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A MEAN VALUE INEQUALITY FOR THE DIGAMMA FUNCTION

Abstract. A recently published result states that for all $x > 0$ the harmonic mean of $\psi(x)$ and $\psi(1/x)$ is greater than or equal to $-\gamma$, that is,

$$-\gamma \leqslant H\big(\psi(x), \psi(1/x)\big) \quad (x > 0).$$

Here, $\psi = \Gamma'/\Gamma$ denotes the digamma function and γ is Euler's constant. We offer a proof for the following refinement:

$$-\gamma H(x, 1/x) \leqslant H\big(\psi(x), \psi(1/x)\big) \quad (x > 0).$$

Keywords: Digamma function, polygamma functions, harmonic mean, Euler's constant, inequality

1. Introduction and main result

The classical harmonic mean of two real numbers a and b (which are not both equal to 0) is defined by

$$H(a,b) = \frac{2ab}{a+b}.$$

This mean value plays a role in various fields of mathematics and it also appears in physics, chemistry and computer science. Many inequalities for the harmonic and other means can be found in the monograph [10].

In 1974, Gautschi [13] published a remarkable mean value inequality for Euler's gamma function. He showed that for all positive real numbers x the harmonic mean of $\Gamma(x)$ and $\Gamma(1/x)$ is greater than or equal to 1, that is,

$$1 \leqslant H\big(\Gamma(x), \Gamma(1/x)\big). \tag{1}$$

Various generalizations, improvements and companions of (1) are given in [2], [3], [4], [5], [7], [8], [14], [15], [16], [17].

In a recent paper, Jameson and the author [9] proved a counterpart of (1) for the digamma function

$$\psi(x) = \frac{d}{dx}\log\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

For all $x > 0$ we have

$$-\gamma \leqslant H\big(\psi(x), \psi(1/x)\big), \tag{2}$$

where the constant lower bound is best possible. Here, as usual, $\gamma = 0.57721...$ denotes Euler's constant.

The digamma function has interesting applications in numerous fields, like, for instance, the theory of special functions, statistics, mathematical physics and number theory. See [11], [12], [18], [19], [21].

The following series and integral representations are valid for $x > 0$:

$$\psi(x) = -\gamma - \frac{1}{x} + \sum_{k=1}^{\infty} \frac{x}{k(x+k)} = \int_0^{\infty} \Big(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}}\Big)dt.$$

ψ has a zero at $x_0 = 1.46163....$ (In what follows, we maintain this notation.) Moreover, ψ is strictly increasing and strictly concave on $(0,\infty)$ and satisfies the limit relations

$$\lim_{x\to 0+} x\psi(x) = -1 \quad \text{and} \quad \lim_{x\to\infty} \frac{\psi(x)}{\log(x)} = 1.$$

These and many other properties of the ψ-function are given, for example, in [1], [20].

It is the aim of this note to offer a refinement of (2). We show that the following mean value inequality holds.

THEOREM. *For all positive real numbers x we have*

$$-\gamma H(x, 1/x) \leqslant H\big(\psi(x), \psi(1/x)\big).$$

The sign of equality holds if and only if $x = 1$.

In order to verify the Theorem we need several lemmas. They are collected in the next section. A proof of the Theorem is given in Section 3.

The numerical values have been calculated via the computer program MAPLE 13.

2. Lemmas

In this section, we present properties of functions which are defined in terms of the ψ-function and its derivatives which are known as polygamma functions. We recall that $(-1)^{n+1}\psi^{(n)}(x) > 0$ for $n \in \mathbb{N}$ and $x > 0$. In particular, ψ' is positive on $(0,\infty)$.

Proofs for the first two lemmas can be found in [6].

LEMMA 1. *Let $k \geqslant 1$ be an integer, c be a real number and*

$$f_{c,k}(x) = x^c|\psi^{(k)}(x)|. \tag{3}$$

(i) $f_{c,k}$ is strictly decreasing on $(0,\infty)$ if and only if $c \leqslant k$.
(ii) $f_{c,k}$ is strictly increasing on $(0,\infty)$ if and only if $c \geqslant k+1$.

LEMMA 2. *Let $k \geqslant 1$ be an integer. The function*

$$g_k(x) = x\frac{\psi^{(k+1)}(x)}{\psi^{(k)}(x)} \tag{4}$$

is strictly increasing on $(0,\infty)$ with

$$\lim_{x\to\infty} g_k(x) = -k.$$

LEMMA 3. *The function*

$$h(x) = \frac{1}{\psi(x)} + \frac{1}{\psi(1/x)} \tag{5}$$

is positive on $(0,1/x_0) \cup (x_0,\infty)$ and negative on $(1/x_0,x_0)$.

Proof. We define for $x \in (0,1)$:

$$u(x) = \psi(x) + \psi(1/x).$$

Then,

$$xu'(x) = f_{1,1}(x) - f_{1,1}(1/x),$$

where $f_{1,1}$ is given in (3). Since $0 < x < 1/x$, we conclude from Lemma 1 (i) that $f_{1,1}(x) > f_{1,1}(1/x)$. It follows that u is strictly increasing on $(0,1)$.
Next, let $0 < x < 1/x_0$. Then,

$$\psi(x) + \psi(1/x) = u(x) < u(1/x_0) = \psi(1/x_0).$$

Hence,
$$0<\psi(1/x)<\psi(1/x_0)-\psi(x).$$
This leads to
$$\frac{1}{\psi(1/x_0)-\psi(x)}<\frac{1}{\psi(1/x)}. \tag{6}$$
Since
$$0<\psi(1/x_0)-\psi(x)<-\psi(x),$$
we obtain
$$-\frac{1}{\psi(x)}<\frac{1}{\psi(1/x_0)-\psi(x)}, \tag{7}$$
so that (6) and (7) imply that h is positive on $(0,1/x_0)$. Since $h(x)=h(1/x)$, we conclude that h is also positive on (x_0,∞).

If $x\in(1/x_0,x_0)$, then $\psi(x)<0$ and $\psi(1/x)<0$. This reveals that h is negative on $(1/x_0,x_0)$. □

LEMMA 4. *The function*
$$v(x)=-x\frac{\psi'(x)}{\psi^2(x)}+\frac{1}{\gamma}x \tag{8}$$
is strictly decreasing on $(1/x_0,x_0)$.

Proof. Let $x\in(1/x_0,x_0)$ and
$$w(x)=-\psi(x)\big(-1-g_1(x)\big)-2f_{1,1}(x),$$
where $f_{1,1}$ and g_1 are defined in (3) and (4), respectively. Then we obtain
$$v'(x)=-\frac{1}{\psi^3(x)}\cdot\psi'(x)\cdot w(x)+\frac{1}{\gamma}. \tag{9}$$
Applying Lemma 1 (i) and Lemma 2 reveals that $f_{1,1}$ is decreasing on $(0,\infty)$ and that $-1-g_1$ is positive and decreasing on $(0,\infty)$. This leads to
$$w(x)\leqslant-\psi(1/x_0)\big(-1-g_1(1/x_0)\big)-2f_{1,1}(x_0)=-2.07\ldots. \tag{10}$$
Let $1/x_0\leqslant r\leqslant x\leqslant s\leqslant x_0$. Since $-1/\psi^3$ is positive and increasing on $(0,x_0)$ and ψ' is positive and decreasing on $(0,\infty)$, we obtain from (9) and (10):
$$v'(x)\leqslant-2.07\cdot\frac{-1}{\psi^3(x)}\cdot\psi'(x)+\frac{1}{\gamma}\leqslant-2.07\cdot\frac{-1}{\psi^3(r)}\cdot\psi'(s)+\frac{1}{\gamma}=y(r,s),\text{ say.} \tag{11}$$
We have
$$y(1/x_0,0.9)=-0.23\ldots\quad\text{and}\quad y(0.9,x_0)=-2.92\ldots,$$
so that (11) reveals that v' is negative on $(1/x_0,x_0)$. □

3. Proof of the Theorem

We define for $x > 0$:

$$H^*(x) = H\big(\psi(x), \psi(1/x)\big).$$

Since $\psi(1) = -\gamma$, we obtain

$$-\gamma H(1,1) = H^*(1) = -\gamma.$$

Thus, it remains to show that

$$-\gamma H(x, 1/x) < H^*(x) \quad \text{for} \quad x \in (0,1). \tag{12}$$

We consider two cases.
Case 1. $0 < x \leqslant 1/x_0$.
Let h be the function defined in (5). Lemma 3 yields

$$H^*(x) = \frac{2}{h(x)} \geqslant 0 > -\gamma H(x, 1/x).$$

Case 2. $1/x_0 < x < 1$.
Using Lemma 3 we obtain that in order to prove (12) it suffices to show that the function

$$z(x) = h(x) + \frac{1}{\gamma}(x + 1/x)$$

is negative on $(1/x_0, 1)$. We have

$$xz'(x) = v(x) - v(1/x), \tag{13}$$

where v is given in (8). Since $1/x_0 < x < 1/x < x_0$, we conclude from Lemma 4 that $v(x) > v(1/x)$. Applying (13) reveals that z is strictly increasing on $(1/x_0, 1)$. Thus,

$$z(x) < z(1) = 0.$$

This completes the proof of the Theorem.

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