], where it is proved that the SSTM holds either the A-stability or the B-stability property for a linear test equation and holds a B-stability property for a general nonlinear equation. Furthermore for a linear test equation with negative drift and positive jump coefficients, we have found that there exists $\theta \leqslant 1/2$ for which the SSTM holds the A-stability property. This overcome the barrier of θ in [4, Theorem 3.4] and [17]. Note that the stochastic theta method (STM) was introduced in [4], where authors proved that the method holds the A-stability property for a linear test equation with positive jump coefficient, but fails for negative jump coefficient. Compensated theta method (CTM) [17] was proved to hold the A-stability property for any jump coefficient; with $\theta \in [1/2, 1]$. But both STM and CTM were proved to convergence under global Lipschitz condition. Here, our SSTM and CSSTM converge strongly under non-global Lipschitz condition and the CSSTM holds the A-stability for any $1/2 \leqslant \theta \leqslant 1$.

The rest of this paper is organized as follows. In Sections 2, 3, 4 we resume the main results reported in [14], related to the convergence, the linear and the non linear stability. We end in Section 5 by giving some numerical experiments illustrating our theoretical finding.

2. Strong convergence of the split-step theta method

Throughout this paper, $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a complete probability space with a filtration $(\mathcal{F}_t)_{t\geqslant 0}$. For all $x, y \in \mathbb{R}^n$, $n \in \mathbb{N}$ we write $\langle x, y\rangle = x_1 y_1 + \cdots x_n y_n$, and $\|x\| := \langle x, x\rangle$. For a matrix $A \in \mathbb{R}^{m\times n}$, $m \in \mathbb{N}$ we denote $\|A\| := \sup_{x\in\mathbb{R}^n, \|x\|\leqslant 1} \|Ax\|$. For two real numbers a and b we denote by $a \vee b := \max(a, b)$ and $a \wedge b := \min(a, b)$.
To ensure the existence and the uniqueness of the solution of (1.1) in the time interval $[0, T]$, we make the following assumptions.

Assumption 2.1. We assume that:

(i) The functions $f, g, h \in C^1(\mathbb{R}^n)$ and for all $p \geqslant 2$ there exits a positive constant L_p

such that $\mathbb{E}\|X_0\|^p \leqslant L_p$.

(ii) The functions g and h satisfy the global Lipschitz condition, i.e., there exists a positive constant K such that

$$\|g(x)-g(y)\|^2 \vee \|h(x)-h(y)\|^2 \leqslant K\|x-y\|, \quad x,y \in \mathbb{R}^n. \tag{2.2}$$

(iii) The function f satisfies the one-sided Lipschitz condition, i.e., there exists a positive constant L such that

$$\langle x-y, f(x)-f(y)\rangle \leqslant L\|x-y\|^2, \quad x,y \in \mathbb{R}^n. \tag{2.3}$$

The proof of the following theorem can be found in [3] and [4, Lemma 1].

THEOREM 1. *Under Assumption 2.1, the SDEwJs (1.1) has a unique solution on $[0,T]$ with all moments bounded, i.e., for all $p \geqslant 2$, there exists a positive constant K_p such that $\mathbb{E}\|X(t)\|^p \leqslant K_p, \quad t \in [0,T]$.*

In the rest of this paper, we take $\theta \in (0,1]$. Following [18], we apply the split-step-theta method to SDEwJs (1.1) and obtain the following scheme called SSTM

$$\begin{cases} Y_m^* = Y_m + \theta f(Y_m^*)\Delta t \\ Y_{m+1} = Y_m + f(Y_m^*)\Delta t + g(Y_m^*)\Delta W_m + h(Y_m^*)\Delta N_m, \end{cases} \tag{2.4}$$

with initial value $Y_0^* := X_0$, where $\Delta t = T/M$ is the time step-size, $M \in \mathbb{N}$ is the number of time subdivisions, $\Delta W_m := W(t_{m+1}) - W(t_m)$ and $\Delta N_m := N(t_{m+1}) - N(t_m)$ are respectively the increment of the Brownian motion and the Poisson process. We recall that the compensated Poisson process $\overline{N}(t) := N(t) - \lambda t$ is a martingale satisfying the following properties

$$\mathbb{E}\left(\overline{N}(t+s) - \overline{N}(t)\right) = 0, \qquad \mathbb{E}|\overline{N}(t+s) - \overline{N}(s)|^2 = \lambda t, \qquad t,s > 0. \tag{2.5}$$

Using the compensated Poisson process, we can rewrite the jump-diffusion SDEs (1.1) in the following equivalent form

$$dX(t) = f_\lambda(X(t^-))dt + g(X(t^-))dW(t) + h(X(t^-))d\overline{N}(t), \tag{2.6}$$

where $f_\lambda(x) := f(x) + \lambda h(x)$. Note that as f, the function f_λ satisfies the one-sided Lipschitz condition (iii) of Assumption 2.1. Applying the compensated split-step theta method as in [7] to (1.1) yields the following scheme for SDEwJs (2.6)

$$\begin{cases} Z_m^* = Z_m + \theta f_\lambda(Z_m^*)\Delta t \\ Z_{m+1} = Z_m + f_\lambda(Z_m^*)\Delta t + g(Z_m^*)\Delta t W_m + h(Z_m^*)\Delta\overline{N}_m, \end{cases} \tag{2.7}$$

where $\Delta\overline{N}_m := \overline{N}(t_{m+1}) - \overline{N}(t_m)$ is the increment of the compensated Poisson process. Let us define the time continuous interpolation of the discrete numerical approximations of (2.4) by

$$\overline{Y}(t) = Y_m + f(Y_m^*)(t-t_m) + g(Y_m^*)(W(t) - W(t_m)) + h(Y_m^*)(N(t) - N(t_m)), \tag{2.8}$$

for $t \in [t_m, t_{m+1})$.

The strong convergence result of the SSTM requires the following local Lipschitz condition.

Assumption 2.2. For any $R > 0$, there exists a positive constant L_R such that

$$\|f(x)-f(y)\|^2 \leqslant L_R\|x-y\|^2, \quad x,y \in \mathbb{R}^d, \quad \|x\| \vee \|y\| \leqslant R. \tag{2.9}$$

The proof of the following theorem is reported in [14].

THEOREM 2. *[Strong convergence result]*
Let $X(t)$ be the exact solution of (1.1). Under Assumptions 2.1 and 2.2, for any $1/2 \leqslant \theta \leqslant 1$ and $0 < \Delta t < \min\left\{1, \dfrac{1}{2\theta L}\right\}$, the following error estimate holds for the time continuous approximation (2.8)

$$\lim_{\Delta t \longrightarrow 0} \mathbb{E}\left[\sup_{t\in[0,T]} \|X(t)-\overline{Y}(t)\|^2\right] = 0. \tag{2.10}$$

If moreover there exist $E > 0$ and $q > 0$ such that f satisfies the following polynomial growth condition

$$\|f(x)-f(y)\|^2 \leqslant E(1+\|x\|^q+\|y\|^q)\|x-y\|^2, \quad x,y \in \mathbb{R}^d,$$

then the numerical scheme (2.8) satisfies

$$\mathbb{E}\sup_{0\leqslant t\leqslant T}\|\overline{Y}(t)-X(t)\|^2 = O(\Delta t).$$

3. Linear mean-square stability

In this section, we consider the following linear test-equation

$$dX(t) = aX(t^-)dt + bX(t^-)dW(t) + cX(t^-)dN(t), \quad X(0) = X_0, \tag{3.11}$$

where a, b, and c are scalar real coefficients. It is well known (see e.g., [4, Section 3.3]) that the exact solution of (3.11) is mean-square stable if and only if

$$l := 2a + b^2 + \lambda c(2+c) < 0. \tag{3.12}$$

The proof of the following two stability results can be found in [14].

THEOREM 3. *Let inequality (3.12) holds. Then for all $\theta \in [1/2, 1]$, the CSSTM (2.7) satisfies the A-stability property, i.e., the numerical method (2.7) is stable for any step-size Δt.*

THEOREM 4. *Let inequality (3.12) holds. Then the SSTM (2.4) applied to the linear test equation (3.11) is mean-square stable if and only if*

$$(a+\lambda c)(a+\lambda c-2a\theta)\Delta t < -l. \tag{3.13}$$

REMARK 1. (i) If $a > 0$, then it follows from (3.13) that the numerical method (2.7) is stable if and only if $\Delta t < \dfrac{-l}{(a+\lambda c)(a+\lambda c-2a\theta)}$.

(ii) If $a < 0$ and $\theta < \min\left\{\dfrac{a+\lambda c}{2a}, 1\right\}$, then from (3.13) the numerical method is stable if and only if $\Delta t < \dfrac{-l}{(a+\lambda c)(a+\lambda c-2a\theta)}$.

(iii) If $a < 0$ and $\dfrac{a+\lambda c}{2a} < 1$, then for any $\dfrac{a+\lambda c}{2a} \leqslant \theta < 1$ the numerical method (2.7) satisfies the A-stability property.

REMARK 2. To the best of our knowledge, there is no theta method satisfying the A-stability for $\theta \in [0, 1/2]$. Note that if $a < 0$ and $c > 0$ then $\frac{a+\lambda c}{2a} < \frac{1}{2}$, since the jump intensity λ is positive. Therefore from Remark 1 (iii), it holds that for all $\frac{a+\lambda c}{2a} \leqslant \theta \leqslant \frac{1}{2}$ the split-step-theta method (2.4) applied to the linear test equation (3.11) is A-stable. This overcomes the barrier of θ given in [4, Theorem 3.4]. To the best of our knowledge, this is the first numerical scheme for SDEwJs which in its non compensated form can satisfy the A-stability property.

4. Nonlinear stability

This section is devoted to analyse the exponential mean-square stability of the CSSTM and the SSTM. We make the following assumption.

Assumption 4.1. There exist constants μ, σ and γ such that the following

$$\langle x-y, f(x)-f(y)\rangle \leqslant \mu\|x-y\|^2,\ \|g(x)-g(y)\|^2 \leqslant \sigma\|x-y\|^2,\ \|h(x)-h(y)\|^2 \leqslant \gamma\|x-y\|^2$$

holds for every $x, y \in \mathbb{R}^n$.

It is well known (see e.g., [5, Theorem 4]) that under Assumptions 4.1, if

$$\alpha := 2\mu + \sigma + \lambda\sqrt{\gamma}(2+\sqrt{\gamma}) < 0, \tag{4.14}$$

then the exact solution of equation (1.1) is exponentially mean-square stable. The proof of the following two stability results can be found in [14].

THEOREM 5. *Let Assumption 4.1 and inequality (4.14) be fulfilled. Then for any $1/2 \leqslant \theta \leqslant 1$, for any step-size $\Delta t > 0$ and for two solutions X_m and Y_m of the CSSTM with initial values X_0 and Y_0 satisfying respectively $\mathbb{E}\|X_0\|^2 < \infty$ and $\mathbb{E}\|Y_0\|^2 < \infty$, the following estimate holds*

$$\mathbb{E}\|X_m - Y_m\|^2 \leqslant \mathbb{E}\|X_0 - Y_0\|^2 e^{\beta(\Delta t,\theta)m\Delta t}, \quad \text{where}$$

$$\beta(\Delta t, \theta) := \frac{1}{\Delta t}\log\left[\frac{1+\left(\alpha-(\mu+\lambda\sqrt{\gamma})\theta\right)\Delta t}{1-(\mu+\lambda\sqrt{\gamma})\theta\Delta t}\right] < 0.$$

THEOREM 6. *Let Assumption 4.1 and inequality (4.14) be fulfilled. Then for any* θ *and* Δt *satisfying*

$$\max\left\{\frac{-4\lambda\sqrt{\gamma}}{\alpha},\frac{1}{2}\right\}<\theta\leqslant 1,\quad \Delta t<\frac{-\left[\alpha+\dfrac{4\lambda\sqrt{\gamma}}{\theta}\right]}{\lambda^2\gamma-2\mu\theta\lambda\sqrt{\gamma}\left(1+\dfrac{1}{\theta}\right)},$$

and for any two solutions X_m *and* Y_m *of the SSTM (2.4) such that* $\mathbb{E}\|X_0\|^2<\infty$ *and* $\mathbb{E}\|Y_0\|^2<\infty$*, the following estimate holds*

$$\mathbb{E}\|X_m-Y_m\|^2\leqslant\mathbb{E}\|X_0-Y_0\|^2e^{\widehat{\beta}(\Delta t,\theta)m\Delta t},\qquad \text{where}$$

$$\widehat{\beta}(\Delta t,\theta):=\frac{1}{\Delta t}\log\left[\frac{1+\left(\alpha+\dfrac{4\lambda\sqrt{\gamma}}{\theta}-2\mu\theta\right)\Delta t+\left(\lambda^2\gamma-2\mu\theta\lambda\sqrt{\gamma}(1+\dfrac{1}{\theta})\right)\Delta t^2}{1-2\mu\theta\Delta t}\right].$$

5. Numerical Experiments

In all simulations, the expectation is obtained by averaging 5×10^3 sample paths.

5.1. Linear case

In this section, we consider the following linear test equation

$$dX(t)=aX(t)dt+bX(t)dW(t)+cX(t)dN(t),\quad X(0)=1,\tag{5.15}$$

with the following coefficients

(i) Example I. $a=-7$, $b=1$, $c=1$ and $\lambda=4$.

(ii) Example II. $a=2$, $b=2$, $c=-0.9$ and $\lambda=9$.

For the two above examples, we can easily check that $l=2a+b^2+\lambda c(2+c)<0$. Therefore, the exact solution is mean-square stable. In example I, we observe that $a<0$ and $c>0$, therefore from Remark 1 (iii) it holds that for all $\frac{a+\lambda c}{2a}=\frac{3}{14}\leqslant\theta\leqslant1$ the SSTM is mean-square stable for all $\Delta t>0$. Therefore for $\theta\in[\frac{3}{14},1/2)$ the SSTM holds the A-stability property. In Figures 1 and 2, we can also observe that our estimate is very sharp. Indeed, for $\theta=0.2145$ the scheme is stable (Figure 1) and for $\theta=0.21$ the scheme is unstable (Figure 2). For $\theta=0.499$, CSSTM scheme is stable if $\Delta t<\frac{-l}{(1-2\theta)(a+\lambda c)^2}=\frac{100}{36}$. In example II, the assumptions made in our theoretical result are not fulfilled for SSTM to ensure the A-stability, since $a>0$. In Figure 3, we illustrate the stability of the CSSTM with differents values of θ. We can observe that for $\theta=0.53$ and $\theta=0.57$, the CSSTM is stable but for $\theta=0.495$, the CSSTM becomes unstable.

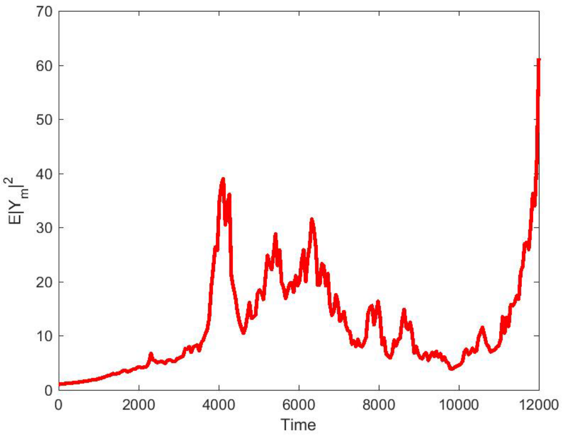

Figure 1: SSTM applied to Example I with $\theta = 0.21$, illustrating the instability of the scheme.

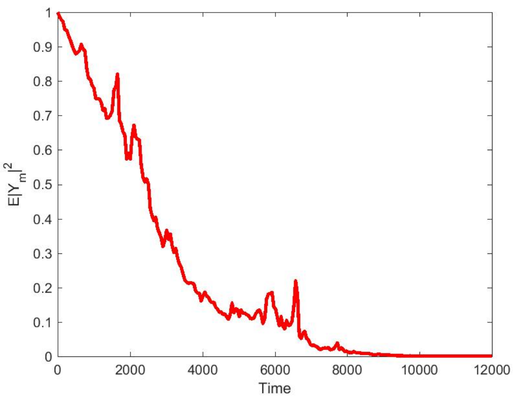

Figure 2: Linear stability of the SSTM for Example I with $\theta = 0.2145$ illustrating the stability of the scheme.

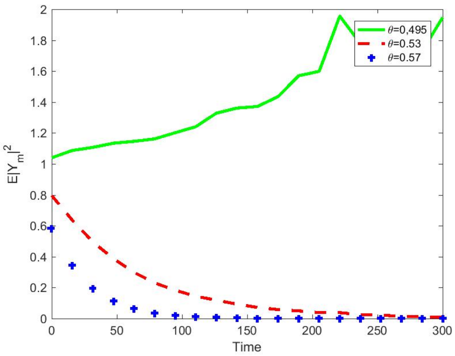

Figure 3: Linear stability of the CSSTM for Example II with different values of θ and $\Delta t = 15$.

5.2. Nonlinear case

To illustrate the stability of the SSTM and CSSTM in the nonlinear case, we consider the following nonlinear SDEwJs. Remember that the nonlinear stability of CSSTM has been studied in [7]. Our goal is to observe the behavior of the two schemes on concrete example.

$$dX(t) = (-4X(t) - X^3(t))dt + X(t)dW(t) + X(t)dN(t), \quad X(0) = 1, \tag{5.16}$$

on the time interval $[0,1]$. The intensity of the Poisson process is $\lambda = 1/2$. So $\mu = -4$, $\sigma = \gamma = 1$ and therefore $\alpha = 2\mu + \sigma + \lambda\sqrt{\gamma}(2+\sqrt{\gamma}) = \frac{-11}{2} < 0$. In this case, CSSTM is A-stable for any $\theta \in [1/2, 1]$. Note that $\dfrac{-4\lambda\sqrt{\gamma}}{\alpha} = \dfrac{4}{11} < 1/2$, therefore for $\theta \in [1/2, 1]$ and for any time step Δt, such that

$$\Delta t < \frac{-\left[\alpha + \dfrac{4\lambda\sqrt{\gamma}}{\theta}\right]}{\lambda^2\gamma - 2\mu\theta\lambda\sqrt{\gamma}\left(1 + \dfrac{1}{\theta}\right)},$$

the SSTM is stable. Hence, for $\theta = 0.6$ the SSTM is stable for any $\Delta t < 0.1985$ and for $\theta = 0.8$ the SSTM is stable if $\Delta t = 0.8649$. Figure 4 illustrates the stability behavior of SSTM and CSSTM for different values of θ and $\Delta t = 0.01$. We can observe that the two methods are stable and have the same behavior, although the SSTM requires many constraints to achieve the stability.

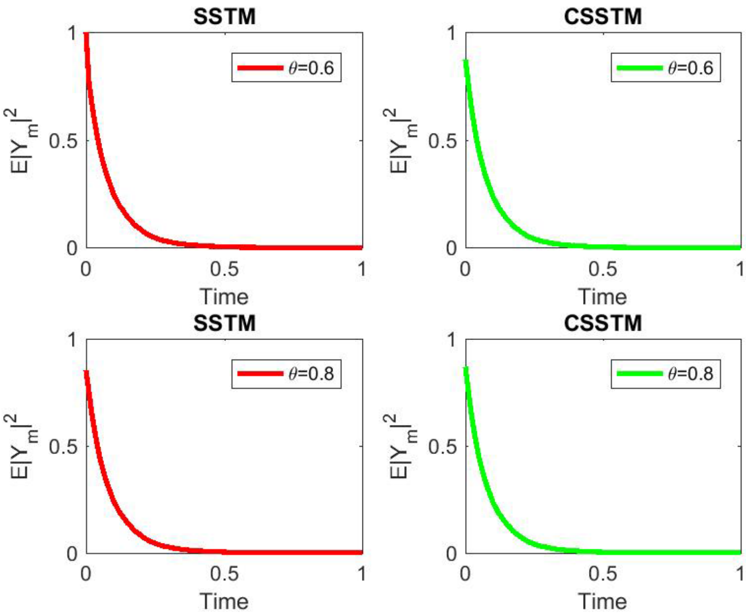

Figure 4: Exponential mean-square stability of SDE (5.16) with $\Delta t = 0.01$, $\theta = 0.6$ and $\theta = 0.8$

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