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# A QUASI-INTERPOLATION SPLINE FOR CAUCHY INTEGRAL EQUATIONS VIA REGULARIZATION

**Abstract.** In this paper, we present some our ongoing researches related to a collocation method based on a double projection scheme via regularization procedure, to numerically solve Cauchy integral equations of the second kind. For the collocation approach, we use spline quasi-interpolating projectors of degree $d$. We prove the existence of the solution for a double projection scheme. Moreover, we give new error bounds.

## 1. Introduction and mathematical background

Singular integral equations play an important roles in many fields of science and engineering such as mathematical physics, aerodynamics, theoretical mechanics, elasticity and chemical engineer. An important class of these equations is Cauchy integral equations. The author of [5] presented several remarks about the Gauss-Radau and Lobatto-Jacobi direct quadrature methods for solving Cauchy-type singular integral equations of the second kind of the form

$$a\varphi(s) + \frac{b}{\pi} \oint_{-1}^{1} \frac{\varphi(t)}{t-s} dt + \int_{-1}^{1} k(s,t)\varphi(t)dt = f(s), \quad -1 < s < 1,$$

with index equal to $-1$.

Several authors studied the above problem. In particular the aim of [6] is to solve the Cauchy singular integral equation of the second kind with constant coefficients via a modified method. The author compared this modified method with the corresponding method resulting from applying the collocation method to the Fredholm integral equation of the second kind equivalent to the Cauchy singular integral equation. Moreover, the author discussed the convergence of the method.

In the same context, the purpose of [4] is to develop a convergence analysis of collocation methods for solving the Cauchy integral equations of the second kind of the above form, under some assumptions on the data. In [8], we proposed a piecewise constant Galerkin approximation to solve the generalized Cauchy integral equation of the second kind.

In this paper, we introduce a spline quasi-interpolant collocation method via regularization to solve a class of singular integral equations.

Let us consider the following integral equation with the Cauchy kernel,

$$\varphi(s) - \oint_{0}^{1} \frac{\varphi(t)}{t-s} dt = f(s), \quad 0 < s < 1,$$

where $f$ is a known function and the integral is understood to be the Cauchy principal value:

$$\oint_0^1 \frac{\varphi(t)}{t-s}dt := \lim_{\varepsilon\to 0}\left(\int_0^{s-\varepsilon}\frac{\varphi(t)}{t-s}dt + \int_{s+\varepsilon}^1 \frac{\varphi(t)}{t-s}dt\right).$$

Let the universe under consideration be the space: $X := L^2([0,1],\mathbb{R})$.

The present problem reads as: Given $f \in X$, find $\varphi \in X$, s.t.

$$\varphi - T\varphi = f,$$

where

$$T\varphi(s) := \oint_0^1 \frac{\varphi(t)}{t-s}dt, \quad 0 < s < 1.$$

We recall that $T$ is linear and bounded from $X$ into itself, moreover $T^* = -T$ and hence its spectrum lies in $\mathrm{i}\mathbb{R}$. So, the equation has one, and only one, solution.

### 1.1. Construction of spline quasi-interpolant projectors

In this section, we consider the construction of spline quasi-interpolant projectors presented in [3].

Define $X_n := S_d^{d-1}([0,1],\mathcal{I}_n)$ the space of splines of class $C^{d-1}$ and degree $d$ on the uniform partition $\mathcal{I}_n := \{l_i = ih \quad i = 0,1,\ldots,n\}$ with $h := \frac{1}{n}$.
Consider the sets

$$\{s_i := \frac{l_{i-1}+l_i}{2}, \quad i = 1,2,\ldots,n\},$$

and

$$\mathcal{X}_n := \{\xi_i\},$$

where

$$\xi_{2i} := l_i \quad \text{for} \quad i = 0,1,\ldots,n,$$

$$\xi_{2i-1} := s_i \quad \text{for} \quad i = 1,2,\ldots,n.$$

The discrete spline quasi-interpolant is the operator

$$\mathcal{Q}_n f = \sum_{k=1}^{N} \lambda_k(f) B_k,$$

whose coefficients are linear combinations of discrete values of $f$ on the set of data points $\mathcal{X}_n$ and $N := n+d$. The $B_k$'s are the B-splines and $\lambda_k$ are continuous linear functionals. ( For details, see [3]).

### 1.2. Regularized and approximate problems

For $\varepsilon > 0$, we consider the following approximate operator

$$T_\varepsilon \varphi(s) := \int_0^1 \frac{(t-s)\varphi(t)}{(t-s)^2+\varepsilon^2} dt, \quad 0 < s < 1,$$

We recall that $T_\varepsilon$ is compact from $X$ into itself.

Let $\varphi_\varepsilon$ be the solution of the following regularized equation:

$$(I - T_\varepsilon)\varphi_\varepsilon = f.$$

Define an approximate operator as follows:

$$T_{\varepsilon,n} := Q_n T_\varepsilon Q_n.$$

Hence, we consider the following approximate equation

$$(I - T_{\varepsilon,n})\,\varphi_{\varepsilon,n} = Q_n f.$$

We can prove the following theorems.

THEOREM 1. *For $n$ large enough, $I - T_{\varepsilon,n}$ is invertible, moreover,*

$$\sup_n \|\,(I - T_{\varepsilon,n})^{-1}\,\| < 1.$$

*Proof.* The proof is similar to the proof of Theorem 1 in [7]. □

THEOREM 2. *The following estimate holds:*

$$\|\varphi_\varepsilon - \varphi_{\varepsilon,n}\|_2 \leqslant C dist(\varphi_\varepsilon, S_d^{d-1}([0,1], \mathcal{I}_n)).$$

*Proof.* It is well known that $x$ is a continuous function (cf. [9]). As in [2] and by using the following classical result (cf. [3])

$$\|x - Q_n x\|_\infty \leqslant C dist(x, S_d^{d-1}([0,1], \mathcal{I}_n)), \quad \text{where} \quad C := 1 + \|Q_n\|_\infty,$$

we get the desired result. □

Since

$$\|\varphi_{\varepsilon,n} - \varphi\|_2 \leqslant \|\varphi_{\varepsilon,n} - \varphi_\varepsilon\|_2 + \|\varphi_\varepsilon - \varphi\|_2,$$

we get

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \varphi_{\varepsilon,n} = \varphi.$$

**References**


[1] AHUES M., LARGILLIER A. AND LIMAYE B.V., *Spectral Computations for Bounded Operators*, CRC, Boca Raton, 2001.

[2] AHUES M. AND MENNOUNI A., *A Collocation Method for Cauchy Integral Equations in $L^2$*, In *Integral Methods in Science and Engineering, C. Constanda and Paul J. Harris (eds.)*, Birkhäuser, New York, NY (2011), pp. 1–5.

[3] DAGNINO C., DALLEFRATE A. AND REMOGNA S., *Projection methods based on spline quasi-interpolation for Urysohn integral equations*, Journal of Computational and Applied Mathematics, (2019) **354**, 360–372.

[4] GOLBERG M. A., *The convergence of a collocation method for a class of Cauchy singular integral equations* , Journal of Mathematical Analysis and Applications, **100** (1984), 500–512.

[5] IOAKIMIDIS N. I., *Some remarks on the numerical solution of Cauchy-type singular integral equations with index equal to* $-1$, Computers & Structures, **14**, (1981), 403–407.

[6] IOAKIMIDIS N. I., *On the numerical solution of Cauchy type singular integral equations by the collocation method*, Applied Mathematics and Computation, 12 (1983), 49-60.

[7] MENNOUNI A., *Two Projection Methods for Skew-Hermitian Operator Equations*, Mathematical and Computer Modelling, **55** (2012) pp. 1649–1654.

[8] MENNOUNI A., *Piecewise constant Galerkin method for a class of Cauchy singular integral equations of the second kind in $L^2$*, Journal of Computational and Applied Mathematics, Volume **326**, pp. 268–272, (2017).

[9] MUSHKELISHVILI N. I., *Singular Integral Equations*, Noordhoff, Groningen, 1953.

[10] VAINIKKO E. AND VAINIKKO G., *A spline product quasi-interpolation method for weakly singular Fredholm integral equations*, SIAM Journal on Numerical Analysis Vol. **46**, No. 4 (2008), pp. 1799–1820.


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