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QUADRATIC EQUATIONS AND EXTENSIONS OF BOOLEAN RINGS

Abstract. We completely characterize those rings whose elements satisfy (one of) the equations $x^2 = x$ or $x^2 = 2 + x$. In particular, we enlarge some recent results due to Koşan-Ying-Zhou published in (Canad. Math. Bull., 2016).

1. Introduction and Fundamentals

Throughout the current paper, let all rings R be associative with the identity element 1 which differs from the zero element 0. Our terminology and notation follow those from [6]. Standardly, $J(R)$ denotes the Jacobson radical of R and $Id(R)$ denotes the set all idempotents in R.

A brief history of the investigated theme on certain quadratic equations in arbitrary rings is as follows: We recall that a ring R is said to be *Boolean* if each its element is a solution of the equation $x^2 = x$. A question which rather naturally arises is what we can say for a ring whose elements are solutions of (one of) the equations $x^2 = x$ or $x^2 = -x$. This was settled independently in [1] and [3] by showing that such a ring is isomorphic to either a Boolean ring B, to $\mathbb{Z}_3$ or to $B \times \mathbb{Z}_3$. It is elementarily seen that $x^2 = x$ or $x^2 = -x$ imply $x^3 = x$. In this way, rings satisfying the equation $x^3 = x$ were examined in [8] and later on in [4] by proving that they are commutative and isomorphic to subdirect products of copies of the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$, meanwhile their elements are sums of two (commuting) idempotents. In this connection, in [2] it was established that $x^3 = x$ is tantamount to $x^2 = xv$, where $v = x^2 + x - 1$ is an involution, i.e., $v^2 = 1$.

The goal of this work is to study a class of rings whose elements satisfy the combination of the equations $x^2 = x$ and $x^2 = 2 + x$. This is a quite logical extension of the classical Boolean rings observing elementarily that the elements of the ring $\mathbb{Z}_4$ are solutions of these equations. All rings' characterizations will be given up to an isomorphism. Besides, we shall demonstrate easier proofs of some well-known results in this subject.

2. Main Result and Questions

We first begin with a refinement of a few results from [4], [8] and [5] by providing a more transparent proof. Specifically, the following holds:

PROPOSITION 1. *For a ring R the next four items are equivalent:*

(i) All elements of R satisfy the equation $x^3 = x$. (cf. [4], [8])

(ii) R is a subdirect product of copies of the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$. (cf. [4], [8])

(iii) R is commutative and all elements of R are sums of two (commuting) idempotents. (cf. [4], [8])

(iv) R is commutative and all elements of R are differences of two (commuting) idempotents. (cf. [5])

Proof. (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii). Since $2^3 = 2$, we have that $6 = 0$ and so that standard Chinese Remainder Theorem allows us to get the decomposition $R \cong R_1 \times R_2$, where R_1 is a ring of characteristic 2 and R_2 is a ring of characteristic 3. We claim that R_1 is Boolean. In fact, the equation $x^3 = x$ continue to hold in R_1 for any its element, so that $(x+1)^3 = x+1$. Consequently, $3x^2 + 3x = 0$ ensuring that $x^2 = x$, as needed. That is why, as it is well-known, R_1 is a subdirect product of copies of the field $\mathbb{Z}_2$. As for the ring R_2, the equation $x^3 = x$ again holds. So, for any its element x, one observes that $-x = (-x-x^2)+x^2$, where both $-x-x^2$ and x^2 are idempotents. But then, the element $-(x-2)$ is also a sum of two idempotents, say e_1 and e_2, respectively. Finally, one sees that $x = (1-e_1)+(1-e_2)$, as pursued. Moreover, R_2 is principally known to be commutative by the famous Jacobson theorem and also regular, thus it is a subdirect product of fields in which $x^3 = x$ is fulfilled as well. Thus $x^3 = x$ reduces to $x^2 = 1$, i.e., to $(x-1)(x+1) = 0$ whence to either $x = -1$ or $x = 1$. And since $3 = 0$, a routine check shows that these fields are just isomorphic to $\mathbb{Z}_3$, as needed. Concerning R, it follows immediately that it is commutative and that each its element is a sum of two idempotents, because these two properties are valid simultaneously in both R_1 and R_2.

(ii) $\Rightarrow$ (i). Since both $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are even perfect fields, that is, $\mathbb{Z}_2^2 = \mathbb{Z}_2$ and $\mathbb{Z}_3^3 = \mathbb{Z}_3$, issue (i) is now obvious.

(iii) $\Longleftrightarrow$ (iv). If an arbitrary $r \in R$ is presentable as $r \in Id(R)+Id(R)$, then again $r+1 \in Id(R)+Id(R)$ and hence $r \in Id(R)-(1-Id(R))$, as required. Conversely, if an arbitrary element $r \in R$ is of the form $r \in Id(R)-Id(R)$, then again $r-1 \in Id(R)-Id(R)$ and so $r \in Id(R)+(1-Id(R))$, as expected.

(iv) $\Rightarrow$ (i). For any $r \in R$ writing $r = e-f$ for some $e, f \in Id(R)$, it is plainly verified that $r^3 = (e-f)^3 = e^3 - 3e^2f + 3ef^2 - f^3 = e-f = r$, as asserted. □

REMARK 1. According to the current proof, and especially to the obvious equivalence of points (iii) and (iv), Proposition 2.1 from [5] is superfluous. Moreover, it is worth to noticing that a variation of Proposition 1 in a slight more weaker form is also given in [7, Exercise 12.0 (3)].

The next technicality is useful.

LEMMA 1. *If $R = R_1 \times R_2$ is a ring whose elements are (either) the sum or the difference of two commuting idempotents, then R_1 or R_2 is a ring whose elements are sums of two commuting idempotents.*

Proof. As observed above in the previous proposition, each element is a sum of two commuting idempotents precisely when each element is a difference of two commuting idempotents. With this at hand, if neither R_1 nor R_2 are such rings, then there is r_1 in

R_1 that is not a sum of two idempotents and there is r_2 in R_2 that is not a difference of two idempotents. We, furthermore, check that the element $r = (r_1, r_2) \in R$ need not be the sum nor the difference of two commuting idempotents, a contradiction, and we are set. □

The following statement is pivotal refining somewhat [5, Theorem 4.3].

PROPOSITION 2. *Let R be a ring. Then the next three items are equivalent:*

(a) For all $x \in R$: $x^2 = x$ *or* $x^2 = 2 + x$ *with* $4 = 0$.

(b) Every element of R is (either) idempotent or minus idempotent or the sum of two commuting idempotents and $4 = 0$.

(c) $R/J(R)$ *is a Boolean ring and either* $J(R) = \{0\}$ *or* $J(R) = \{0, 2\}$.

Proof. (a) ⇒ (b). Let $y \in R$ be an element which is not an idempotent, i.e., $y^2 \neq y$. Hence $y^2 = 2 + y$. Consider the element $y + 1$. If $(y+1)^2 = y + 1$, then $y^2 = -y$ and thus $y = -y^2$ is minus the idempotent y^2. But if $(y+1)^2 = 2 + (y+1)$, we then have that $y^2 + y - 2 = 0$. However, combining this with $y^2 - y - 2 = 0$, one infers that $2y = 0$. Multiplying $y^2 - y - 2 = 0$ by y, we get that $y^3 = y^2$ whence $y^4 = y^2$ showing that y^2 is an idempotent. We finally write $y = 2 - y^2 = 1 + (1 - y^2)$, that is the sum of two idempotents, as required.

It is worthwhile to notice that we have not used here the limitation $4 = 0$.

Also, if x is not of the form $\pm e$ with e an idempotent, then $x = 1 + (x - 1)$ has the required form as the sum of two idempotents, because $(x-1)^2 = x - 1$ whenever $x^2 + x - 2 = 0$ provided $4 = 0$.

(b) ⇒ (c). Since each element is obviously the sum or the difference of two idempotents, then we may employ [5] to get the claim.

(c) ⇒ (a). Clearly, $2 \in J(R)$ and $2^2 = 4 \in J(R)$. Hence $4 = 0$ or $4 = 2$, i.e., $4 = 0$ or $2 = 0$ holds. Moreover, for an arbitrary $r \in R$, we have $r^2 - r \in J(R)$ and so $r^2 = r$ or $r^2 - r = 2$, that is, $r^2 = 2 + r$, as stated.

We shall also add here a direct proof of the implication (b) ⇒ (a) which raises some new approaches other than the given ones so far: We claim that R is commutative. Indeed, $x^2 = \pm x$ yields $x^3 = x$ whence $(x-1)^3 = x - 1$ or $(x-1)^3 = 7(x-1) + 6$, respectively, while $x = e + f$ for some commuting $e, f \in Id(R)$ implies $x - 1 = e - (1 - f)$ and thus $(x-1)^3 = x - 1$. A variation of the paramount Jacobson Theorem for Commutativity now insures that R has to be commutative, as claimed. We now assert that $J(R) = \{0, 2\}$ and, in particular, that for any $h \in Id(R)$: $2h = 0$ or $2h = 2$. To that aim, given an arbitrary $z \in J(R)$ we have $z^2 = z$, $z^2 = -z$ or $z^2 = e + f$ for e, f as above. In the first two cases, $z(1 \pm z) = 0$ assures $z = 0$ because $1 \pm z$ is always invertible. If now $z = e + f$, then $z - 1 = e - (1 - f)$ whence it is easily checked that $(z-1)^3 = z - 1$. But $z - 1$ is always a unit, so that $(z-1)^2 = 1$ forcing that $z^2 = 2z$ and that $2z^2 = 0$. On the other hand, $z^2 = (e+f)^2 = z + 2ef$ and thus $2z^2 = 2z = z^2 = 0$. Consequently, $z = 2ef = e + f$. Multiplying by $1 - f$ on the right, we deduce $e(1 - f) = 0$, i.e., $e = ef$. Similarly, multiplying by $1 - e$ again on the right, we derive $f(1 - e) = 0$, i.e., $f = fe$.

Finally, $e = f$ and $z = 2e$. If $2e \neq 0$ and $2e \neq 2$, whence $2(1-e) \neq 0$, we see that $J(eR) \neq 0$ and $J((1-e)R) \neq 0$. Therefore, consulting with Proposition 1, we conclude that eR and $(1-e)R$ are rings whose elements are not the sum of two idempotents. But then $R \cong eR \times (1-e)R$ does not have elements of the given kind, thus contradicting Lemma 1.

Hereafter, if $2 = 0$, we are finished. So, assume that $2 \neq 0$. Suppose also that $y \in R$ with $y^2 \neq y$, whence $y^2 = -y$ or $y = e + f$. Considering the element $y + 2$, we have three cases.

Case 1: $(y+2)^2 = y+2$. Thus $y^2 = -3y-2$ amounting to $y^2 = y+2$, so we are set.

Case 2: $(y+2)^2 = -(y+2) = -y-2$. Thus $y^2 = -y+2$. If we combine this with $y^2 = -y$, we get $2 = 0$ which is contrary to our assumption. We, therefore, combine it with $y = e+f$. The last by squaring leads to $y^2 = y+2ef$. Since $ef \in Id(R)$, by what we have just shown above, either $2ef = 0$ or $2ef = 2$. In the first situation, we get the contradictive $y^2 = y$, so that what remains is $y^2 = y+2$, as promised. Notice that the desired equation comes just from the equality $y = e+f$.

Case 3: $y+2$ is a sum of two commuting idempotents. As in Case 2, one has that either $(y+2)^2 = y+2$ or that $(y+2)^2 = (y+2)+2 = y$. Hence $y^2 = y+2$, as needed, or $y^2 = y$ which is impossible. □

Combining the preceding proposition together with the corresponding results from [5], one obtains the surprising fact:

REMARK 2. For a ring R in which $4 = 0$ (or even for which $2 \in J(R)$), every element is a sum or a difference of two commuting idempotents exactly when every element is (either) idempotent or minus idempotent or the sum of two commuting idempotents. As already indicated above, this somewhat refines Theorem 4.3 in [5].

An other direct verification of the implication (b) ⇒ (a) could also be as follows: Namely, let us first show that in this ring R we have that $2e \in \{0,2\}$ for any idempotent $e \in R$. To that goal, let us look at the element $x = 1+2e$. Since $4 = 0$, we have that $x^2 = 1$. So, if $x = \pm f$, for an idempotent f, we get that $x = \pm 1$ and thus $2e \in \{0,2\}$, indeed. However, if $x = f+g$, for two commuting idempotents f and g, we obtain after squaring that $1 = f+g+2fg$, so that $2e = 2fg$. Furthermore, from the equality $1+2e = f+g$, after multiplication by f, we get $2ef = fg$. So, $2e = 4ef = 0$ and we are done with the proof of the assertion.

In order to show that for every x we have $x^2 - x \in \{0,2\}$, we just must to check two cases (one sees that if x is an idempotent, we have nothing to verify). If $x = -e$, for an idempotent e, then $x^2 - x = 2e \in \{0,2\}$ by what we have already established above. If now $x = f+g$, for commuting idempotents f, g, we deduce that $x^2 - x = 2fg \in \{0,2\}$ again, because fg is obviously an idempotent. So, we are finally finished after all.

We now arrive at our central result.

THEOREM 1. *Suppose R is a ring. Then each its element satisfies (one of) the equations $x^2 = x$ or $x^2 = 2+x$ if, and only if, exactly one of the next two items is valid:*

(1) $R \cong B$ or $R \cong \mathbb{Z}_3$ or $R \cong B \times \mathbb{Z}_3$, where B is a Boolean ring.

(2) $R/J(R) \cong B$ and either $J(R) = \{0\}$ or $J(R) = \{0,2\}$, where B is a Boolean ring.

Proof. "**Necessity**". Since these equations are trivially satisfied for the elements 0, 1 and 2, one writes that $3^2 = 3$ or $3^2 = 2+3$. So, $6 = 0$ or $4 = 0$. We thus come to two different basic cases:

Case: $6 = 0$**.** Utilizing the classical Chinese Reminder Theorem, we write that $R \cong R_1 \times R_2$, where R_1, R_2 are rings of characteristic 2 and 3, respectively. Thus, since $2 = 0$ in R_1, the equations $x^2 = x$ and $x^2 = 2+x$ are equivalent. Hence R_1 is Boolean.

As for the other direct factor R_2, we consider the element $x+2$. If $(x+2)^2 = x+2$, then $x^2 = 1$. This combined with $x^2 = x$ gives that $x = 1$, whereas combined with $x^2 = x+2$ forces that $x = 2$. But if now $(x+2)^2 = 2+(x+2)$, then $x^2 = 0$. Thus either $x = 0$ or $x+2 = 0$ which yields that $x = 1$, and it follows finally that either $R_2 = \{0\}$ or $R_2 = \{0,1,2\} \cong \mathbb{Z}_3$, as required.

Case: $4 = 0$**.** This follows immediately from Proposition 2. An independent proof, however, is like this: Namely, 2 is then nilpotent and so $2 \in J(R)$. Therefore, $R/J(R)$ is of characteristic 2 and since it satisfies the same two conditions as these for R, it is necessarily a Boolean ring. If now $J(R) \neq \{0\}$, we take $z \in J(R)\backslash\{0\}$. So, $u = 1+z \in U(R)\backslash\{1\}$. Thus, $u^2 = u+2$ and, since $u+2$ is also invertible, we have that $(u+2)^2 = u+2$, because the characteristic is 4. Hence, $u+2 = 1$ and thus $u = -1$ yields that $z = 2$, as expected.

"**Sufficiency**". (1) A plain direct check unambiguously shows that all of the elements in B, $\mathbb{Z}_3$ and $B \times \mathbb{Z}_3$ satisfy the desired two equations, so we omit the details.

(2) This follows directly by virtue of Proposition 2. □

There are various concrete examples of rings from the point (2) such as: $\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$. However, $\mathbb{Z}_4 \times \mathbb{Z}_4$ need *not* be of this type as simple calculations show.

REMARK 3. Certainly, the theorem can also be written as follows: All elements of a ring R satisfy (one of) the equations $x^2 = x$ or $x^2 = 2+x$ if, and only if, $R \cong B$ or $R \cong \mathbb{Z}_3$ or $R \cong B \times \mathbb{Z}_3$ or $R/J(R) \cong B$ with $J(R) = \{0,2\}$, where B is a Boolean ring.

The following consequence follows by a direct application of Theorem 1, but we shall give here a more direct and conceptual proof.

COROLLARY 1. *Let R be a ring. Then, for any of its elements, the equations $x^2 = x$ or $x^2 = -x$ are true $\iff$ for any of its elements the equations $x^2 = x$ or $x^2 = 2+x$ are true and $6 = 0$.*

Proof. Observe that if $2 = 0$ we are done, so assume that $2 \neq 0$.

"$\Rightarrow$". Since $2^2 = -2$ (the other possibility $2^2 = 2$ is now unavailable), we have $6 = 0$. Let $y \in R$ such that $y^2 \neq y$. Hence $y^2 = -y$. Consider the element $y+2$. If first $(y+2)^2 = -y-2$, then it follows that $y^2+5y+6=0$, i.e., $y^2 = -5y = y$ which is against our assumption. We therefore may write $(y+2)^2 = y+2$ which implies that $y^2+3y+2=0$. Comparing this with $y^2 = -y$, one deduces that $-y = y+2$ whence $y^2 = y+2$, as desired.

"$\Leftarrow$". Let now $y \in R$ with $y^2 \neq y$. Thus $y^2 = 2+y$. Consider the element $y+4$. If $(y+4)^2 = y+4$, we obtain that $y^2+7y+12=0$, that is, $y^2 = -7y = -y$ and we are set. Otherwise, if $(y+4)^2 = 2+(y+4)$, we derive that $y^2+7y+16=0$ amounting to $y^2+y-2=0$. Comparing that with $y^2 = 2+y$, one concludes that $2y=0$. Furthermore, multiplying both sides of $y^2 = y+2$ by 2, it follows that $0=4$ which along with $6=0$ ensures that $2=0$, again a contradiction with our assumption. $\square$

We finish off with some comments and two problem of interest.

PROBLEM 1. *Find what equation holds for a ring of which each element is a sum or a difference of two commuting idempotents, and vice versa. Same problem exists also for a ring of which each element is a sum of three commuting idempotents.*

Notice that the sum of three commuting idempotents always satisfies the equation $x^5 = x$ whenever $30 = 0$.

In the spirit of Proposition 1, it is adequate to state the following.

PROBLEM 2. *Describe those rings having all elements as solutions of (one of) the equations $x^3 = x$ or $x^3 = -x$.*

Note that it is trivially seen that the field $\mathbb{Z}_5$ satisfies the above two equalities; they both also imply that $x^5 = x$.

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References

[1] M.-S. Ahn and D.D. Anderson, *Weakly clean rings and almost clean rings*, Rocky Mountain J. Math. **36** (2006), 783–798.

[2] P.V. Danchev, *On weakly clean and weakly exchange rings having the strong property*, Publ. Inst. Math. (Beograd) (N.S.) (1) **101** (2017), 135–142.

[3] P.V. Danchev and W.Wm. McGovern, *Commutative weakly nil clean unital rings*, J. Algebra (5) **425** (2015), 410–422.

[4] Y. Hirano, H. Tominaga, *Rings in which every element is the sum of two idempotents*, Bull. Austral. Math. Soc. **37** (1988), 161–164.

[5] T. Koşan, Z. Ying and Y. Zhou, *Rings in which every element is a sum of two tripotents*, Can. Math. Bull. (3) **59** (2016), 661–672.

[6] T.Y. Lam, A First Course in Noncommutative Rings, Second Edition, Graduate Texts in Math., Vol. **131**, Springer-Verlag, Berlin-Heidelberg-New York, 2001.

[7] T.Y. Lam, Exercises in Classical Ring Theory, Second Edition, Problem Books in Math., Springer-Verlag, Berlin-Heidelberg-New York, 2003.

[8] H. Tominaga, *On anti-inverse rings*, Publ. Inst. Math. (Beograd) (N.S.) **33 (47)** (1983), 225.

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