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B. Cahen

WEYL CALCULUS ON THE FOCK SPACE AND STRATONOVICH-WEYL CORRESPONDENCE FOR HEISENBERG MOTION GROUPS

Abstract. We construct a Stratonovich-Weyl correspondence for each generic representation of a Heisenberg motion group by using the Weyl calculus on the Fock space.

1. Introduction

The notion of Stratonovich-Weyl correspondence was first introduced in [38] as a generalization of the classical Weyl correspondence between functions on $\mathbb{R}^{2n}$ and operators on $L^2(\mathbb{R}^n)$ to the setting of a Lie group acting on a homogeneous space. The systematic study of Stratonovich-Weyl correspondences began with the work of J.M. Gracia-Bondìa, J.C. Vàrilly and their co-workers, see in particular [31], [28], [26] and [13]. A good review of Stratonovich-Weyl correspondences and related topics can be found in [30].

The following definition is taken from [30], see also [31].

DEFINITION 1. *Let G be a Lie group and π be a unitary representation of G on a Hilbert space $\mathcal{H}$. Let M be a homogeneous G-space and let μ be a G-invariant measure on M. Then a Stratonovich-Weyl correspondence for the triple (G,π,M) is an isomorphism $\mathcal{W}$ from a vector space of operators on $\mathcal{H}$ to a vector space of functions on M satisfying the following properties:*

1. *The function $\mathcal{W}(A^*)$ is the complex-conjugate of $\mathcal{W}(A)$;*
2. *Covariance: we have $\mathcal{W}(\pi(g)A\,\pi(g)^{-1})(x) = \mathcal{W}(A)(g^{-1}\cdot x)$;*
3. *Traciality: we have*

$$\int_M \mathcal{W}(A)(x)\,\mathcal{W}(B)(x)\,d\mu(x) = \mathrm{Tr}(AB).$$

Note that Stratonovich-Weyl correspondences are particular cases of the invariant symbolic calculi introduced and studied by J. Arazy and H. Upmeir, especially for symmetric domains, see [2], [3], [4].

In the previous definition, M is generally taken to be a coadjoint orbit of G which is associated with π by the Kirillov-Kostant method of orbits [33], [34]. A basic example is then the case when G is the $(2n+1)$-dimensional Heisenberg group. Each non-degenerate coadjoint orbit M of G is diffeomorphic to $\mathbb{R}^{2n}$ and is associated with a unitary irreducible representation π of G on $L^2(\mathbb{R}^n)$. In this case, it is well-known that the classical Weyl correspondence gives a Stratonovich-Weyl correspondence for the triple (G,π,M) [29], [30]. More generally, the Pedersen-Weyl calculus on the flat coadjoint orbits of a nilpotent Lie group G also provides a Stratonovich-Weyl correspondence for the unitay irreducible representations of G which are square-integrable modulo the center of G [37], [8].

On the other hand, Stratonovich-Weyl correspondences for the holomorphic representations of Hermitian Lie groups were obtained by taking the isometric part in the polar decomposition of the Berezin quantization map, see [18], [16], [17], [28].

In [22], we considered the case where G is a Heisenberg motion group as a non-trivial example beyond the nilpotent and Hermitian cases. A Heisenberg motion group is the semidirect product of the $(2n+1)$-dimensional Heisenberg group H_n by a compact subgroup K of the unitary group $U(n)$. Note that the Heisenberg motion groups play an important role in the theory of Gelfand pairs [9], [10].

Each generic unitary irreducible representation π of $G = H_n \rtimes K$ is holomorphically induced from the tensor product of a character of the center of H_n by a unitary irreducible representation ρ of K [9]. Then π can be associated, in the Kirillov-Kostant method of orbits, with a coadjoint O of G which is diffeomorphic to $\mathbb{C}^n \times o$ where o is the coadjoint orbit of K associated with ρ.

In [22], we used the Bargmann transform in order to get a Schrödinger realization of π in a space of square-integrable functions on $\mathbb{R}^{2n}$ and we showed that the usual Weyl correspondence then gives a Stratonovich-Weyl correspondence for π. Note that, in [23], we used a similar method to get an adapted Weyl correspondence in the sense of [14]. However, in the Schrödinger realization, it is difficult to obtain explicit formulas for the representation operators.

So, we propose here a slightly different method based on the Bargmann-Fock version of the Weyl calculus which appears as a particular case of the Weyl calculus on symmetric domains [2], [4]. Combining the Weyl calculus on $\mathbb{C}^n$ with a Stratonovich-Weyl correspondence on o [28], [16], we obtain a Stratonovich-Weyl correspondence for π. In particular, we exhibit the Stratonovich-Weyl quantizer and show that the Stratonovich-Weyl correspondence can also be obtained by using the general method of [30] and [6].

The paper is organized as follows. First, we recall the Berezin calculus (Section 2) and the complex Weyl calculus (Section 3) for the (non-degenerate) unitary irreducible representations of the Heisenberg group. In Section 4 and 5, we introduce the Heisenberg motion groups, their generic representations and the associated Berezin calculus. The Stratonovich-Weyl correspondence for a generic representation π is constructed in Section 6. In particular, we compute the Stratonovich-Weyl symbol of $d\pi(X)$ for each X in the Lie algebra of G.

2. Heisenberg group: Berezin quantization

In this section, we review some well-known results about the the Bargmann-Fock model for the unitary irreducible (non-degenerated) representations of the Heisenberg group. We follow the presentation of [20] and [22].
For each $z,\, w \in \mathbb{C}^n$, we denote $zw := \sum_{k=1}^n z_k w_k$. For each $z, z' \in \mathbb{C}^n$, let us define

$$\omega(z,z') = \frac{i}{2}(z\bar{z}' - \bar{z}z').$$

Then the Heisenberg group G_0 of dimension $2n+1$ is $\mathbb{C}^n \times \mathbb{R}$ equipped with the multiplication

$$(z,c)\cdot(z',c') = (z+z', c+c' + \tfrac{1}{2}\omega(z,z')).$$

The Lie algebra $\mathfrak{g}_0$ of G_0 can be identified to $\mathbb{C}^n \times \mathbb{R}$ with the Lie brackets

$$[(v,c),(v',c')] = (0, \omega(v,v')).$$

Let $(e_k)_{1\leqslant k\leqslant n}$ be the canonical basis of $\mathbb{C}^n$ and let $\{X_1,\ldots,X_n,Y_1,\ldots,Y_n,Z\}$ be the basis of $\mathfrak{g}_0$ defined by $X_k = (e_k,0), Y_k = (ie_k,0)$ for $k = 1,2,\ldots,n$ and $Z = (0,1)$. Then the only non trivial brackets in this basis are $[X_k\,,\, Y_k] = Z$ for $k = 1,2,\ldots,n$. In particular, we have, for each $(v,c) \in \mathfrak{g}_0$,

$$(v,c) = \sum_{k=1}^{n} (\operatorname{Re} v_k) X_k + (\operatorname{Im} v_k) Y_k + cZ.$$

Also, we denote by $\{X_1^*,\ldots,X_n^*,Y_1^*,\ldots,Y_n^*,Z^*\}$ be the corresponding dual basis of $\mathfrak{g}_0^*$. For each $\alpha \in \mathbb{C}^n$ and $\gamma \in \mathbb{R}$, let $(\alpha,\gamma)_*$ denote the element of $\mathfrak{g}_0^*$ defined by

$$\langle(\alpha,\gamma)_*, (v,c)\rangle := \omega(\alpha, v) + \gamma c$$

for each $(v,c) \in \mathfrak{g}_0$. Then we have

$$(\alpha,\gamma)_* = \sum_{k=1}^{n} (-\operatorname{Im}\alpha_k) X_k^* + (\operatorname{Re}\alpha_k) Y_k^* + \gamma Z^*.$$

For each real number $\lambda > 0$, we denote by $\mathcal{O}_\lambda$ the orbit of the element $(0,\lambda)_* = \lambda Z^*$ of $\mathfrak{g}_0^*$ under the coadjoint action of G_0 (the case $\lambda < 0$ can be treated similarly). By the Stone-von Neumann theorem, there exists a unique (up to unitary equivalence) unitary irreducible representation of G_0 whose restriction to the center of G_0 is the character $(0,c) \rightarrow e^{i\lambda c}$ [7], [29]. In fact, this representation is associated with the coadjoint orbit $\mathcal{O}_\lambda$ by the Kirillov-Kostant method of orbits [33], [34]. Indeed, if we take the polarization at λZ^* to be the space spanned by the elements $X_k + iY_k$ for $k = 1,2,\ldots,n$ and Z then the method of orbits leads to the Bargmann-Fock representation π_0 defined as follows. Note that, for our method, we take here a complex polarization at λZ^* instead of a real one.

Let $\mathcal{H}_0$ be the Hilbert space of holomorphic functions f on $\mathbb{C}^n$ such that

$$\|f\|^2_{\mathcal{H}_0} := \int_{\mathbb{C}^n} |f(z)|^2 e^{-|z|^2/2\lambda} d\mu_\lambda(z) < +\infty$$

where $d\mu_\lambda(z) := (2\pi\lambda)^{-n} dx dy$. Here $z = x + iy$ with x and y in $\mathbb{R}^n$.
Then we have

$$(\pi_0(g)f)(z) = \exp\left(i\lambda c_0 + \tfrac{1}{2} i\bar{z}_0 z - \tfrac{\lambda}{4}|z_0|^2\right) f(z + i\lambda z_0)$$

for each $g = (z_0, c_0) \in G$ and $z \in \mathbb{C}^n$.
The differential of π_0 is thus given by

$$(d\pi_0(v,c)f)(z) = i(\lambda c + \tfrac{1}{2}\bar{v}z)f(z) + df_z(i\lambda v)$$

or, equivalently, by

$$\begin{cases} d\pi_0(X_k)f(z) = \dfrac{1}{2} iz_k f(z) + \lambda i \dfrac{\partial f}{\partial z_k} \\ d\pi_0(Y_k)f(z) = \dfrac{1}{2} z_k f(z) - \lambda \dfrac{\partial f}{\partial z_k} \\ d\pi_0(Z)f(z) = i\lambda f(z). \end{cases}$$

Now, we recall the definition and some of the properties of the Berezin quantization map [11], [12], [20].
For each $z \in \mathbb{C}^n$, consider the coherent state $e_z(w) = \exp(\bar{z}w/2\lambda)$. Then we have the reproducing property $f(z) = \langle f, e_z\rangle_{\mathcal{H}_0}$ for each $f \in \mathcal{H}_0$.
Let $\mathcal{C}_0$ be the space of all operators (not necessarily bounded) A on $\mathcal{H}_0$ whose domain contains e_z for each $z \in \mathbb{C}^n$. Then the Berezin symbol of $A \in \mathcal{C}_0$ is the function $S_0(A)$ defined on $\mathbb{C}^n$ by

$$S_0(A)(z) := \frac{\langle A\, e_z\, ,\, e_z\rangle_{\mathcal{H}_0}}{\langle e_z\, ,\, e_z\rangle_{\mathcal{H}_0}}.$$

Let us consider now the action of G_0 on $\mathbb{C}^n$ defined by $g \cdot z = z - i\lambda z_0$ for $g = (z_0, c_0)$.
We have the following result, see for instance [20].

PROPOSITION 1. *1. Each $A \in \mathcal{C}_0$ is determined by $S_0(A)$;*

2. *For each $A \in \mathcal{C}_0$ and each $z \in \mathbb{C}^n$, we have $S_0(A^*)(z) = \overline{S_0(A)(z)}$;*
3. *For each $z \in \mathbb{C}^n$, we have $S_0(\mathrm{id}_{\mathcal{H}_0})(z) = 1$. Here $\mathrm{id}_{\mathcal{H}_0}$ denotes the identity operator of $\mathcal{H}_0$;*
4. *For each $A \in \mathcal{C}_0$, $g \in G_0$ and $z \in \mathbb{C}^n$, we have $\pi_0(g)^{-1}A\pi_0(g) \in \mathcal{C}_0$ and*

$$S_0(A)(g \cdot z) = S_0(\pi_0(g)^{-1}A\pi_0(g))(z);$$

5. *The map S_0 is a bounded operator from $\mathcal{L}_2(\mathcal{H}_0)$ (endowed with the Hilbert-Schmidt norm) to $L^2(\mathbb{C}^n, \mu_\lambda)$ which is one-to-one and has dense range.*

The map $\Phi_\lambda : \mathbb{C}^n \to \mathcal{O}_\lambda, z \to (-iz, \lambda)_*$ is clearly a diffeomorphism satisfying

$$\Phi_\lambda(g \cdot z) = \mathrm{Ad}^*(g)\,\Phi_\lambda(z)$$

for each $g \in G_0$ and $z \in \mathbb{C}^n$. Moreover, we have, for each $X \in \mathfrak{g}_0$ and each $z \in \mathbb{C}^n$,

$$S_0(d\pi_0(X))(z) = i\langle \Phi_\lambda(z), X \rangle.$$

This property gives a connection between π_0 and $\mathcal{O}_\lambda$.
Note that the map $z \to g_z := (\lambda^{-1} iz, 0)$ is a section for the action of G on $\mathbb{C}^n$, that is, we have $g_z \cdot 0 = z$ for each $z \in \mathbb{C}^n$.

3. Heisenberg group: Weyl quantization

Here we apply the general method for constructing Stratonovich-Weyl correspondence [30], [6] and then recover the Bargmann-Fock version of the Weyl calculus, see [2], Example 2.2 and Example 4.2.
We start from the so-called Stratonovich-Weyl quantizer Ω. Here it is generated by the parity operator R_0 of $\mathcal{H}_0$ defined by

$$(R_0 f)(z) = 2^n f(-z).$$

More precisely, we define

$$\Omega_0(z) := \pi_0(g_z) R_0 \pi_0(g_z)^{-1}$$

for each $z \in \mathbb{C}^n$. Then we get immediately

$$(\Omega_0(z) f)(w) = 2^n \exp\left(\tfrac{1}{\lambda}(w\bar{z} - |z|^2)\right) f(2z - w) \tag{1}$$

for each $z, w \in \mathbb{C}^n$. Thus Ω_0 satisfies the covariance property

$$\Omega_0(g \cdot z) = \pi_0(g) \Omega_0(z) \pi_0(g)^{-1} \tag{2}$$

for each $g \in G_0$ and $z \in \mathbb{C}^n$.
We are now in position to recover the complex Weyl calculus. For each trace-class operator A on $\mathcal{H}_0$, let $W_0(A)$ be the function on $\mathbb{C}^n$ defined by

$$W_0(A)(z) := \mathrm{Tr}(A\Omega_0(z))$$

for each $z \in \mathbb{C}^n$.
We can give an integral expression for $W_0(A)$ as follows. For each trace class operator A on $\mathcal{H}_0$, let $k_A(z, w)$ be the kernel of A, that is, for each $f \in \mathcal{H}_0$ and $z \in \mathbb{C}^n$, we have

$$(Af)(z) = \int_{\mathbb{C}^n} k_A(z, w) f(w)\, e^{-|w|^2/2\lambda} d\mu_\lambda(w).$$

Note that, by the reproducing property, we have, for each $f \in \mathcal{H}_0$ and $z \in \mathbb{C}^n$

$$\begin{aligned}(Af)(z) =&\langle Af, e_z\rangle_{\mathcal{H}_0} = \langle f, A^* e_z\rangle_{\mathcal{H}_0}\\ =&\int_{\mathbb{C}^n} f(w)\overline{(A^*e_z)(w)}\,e^{-|w|^2/2\lambda}d\mu_\lambda(w)\end{aligned}$$

so that we get

$$k_A(z,w) = \overline{(A^*e_z)(w)} = \overline{\langle A^*e_z, e_w\rangle_{\mathcal{H}_0}} = \langle Ae_w, e_z\rangle_{\mathcal{H}_0}$$

for each $z,w \in \mathbb{C}^n$. This shows that $k_A(z,w)$ is holomorphic in z and anti-holomorphic in w.

PROPOSITION 2. *For each trace-class operator A on $\mathcal{H}_0$ and each $z \in \mathbb{C}^n$, we have*

$$W_0(A)(z) = 2^n \int_{\mathbb{C}^n} k_A(w, 2z-w)\exp\left(\tfrac{1}{\lambda}\left(-z\bar{z} + z\bar{w} - \tfrac{1}{2}w\bar{w}\right)\right)d\mu_\lambda(w).$$

Proof. By Eq. 1, we have for each $f \in \mathcal{H}_0$ and $z, z' \in \mathbb{C}^n$

$$\begin{aligned}(A\Omega_0(z)f)(z') =& \int_{\mathbb{C}^n} k_A(z',w)(\Omega_0(z)f)(w)e^{-|w|^2/2\lambda}d\mu_\lambda(w)\\ =&2^n\int_{\mathbb{C}^n} k_A(z',w)f(2z-w)\exp\left(\tfrac{1}{\lambda}\left(\bar{z}w - z\bar{z} - \tfrac{1}{2}w\bar{w}\right)\right)d\mu_\lambda(w)\\ =&2^n\int_{\mathbb{C}^n} k_A(z',2z-w)f(w)\exp\left(\tfrac{1}{\lambda}\left(-z\bar{z} + z\bar{w}\right)\right)e^{-|w|^2/2\lambda}d\mu_\lambda(w).\end{aligned}$$

This shows that the kernel of $A\Omega_0(z)$ is

$$k_{A\Omega_0(z)}(z',w) = 2^n k_A(z', 2z-w)\exp\left(\tfrac{1}{\lambda}\left(-z\bar{z} + z\bar{w}\right)\right).$$

Then, by applying Mercer's theorem, we get

$$\begin{aligned}W_0(A)(z) =& \int_{\mathbb{C}^n} k_{A\Omega_0(z)}(w,w)e^{-|w|^2/2\lambda}d\mu_\lambda(w)\\ =& 2^n\int_{\mathbb{C}^n} k_A(w,2z-w)\exp\left(\tfrac{1}{\lambda}\left(-z\bar{z} + z\bar{w}\right)\right)e^{-|w|^2/2\lambda}d\mu_\lambda(w).\end{aligned}$$

□

Let us recall the usual Weyl correspondence and briefly describe its connection with W_0. Let G_0 act on $\mathbb{R}^{2n}$ by

$$(z_0, c_0)\cdot(p,q) := (p + \operatorname{Re} z_0, q + \lambda \operatorname{Im} z_0).$$

This action corresponds to the preceding action of G_0 on $\mathbb{C}^n$ via the identification $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ given by $(p,q) \leftrightarrow z = q - i\lambda p$.

Let R_1 be the operator on $L^2(\mathbb{R}^n)$ defined by

$$(R_1\phi)(x) = 2^n\phi(-x)$$

and let π_1 be the Schrödinger representation of G_0 on $L^2(\mathbb{R}^n)$ is defined by

$$(\pi_1(a+ib,c)\phi)(x) = e^{i\lambda(c-bx+\frac{1}{2}ab)}f(x-a).$$

We consider the Stratonovich-Weyl quantizer Ω_1 on $\mathbb{R}^{2n}$ given by

(3) $$\Omega_0(g\cdot(0,0)) := \pi_1(g)R_1\pi_1(g)^{-1}$$

for each $g \in G$. Then it is well-known that the corresponding Stratonovich-Weyl correspondence W_1 defined on $\mathbb{R}^{2n}$ by

$$W_1(A)(p,q) := \mathrm{Tr}(A\Omega_1(p,q))$$

is exactly the inverse map of the classical Weyl correspondence W^1 which is usually defined by the formula

$$(W^1(F)\phi)(p) = (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{isq}\,F(p+\tfrac{1}{2}s,q)\,\phi(p+s)\,ds\,dq,$$

see, for instance, [1], [30], [24].

The connection between W_0 and W_1 (hence W^1) is given by the Bargmann transform which is the unitary operator $B: L^2(\mathbb{R}^n) \to \mathcal{H}_0$ defined by

$$(B\phi)(z) = (\lambda/\pi)^{n/4}\int_{\mathbb{R}^n} e^{(1/4\lambda)z^2+ixz-(\lambda/2)x^2}\,\phi(x)\,dx.$$

Indeed, it is well-known that B intertwines π_0 and π_1, that is, we have for each $g \in G$, $\pi_1(g) = B^{-1}\pi_0(g)B$ [29], [27]. Moreover, we can verify that $R_1 = B^{-1}R_0B$. Taking Eq. 2 and Eq. 3 into account, this implies that, for each trace-class operator A on $\mathcal{H}_0$ and each $(p,q) \in \mathbb{R}^{2n}$, we have

(4) $$W_0(A)(q-i\lambda p) = W_1(B^{-1}AB)(p,q).$$

PROPOSITION 3. *W_0 is a Stratonovich-Weyl correspondence for $(G_0,\pi_0,\mathbb{C}^n)$.*

Proof. For each trace-class operator A on $\mathcal{H}_0$ and each $z \in \mathbb{C}^n$, we have

$$W_0(A^*)(z) = \mathrm{Tr}(A^*\Omega_0(z)) = \mathrm{Tr}(\Omega_0(z)A)^* = \overline{\mathrm{Tr}(\Omega_0(\xi)A} = \overline{W_0(A)(z)}.$$

Moreover, W_0 is covariant by construction. Finally, by Eq. 4, W_0 is unitary since W_1 is, see [29]. $\square$

In particular, since W_0 is unitary, we have that $W_0^{-1} = W_0^*$. Then we get, for each $F \in L^2(\mathbb{C}^n,\mu_\lambda)$

$$W_0^{-1}(F) = \int_{\mathbb{C}^n} \Omega(z)F(z)\,d\mu_\lambda(z).$$

Such a formula is particularly suitable for extending the Weyl calculus to the symmetric domains, see [2]. However, the generalized Weyl calculus is, in general, no longer unitary [3].

Let us also mention that that the unitary part in the polar decomposition of the Berezin correspondence S_0 introduced in Section 2 is W_0, see [35], Theorem 6 and, for a slightly different proof based on covariance, [20].

On the other hand, it is also known that the classical Weyl correspondence can be extended to differential operators on $\mathbb{R}^n$, see for instance [32], [40]. Here, we can similarly extend W_0 to operators of the form $A=\sum_{p,q} a_{pq} z^p (\frac{\partial}{\partial z})^q$ by using the integral formula for $W_0(A)$ given in Proposition 2.

Here, we use the standard multi-index notation. If $p=(p_1,p_2,\ldots,p_n)\in\mathbb{N}^n$, we set $z^p=z_1^{p_1}z_2^{p_2}\ldots z_n^{p_n}$, $|p|=p_1+p_2+\cdots+p_n$, $p!=p_1!p_2!\ldots p_n!$. Also, we say that $p\leqslant q$ if $p_k\leqslant q_k$ for each $k=1,2,\ldots,n$ and, in this case, we denote $\binom{q}{p}=\frac{q!}{p!(q-p)!}$.

PROPOSITION 4. *For each $p,q\in\mathbb{N}^n$, let $A_{pq}:=z^p(\frac{\partial}{\partial z})^q$. Then we have*

$$W_0(A_{pq})(z)=(2\lambda)^{-|q|}\sum_{k\leqslant p,q}(-\lambda)^{|k|}\frac{p!q!}{k!(p-k)!(q-k)!}z^{p-k}\bar{z}^{q-k}.$$

Proof. Let $p,q\in\mathbb{N}^n$. By differentiating, for each $f\in\mathcal{H}_0$, the reproducing property

$$f(z)=\int_{\mathbb{C}^n} f(w)e^{z\bar{w}/2\lambda}e^{-|w|^2/2\lambda}d\mu_\lambda(w)$$

with respect to z, we get

$$z^p\left(\frac{\partial}{\partial z}\right)^q f=(2\lambda)^{-|q|}z^p\int_{\mathbb{C}^n} f(w)\bar{w}^q e^{z\bar{w}/2\lambda}e^{-|w|^2/2\lambda}d\mu_\lambda(w).$$

Hence A_{pq} has kernel

$$k_{A_{pq}}(z,w)=(2\lambda)^{-|q|}z^p\bar{w}^q e^{z\bar{w}/2\lambda}.$$

Consequently, by using the integral formula of Proposition 2, we have that

$$\begin{aligned}W_0(A_{pq})(z)&=2^n(2\lambda)^{-|q|}\int_{\mathbb{C}^n} w^p(2\bar{z}-\bar{w})^q e^{-|w-z|^2/\lambda}d\mu_\lambda(w)\\&=2^n(2\lambda)^{-|q|}\int_{\mathbb{C}^n}(w+z)^p(\bar{z}-\bar{w})^q e^{-|w|^2/\lambda}d\mu_\lambda(w).\end{aligned}$$

Now, by the binome formula, we can write

$$\begin{aligned}(w+z)^p&=\sum_{k\leqslant p}\binom{p}{k}z^{p-k}w^k;\\(\bar{z}-\bar{w})^q&=\sum_{l\leqslant q}\binom{q}{l}(-\bar{w})^l\bar{z}^{q-l}\end{aligned}$$

and we can remark that for each $k,l \in \mathbb{N}^n$, we have

$$\int_{\mathbb{C}^n} w^k \bar{w}^l e^{-|w|^2/\lambda} d\mu_\lambda(w) = 2^{-n}\lambda^{|k|}k!\delta_{kl}.$$

By replacing in the previous expression of $W_0(A_{pq})(z)$, we obtain the desired result. □

By computing in particular $W_0(d\pi_0(X))$ for $X \in \mathfrak{g}_0$, we can verify that

$$W_0(d\pi_0(X))(z) = i\langle \Phi_\lambda(z), X\rangle.$$

for each $X \in \mathfrak{g}_0$ and $z \in \mathbb{C}^n$. This implies that W_0 is also an adapted Weyl correspondence in the sense of [14].

4. Orbits and representations of Heisenberg motion groups

We fix a closed subgroup K of $U(n)$. Then K acts on G_0 by $k \cdot (z,c) = (kz,c)$ and we can form the semidirect product $G := G_0 \rtimes K$ which is called a Heisenberg motion group. The elements of G can be written as (z,c,k) where $z \in \mathbb{C}^n$, $c \in \mathbb{R}$ and $k \in K$. The multiplication of G is then given by

$$(z,c,k)\cdot(z',c',k') = (z+kz', c+c'+\tfrac{1}{2}\omega(z,kz'), kk').$$

We denote by $\mathfrak{k}$ and $\mathfrak{g}$ the Lie algebras of K and G. The Lie brackets of $\mathfrak{g}$ are given by

$$[(v,c,A),(v',c',A')] = (Av' - A'v, \omega(v,v'), [A,A']).$$

We compute the adjoint and coadjoint actions of G. Consider $g = (z_0,c_0,k_0) \in G$ and $X = (v,c,A) \in \mathfrak{g}$. We can verify that

$$\begin{aligned}\mathrm{Ad}(g)X &= \frac{d}{dt}(g\exp(tX)g^{-1})|_{t=0}\\ &= \big(k_0v - (\mathrm{Ad}(k_0)A)z_0, c+\omega(z_0,k_0v) - \tfrac{1}{2}\omega(z_0,(\mathrm{Ad}(k_0)A)z_0), \mathrm{Ad}(k_0)A\big).\end{aligned}$$

Let us denote by $\xi = (u,d,\phi)_*$, where $u \in \mathbb{C}^n$, $d \in \mathbb{R}$ and $\phi \in \mathfrak{k}^*$, the element of $\mathfrak{g}^*$ defined by

$$\langle \xi, (v,c,A)\rangle = \omega(u,v) + dc + \langle \phi, A\rangle.$$

Also, for $u,v \in \mathbb{C}^n$, we denote by $v \times u$ the element of $\mathfrak{k}^*$ defined by $\langle v\times u, A\rangle := \omega(u,Av)$ for $A \in \mathfrak{k}$. Then, from the formula for the adjoint action, we deduce that for each $\xi = (u,d,\phi)_* \in \mathfrak{g}^*$ and $g = (z_0,c_0,k_0) \in G$ we have

$$\mathrm{Ad}^*(g)\xi = \big(k_0u - dz_0, d, \mathrm{Ad}^*(k_0)\phi + z_0 \times (k_0u - \tfrac{1}{2}dz_0)\big)_*.$$

From this, we deduce that if a coadjoint orbit of G contains a point $(u,d,\phi)_*$ with $d \neq 0$ then it also contains a point of the form $(0,d,\phi_0)_*$. Such an orbit is called *generic*.

Now we recall the definition of the *generic* unitary representations of G which are associated with the generic coadjoint orbits of G [22]. Note that these representations can be obtained as holomorphically induced representations by the general method of [36], since G is a quasi-Hermitian Lie group [21].

Let ρ be a unitary irreducible representation of K on a (finite-dimensional) Hilbert space V and let $\lambda > 0$. Let $\mathcal{H}$ the Hilbert space of all holomorphic functions $f : \mathbb{C}^n \to V$ such that

$$\|f\|^2_{\mathcal{H}} := \int_{\mathbb{C}^n} \|f(z)\|^2_V \, e^{-|z|^2/2\lambda} \, d\mu_\lambda(z) < +\infty.$$

Then we can consider the representation π of G on $\mathcal{H}$ defined by

$$(\pi(g)f)(z) = \exp\left(i\lambda c_0 + \tfrac{1}{2} i\bar{z}_0 z - \tfrac{\lambda}{4}|z_0|^2\right) \rho(k) \, f(k^{-1}(z + i\lambda z_0))$$

for each $g = (z_0, c_0, k) \in G$ and $z \in \mathbb{C}^n$.

Clearly, we have that $\mathcal{H} = \mathcal{H}_0 \otimes V$. For $f_0 \in \mathcal{H}_0$ and $v \in V$, we denote by $f_0 \otimes v$ the function $z \to f_0(z)v$. Moreover, if A_0 is an operator on $\mathcal{H}_0$ and A_1 is an operator on V then we denote by $A_0 \otimes A_1$ the operator on $\mathcal{H}$ defined by $(A_0 \otimes A_1)(f_0 \otimes v) = A_0 f_0 \otimes A_1 v$.

Also, we denote by τ the left-regular representation of K on $\mathcal{F}_0$, that is, one has $(\tau(k)f_0)(z) = f_0(k^{-1}z)$. Then we have

$$\pi(z_0, c_0, k) = \pi_0(z_0, c_0)\tau(k) \otimes \rho(k) \tag{5}$$

for each $z_0 \in \mathbb{C}^n$, $c_0 \in \mathbb{R}$ and $k \in K$. This is precisely Formula (3.18) in [9].

We can easily compute the differential of π:

PROPOSITION 5. *Let $X = (v, c, A) \in \mathfrak{g}$. Then, for each $f \in \mathcal{H}$ and each $z \in \mathbb{C}^n$, we have*

$$(d\pi(X)f)(z) = d\rho(A)f(z) + i(\lambda c + \tfrac{1}{2}\bar{v}z)f(z) + df_z(-Az + i\lambda v)$$

or, equivalently,

$$d\pi(X) = (d\pi_0(v, c) + d\tau(A)) \otimes \mathrm{id}_V + \mathrm{id}_{\mathcal{H}_0} \otimes d\rho(A).$$

In the rest of the paper, we fix ϕ_0 such that ρ is associated with the orbit $o(\phi_0)$ for the coadjoint action of K as in [21], [41]. Then the orbit $O(\xi_0)$ of $\xi_0 = (0, \lambda, \phi_0)_* \in \mathfrak{g}^*$ for the coadjoint action of G is associated with π, see [22].

5. Berezin quantization for π

In this section, we show that the Berezin calculus on $\mathcal{H}$ naturally provides a diffeomorphism $\Psi : \mathbb{C}^n \times o(\phi_0) \to O(\xi_0)$. We begin with the Berezin calculus on $o(\phi_0)$.

The Berezin calculus on $o(\phi_0)$ associates with each operator A_1 on V a complex-valued function $s_1(A_1)$ on the orbit $o(\phi_0)$ which is called the symbol of the operator A_1 (see [11]). We denote by $Sy(o(\phi_0))$ the space of all such symbols.

The following proposition summarizes some well-known properties of the Berezin calculus, see for instance [5], [25], [15] and [41].

PROPOSITION 6. *1. The map $A_1 \to s_1(A_1)$ is injective.*

2. *We have $s_1(\mathrm{id}_V) = 1$.*
3. *For each operator A_1 on V, we have $s_1(A_1^*) = \overline{s_1(A_1)}$.*
4. *For each operator A_1 on V, $k \in K$ and $\phi \in o(\phi_0)$, we have*

$$s_1(A_1)(\mathrm{Ad}^*(k)\phi) = s_1(\rho(k)^{-1}A_1\rho(k))(\phi).$$

The Berezin calculus S for π can be defined as follows, [22], [23]. For each operator A_0 on $\mathcal{H}_0$ and each operator A_1 on V, we set

$$S(A_0 \otimes A_1) := S_0(A_0) \otimes s_1(A_1)$$

and then we extend S by linearity to operators on $\mathcal{H}$.
Consider the action of G on $\mathbb{C}^n \times o(\varphi_0)$ defined by

$$g \cdot (z, \phi) := (kz - i\lambda z_0, \mathrm{Ad}^*(k)\phi)$$

for $g = (z_0, c_0, k) \in G$, $z \in \mathbb{C}^n$ and $\phi \in o(\phi_0)$. Then we can show that S is G-covariant with respect to π [22]. Moreover, we have the following result, see [21], [22].

PROPOSITION 7. *1. For each $X = (v, c, A) \in \mathfrak{g}$, $z \in \mathbb{C}^n$ and $\phi \in o(\phi_0)$, we have*

$$S(d\pi(X))(z, \phi) = i\lambda c + \frac{i}{2}(\bar{v}z + v\bar{z}) - \frac{1}{2\lambda}\bar{z}(Az) + s(d\rho(A))(\phi).$$

2. *For each $X \in \mathfrak{g}$, $z \in \mathbb{C}^n$ and $\phi \in o(\phi_0)$, we have*

$$S(d\pi(X))(z, \phi) = i\langle \Psi(z, \phi), X \rangle$$

where the map $\Psi : \mathbb{C}^n \times o(\phi_0) \to \mathfrak{g}^*$ is defined by

$$\Psi(z, \phi) = \left(-iz, \lambda, \phi - \tfrac{1}{2\lambda} z \times z\right)_* .$$

3. *Ψ is a diffeomorphism from $\mathbb{C}^n \times o(\phi_0)$ onto $O(\xi_0)$.*
4. *We have, for each $g \in G$, $z \in \mathbb{C}^n$ and $\phi \in o(\phi_0)$,*

$$\Psi(g \cdot (z, \phi)) = \mathrm{Ad}^*(g)\Psi(z, \phi).$$

6. Stratonovich-Weyl correspondence for π

Here we construct a Stratonovich-Weyl correspondence for π by combining the Weyl calculus on $\mathbb{C}^n$ and a Stratonovich-Weyl correspondence for ρ.

We fix an invariant measure ν on $o(\phi_0)$ and consider $Sy(o(\phi_0))$ as a (finite dimensional) subspace of $L^2(o(\phi_0), \nu)$. On the other hand, we can equip $\mathrm{End}(V)$ with the Hilbert-Schmid norm. Then we can consider the unitary part w_1 in the polar decomposition of $s_1 : \mathrm{End}(V) \to Sy(o(\phi_0))$. We immediately see that w_1 inherits some properties from s_1 and that w_1 is a Stratonovich-Weyl correspondence for $(K, \rho, o(\phi_0))$ [28], [18]. Moreover, for each $\phi \in o(\phi_0)$, there exists a unique $\omega_1(\phi) \in \mathrm{End}(V)$ such that

$$w_1(A_1)(\phi) = \mathrm{Tr}(A_1\omega_1(\phi))$$

for each $A_1 \in \mathrm{End}(V)$.

Recall that such a map $\phi \to \omega_1(\phi)$ is called a Stratonovich-Weyl quantizer and that the properties of w_1 are reflected by similar properties of ω_1, see for instance [30]. In particular, the covariance property of w_1 is equivalent to the fact that for each $k \in K$ and $\phi \in o(\phi_0)$, we have

$$\omega_1(\mathrm{Ad}^*(k)\phi) = \rho(k)\omega_1(\phi)\rho(k)^{-1}.$$

In the rest of this paper, we fix a section (defined on a dense open subset of $o(\phi_0)$) $\phi \to k_\phi$ for the action of K on $o(\phi_0)$, see [19]. Then we have

$$\omega_1(\phi) = \rho(k_\phi)\omega_1(\phi_0)\rho(k_\phi)^{-1}.$$

By analogy with the Berezin calculus S, we define for each (suitable) operator A_0 on $\mathcal{H}_0$ and each operator A_1 on V, the function $W(A_0 \otimes A_1)$ on $\mathbb{C}^n \times o(\phi_0)$ by

$$W(A_0 \otimes A_1)(z, \phi) := W(A_0)(z)w_1(A_1)(\phi).$$

By Section 3, this definition works in particular when A_0 is trace-class (or more generally Hilbert-Schmidt) and also when A_0 is a differential operator with polynomial coefficients. Of course, we can extend W to finite sums of operators of the form $A_0 \otimes A_1$. In order to prove that W is a Stratonovich-Weyl correspondence, we need the following lemma.

LEMMA 1. *For each trace-class operator A_0 on $\mathcal{H}_0$, $k \in K$ and $z \in \mathbb{C}^n$, we have*

$$W_0(\tau(k)^{-1}A_0\tau(k))(z) = W(A_0)(kz).$$

Proof. For each trace-class operator A_0 on $\mathcal{H}_0$ and each $k \in K$, we can verify that the kernel of $\tau(k)^{-1}A_0\tau(k)$ is $(z, w) \to k_A(kz, kw)$. The result then follows from Proposition 2. □

Now we have the following result.

PROPOSITION 8. *W is a Stratonovich-Weyl correspondence for $(G, \pi, \mathbb{C}^n \times o(\phi_0))$.*

Proof. From the properties of W_0 and w_1, we see immediately that W is unitary and satisfies the property that $W(A^*) = \overline{W(A)}$ for A operator on $\mathcal{H}$. It remains to prove that W is covariant with respect to π. We have just to consider the case where $A = A_0 \otimes A_1$ with A_0 operator on $\mathcal{H}$ and A_1 operator on V. Let $g = (z_0, c_0, k) \in G$. By using Eq. 5, we see that

$$W(\pi(g)^{-1}A\pi(g)) = W_0(\tau(k)^{-1}\pi_0(z_0, c_0)^{-1}A_0\pi_0(z_0, c_0)\tau(k)) \otimes w_1(\rho(k)^{-1}A_1\rho(k)).$$

Then, by using Lemma 1 and the covariance of W_0 and w_1, we have, for each $(z,\phi)\in\mathbb{C}^n\times o(\phi_0)$,

$$\begin{aligned}W(\pi(g)^{-1}A\pi(g))(z,\phi) &= W_0(\pi_0(z_0,c_0)^{-1}A_0\pi_0(z_0,c_0))(kz)w_1(A_1)(\mathrm{Ad}^*(k)\phi)\\ &= W_0(A_0)((z_0,c_0)\cdot kz)w_1(A_1)(\mathrm{Ad}^*(k)\phi) = W(A)(g\cdot(z,\phi)).\end{aligned}$$

This ends the proof. □

Now, we aim to identify the Stratonovich-Weyl quantizer associated with W. We need the section for the action of G on $\mathbb{C}^n\times o(\phi_0)$ defined by

$$(z,\phi)\to g_{(z,\phi)} := (\lambda^{-1}iz,0,k_\phi)\in G.$$

Recall that $\phi\to k_\phi$ is a section for the action of K on $o(\phi_0)$ and that $z\to g_z=(\lambda^{-1}iz,0)\in G_0$ is a section for the action of G_0 on $\mathbb{C}^n$.

Recall also that R_0 denotes the parity operator on $\mathcal{H}_0$, that is, $(R_0f_0)(z)=2^nf_0(-z)$. Then we introduce the operator R on $\mathcal{H}$ defined by $R:=R_0\otimes\omega_1(\phi_0)$ and we define the Stratonovich-Weyl quantizer Ω by

$$\Omega(g\cdot(0,\phi_0)) := \pi(g)R\pi(g)^{-1}.$$

This definition makes sense since R commutes with $\pi(g)$ for each g in the stabilizer $G_{(0,\phi_0)}$ of $(0,\phi_0)$ in G, that is, for each g of the form $(0,c_0,k)$ with $c_0\in\mathbb{R}$ and k in the stabilizer of ϕ_0 in K. Note that $G_{(0,\phi_0)}$ is also the stabilizer of ξ_0 for the coadjoint action of G.

In particular, we have, for each $(z,\phi)\in\mathbb{C}^n\times o(\phi_0)$,

$$\Omega(z,\phi)=\pi(g_{(z,\phi)})R\pi(g_{(z,\phi)})^{-1}.$$

PROPOSITION 9. *1. For each $(z,\phi)\in\mathbb{C}^n\times o(\phi_0)$, we have*

$$\Omega(z,\phi)=\Omega_0(z)\otimes\omega_1(\phi);$$

2. Ω is the quantizer associated with W, that is, for each trace-class operator A on $\mathcal{H}$ and each $(z,\phi)\in\mathbb{C}^n\times o(\phi_0)$, we have

$$W(A)(z,\phi)=\mathrm{Tr}(A\Omega(z,\phi)).$$

Proof. (1) First, note that for each $k\in K$, we have

$$\tau(k)^{-1}\Omega_0(0)\tau(k)=\Omega_0(0).$$

Now, let $(z,\phi)\in\mathbb{C}^n\times o(\phi_0)$. We have

$$\begin{aligned}\Omega(z,\phi) &= \pi(g_{(z,\phi)})R\pi(g_{(z,\phi)})^{-1}\\ &= (\pi_0(g_z)\tau(k_\phi)\otimes\rho(k_\phi))(\Omega_0(0)\otimes\omega_1(\phi_0))(\tau(k_\phi)^{-1}\pi_0(g_z)^{-1}\otimes\rho(k_\phi)^{-1})\\ &= (\pi_0(g_z)\tau(k_\phi)\Omega_0(0)\tau(k_\phi)^{-1}\pi_0(g_z)^{-1})\otimes(\rho(k_\phi)\omega_1(\phi_0)\rho(k_\phi)^{-1})\\ &= (\pi_0(g_z)\Omega_0(0)\pi_0(g_z)^{-1})\otimes\omega_1(\phi)\\ &= \Omega_0(z)\otimes\omega_1(\phi).\end{aligned}$$

(2) We have just to consider the case where $A = A_0 \otimes A_1$ where A_0 is a trace-class operator on $\mathcal{H}_0$. Then we have

$$\begin{aligned} W(A)(z,\phi) &= \mathrm{Tr}(A_0 \otimes A_1)\,\mathrm{Tr}(\Omega_0(z) \otimes \omega_1(\phi)) \\ &= \mathrm{Tr}(A_0\Omega_0(z) \otimes A_1\omega_1(\phi)) \\ &= \mathrm{Tr}(A_0\Omega_0(z))\,\mathrm{Tr}(A_1\omega_1(\phi)) \\ &= W_0(A_0)(z)w_1(A_1)(\phi) \\ &= W(A)(z,\phi). \end{aligned}$$

This ends the proof. □

Note that Proposition 9 also gives a construction of the Stratonovich-Weyl correspondence for π in the spirit of the general method of [30] and [6]. We can also give an integral expression of $W(A)$ and then extend Proposition 2. Note that each Hilbert-Schmidt operator A on $\mathcal{H}$ has a kernel $K_A : (z,w) \to K_A(z,w) \in \mathrm{End}(V)$ so that, for each $f \in \mathcal{H}$ and each $z \in \mathbb{C}^n$, we have

$$(Af)(z) = \int_{\mathbb{C}^n} K_A(z,w) f(w)\, e^{-|w|^2/2\lambda} d\mu_\lambda(w).$$

PROPOSITION 10. *For each trace-class operator A on $\mathcal{H}$ and each $(z,\phi) \in \mathbb{C}^n \times o(\phi_0)$, we have*

$$W(A)(z,\phi) = 2^n \int_{\mathbb{C}^n} w_1(K_A(w, 2z-w))(\phi) \exp\left(\tfrac{1}{\lambda}\left(-z\bar{z} + z\bar{w} - \tfrac{1}{2}w\bar{w}\right)\right) d\mu_\lambda(w).$$

Proof. As usual, it is sufficient to consider the case where $A = A_0 \otimes A_1$ where A_0 is a trace-class operator on $\mathcal{H}_0$ and A_1 an operator on V. Clearly, we have $K_A(z,w) = k_{A_0}(z,w)A_1$ and then for each $(z,\phi) \in \mathbb{C}^n \times o(\phi_0)$ we have

$$W(A)(z,\phi) = W_0(A_0)(z)w_1(A_1)(\phi).$$

Hence the result follows from Proposition 2. □

Note that, since W_0 is the unitary part in the polar decomposition of $S_0 : \mathcal{L}_2(\mathcal{H}_0) \to L^2(\mathbb{C}^n, \mu_\lambda)$, see Section 3, we have that W is the unitary part in the polar decomposition of $S : \mathcal{L}_2(\mathcal{H}) \to L^2(\mathbb{C}^n, \mu_\lambda) \otimes V$. It is also clear that since the Bargmann transform connects W_0 to W_1 (see Section 3), it also connects W to the Stratonovich-Weyl correspondence constructed in [22].

We can immediately deduce a Stratonovich-Weyl correspondence for $(G, \pi, \mathcal{O}(\xi_0))$ from W. Indeed, let $\mathcal{W}$ be the map defined by $\mathcal{W}(A) = W(A) \circ \Psi^{-1}$ for each trace-class operator A on $\mathcal{H}$. Then, as an immediate consequence of Proposition 8, we have the following result.

PROPOSITION 11. *$\mathcal{W}$ is a Stratonovich-Weyl correspondence for $(G, \pi, \mathcal{O}(\xi_0))$.*

We finish by giving an explicit expression for $W(d\pi(X), X \in \mathfrak{g}$ in the spirit of Proposition 7.

PROPOSITION 12. *For each* $X = (v,c,A) \in \mathfrak{g}$ *and each* $(z,\phi) \in \mathbb{C}^n \times o(\phi_0)$, *we have*

$$W(d\pi(X))(z,\phi) = i\lambda c + \frac{i}{2}(\bar{v}z + v\bar{z}) - \frac{1}{2\lambda}\bar{z}(Az) + \frac{1}{2\lambda}\operatorname{Tr}(A) + w_1(d\rho(A))(\phi).$$

Proof. Let $X = (v,c,A) \in \mathfrak{g}$. Recall that (Proposition 5)

$$d\pi(X) = (d\pi_0(v,c) + d\tau(A)) \otimes \mathrm{id}_V + \mathrm{id}_{\mathcal{H}_0} \otimes d\rho(A).$$

This implies that

$$W(d\pi(X))(z,\phi) = W_0(d\pi_0(v,c))(z) + W_0(d\tau(A))(z) + w_1(d\rho(A))(\phi). \tag{6}$$

But by Proposition 4 we have

$$W_0(d\pi_0(v,c))(z) = i\lambda c + \frac{i}{2}(\bar{v}z + v\bar{z}).$$

Moreover, writing $A = (a_{ij})$, we get

$$(d\tau(A)f_0)(z) = -df_0(z)(Az) = -\sum_{ij} a_{ij} z_j \frac{\partial f_0}{\partial z_i}$$

for each $f_0 \in \mathcal{H}$. Then, applying Proposition 4 again we get

$$\begin{aligned} W_0(d\tau(A))(z) &= -\frac{1}{2\lambda}\sum_{ij} a_{ij}\bar{z}_i z_j + \sum_i a_{ii} \\ &= -\frac{1}{2\lambda}\bar{z}(Az) + \frac{1}{2\lambda}\operatorname{Tr}(A). \end{aligned}$$

Hence, by replacing in Eq. 6, we obtain the desired result. □

Note that if $K \subset SU(n)$, one has $\operatorname{Tr}(A) = 0$ for each $A \in \mathfrak{k}$ and the preceding formula for $W_0(d\pi(X))$ is very close to the formula for $S(d\pi(X))$, see Proposition 7.

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Benjamin CAHEN,
Université de Lorraine, Site de Metz, UFR-MIM, Département de mathématiques,
Bâtiment A, 3 rue Augustin Fresnel, BP 45112, 57073 METZ Cedex 03, FRANCE
e-mail: `benjamin.cahen@univ-lorraine.fr`

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