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RENORMALIZED SOLUTIONS TO A FOURTH ORDER NLS IN THE MASS SUBCRITICAL REGIME

Abstract. We study the mixed dispersion fourth order nonlinear Schrödinger equation

$$i\partial_t\psi - \gamma\Delta^2\psi + \beta\Delta\psi + |\psi|^{2\sigma}\psi = 0 \text{ in } \mathbb{R}\times\mathbb{R}^N,$$

where $\gamma, \sigma > 0$ and $\beta \in \mathbb{R}$. We focus on standing wave solutions, namely solutions of the form $\psi(x,t) = e^{i\alpha t}u(x)$, for some $\alpha \in \mathbb{R}$. This ansatz yields the fourth-order elliptic equation

$$\gamma\Delta^2 u - \beta\Delta u + \alpha u = |u|^{2\sigma}u.$$

We consider an associated mass constrained minimization problem in the case $\sigma N < 4$. Under suitable conditions, we establish existence of minimizers and we investigate their qualitative properties. Based on a joint work with D. Bonheure, E.M. dos Santos and R. Nascimento [5].

1. Introduction

We consider the following mixed dispersion fourth order nonlinear Schrödinger equation

$$\text{(Mixed 4NLS)}\quad i\partial_t\psi - \gamma\Delta^2\psi + \Delta\psi + |\psi|^{2\sigma}\psi = 0,\ \psi(0,x) = \psi_0(x),\ (t,x) \in \mathbb{R}\times\mathbb{R}^N,$$

for $\sigma > 0$ and some $\gamma > 0$. The fourth order term has been introduced by Karpman and Shagalov (see [15] and the references therein) to regularize and stabilize solutions to the standard nonlinear Schrödinger equation

$$\text{(NLS)}\qquad i\partial_t\psi + \Delta\psi + |\psi|^{2\sigma}\psi = 0,\ \psi(0,x) = \psi_0(x),\ (t,x) \in \mathbb{R}\times\mathbb{R}^N.$$

Indeed, it is well-known that, when $\sigma N < 2$, all solutions to (NLS) exist globally in time and standing waves (solutions of the form $\psi(t,x) = e^{i\alpha t}u(x)$ for some $\alpha \in \mathbb{R}$) are orbitally stable. Whereas if $\sigma N \geqslant 2$, then finite time blow-up may appear and standing wave solutions become unstable. We refer for instance to [10, 19]. Observe that for $N = 2$ and $N = 3$, the Kerr nonlinearity ($\sigma = 1$) is respectively critical and supercritical. Using a combination of stability analysis and numerical simulations, Karpman and Shagalov (see also [12]) showed that when $0 < N\sigma < 4$ and (γ is small enough if $2 \leqslant N\sigma < 4$), standing wave solutions to (Mixed 4NLS) exist globally in time and are stable and when $N\sigma > 4$, they become unstable. Notice that the Kerr nonlinearity is now subcritical in dimension 2 and 3 in this extended model.

The equation (Mixed 4NLS) has attracted less attention than its classical counterpart (NLS) though with an increasing interest more recently. We refer to the works

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by Ben-Artzi-Koch-Saut [2] and Pausader [17] for well-posedness and scattering, see also [13, 18] and to the recent work of Boulenger-Lenzmann [7] and the references therein concerning finite-time blow-up. We also mention that the one-dimensional stationary mixed dispersion NLS has been studied in [1] and [8].

Here, we focus on standing wave solutions of (Mixed 4NLS). The ansatz $\psi(t,x) = e^{i\alpha t}u(x)$ yields the fourth-order semilinear elliptic equation

$$\gamma\Delta^2 u - \Delta u + \alpha u = |u|^{2\sigma}u \text{ in } \mathbb{R}^N. \tag{1.1}$$

Observe that two constrained minimization problems naturally arise as for (NLS). Indeed, if one looks for time independent solutions, it is natural to consider the following problem

$$m = \inf_{u\in\tilde{M}} J_{\gamma,\alpha}(u), \tag{1.2}$$

where $J_{\gamma,\alpha} : H^2(\mathbb{R}^N) \to \mathbb{R}$ is the quadratic form defined by

$$J_{\gamma,\alpha}(u) = \gamma\int_{\mathbb{R}^N} |\Delta u|^2\,dx + \int_{\mathbb{R}^N} |\nabla u|^2\,dx + \alpha\int_{\mathbb{R}^N} |u|^2\,dx, \tag{1.3}$$

and

$$\tilde{M} = \{u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^{2\sigma+2}dx = 1\}. \tag{1.4}$$

Notice that if m is achieved by some $u\in\tilde{M}$, then $v = m^{\frac{1}{2\sigma}}u$ is a solution to (1.1). The following result is proved in [6].

THEOREM 1.1 ([6, Theorem 1.1]). Assume $\alpha,\gamma > 0$ and $2 < 2\sigma+2 < 2N/(N-4)$ if $N \geqslant 5$. Then problem (1.2) has a nontrivial solution. If $\alpha \leqslant 1/(4\gamma)$, then any least energy solution does not change sign, is radially symmetric around some point and strictly radially decreasing.

We now turn to the second natural variational problem associated with (Mixed 4NLS) which will be our main focus. Since the L^2-norm is conserved along the flow for (Mixed 4NLS), it is natural to look for standing waves having a prescribed L^2-norm. Such solutions were built by Cazenave and Lions [11] for (NLS). Their construction consists in minimizing the functional $E_0 : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$E_0(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \frac{1}{2\sigma+2}\int_{\mathbb{R}^N} |u|^{2\sigma+2}\,dx \tag{1.5}$$

under the constraint $\|u\|_{L^2}^2 = \mu$. If $0 < N\sigma < 2$, E_0 achieves its infimum and any associated minimizer solves

$$-\Delta u + \alpha u = |u|^{2\sigma}u \text{ in } \mathbb{R}^N, \tag{1.6}$$

with the Lagrange multiplier

$$\alpha = \frac{1}{\mu}\left(\int_{\mathbb{R}^N} |u|^{2\sigma+2}\,dx - \int_{\mathbb{R}^N} |\nabla u|^2\,dx\right). \tag{1.7}$$

Moreover, Cazenave and Lions [11, Theorem II.2] showed that those standing waves minimizing E_0 are orbitally stable for (NLS) whereas standing waves built for instance in [3, 4] are unstable for $N/2 < \sigma < 2/(N-2)$ as arbitrarily close initial conditions lead to blowing up solutions, see [11, Remark II.2] .

For (Mixed 4NLS), we obtain the following counterpart. Define

$$I_\gamma(\mu) = \inf_{u\in M_\mu} E_\gamma(u) \tag{1.8}$$

where

$$M_\mu = \{u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2\,dx = \mu\} \tag{1.9}$$

and

$$E_\gamma(u) = \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \frac{1}{2\sigma+2}\int_{\mathbb{R}^N} |u|^{2\sigma+2}\,dx. \tag{1.10}$$

If $I_\gamma(\mu)$ is achieved, then any associated minimizer solves (1.1) with the Lagrange multiplier

$$\alpha = \frac{1}{\mu}\left(\int_{\mathbb{R}^N} |u|^{2\sigma+2}\,dx - \gamma\int_{\mathbb{R}^N} |\Delta u|^2\,dx - \int_{\mathbb{R}^N} |\nabla u|^2\,dx\right). \tag{1.11}$$

Our main result is the following:

THEOREM 1.2. Assume $\gamma > 0$. If $0 < \sigma < 2/N$, then $I_\gamma(\mu)$ is achieved for every $\mu > 0$. If $2/N \leqslant \sigma < 4/N$, then there exists a critical mass $\mu_c(\gamma,\sigma)$ such that

1. $I_\gamma(\mu)$ is not achieved if $\mu < \mu_c$;
2. $I_\gamma(\mu)$ is achieved if $\mu > \mu_c$ and $\sigma = 2/N$;
3. $I_\gamma(\mu)$ is achieved if $\mu \geqslant \mu_c$ and $\sigma \neq 2/N$;

As far as we know, it is the first result in the literature concerning the existence of standing waves of (Mixed 4NLS) with a prescribed L^2 mass. Observe that the mass threshold for existence is due to a lack of homogeneity. Indeed, all the terms of the functional to be minimized scale differently. Such a behaviour is present in other models like the Schrödinger-Poisson equation, see [9, 14].

The plan of this paper is the following : in a first time, we sketch the proof of Theorem 1.2. Then, we discuss briefly the qualitative properties of solutions such as positivity, symmetry and stability.

2. Existence of standing waves with a prescribed mass

In all the following, to avoid technical issues and simplify notations, we restrict ourselves to the case $N > 4$ and we assume that $\sigma \neq 2/N$.

2.1. Gagliardo-Nirenberg interpolation inequalities

We begin by recalling two well-known Gagliardo-Nirenberg interpolation inequalities for functions $u \in H^2(\mathbb{R}^N)$, namely, for $0 \leqslant \sigma < \frac{4}{N-4}$,

$$\|u\|_{L^{2\sigma+2}}^{2\sigma+2} \leqslant B_N(\sigma)\|\Delta u\|_{L^2}^{\frac{\sigma N}{2}} \|u\|_{L^2}^{2+2\sigma-\frac{\sigma N}{2}}, \tag{2.1}$$

and, for $0 \leqslant \sigma < \frac{2}{N-2}$,

$$\|u\|_{L^{2\sigma+2}}^{2\sigma+2} \leqslant C_N(\sigma)\|\nabla u\|_{L^2}^{\sigma N} \|u\|_{L^2}^{2+\sigma(2-N)}. \tag{2.2}$$

The constants $B_N(\sigma)$ and $C_N(\sigma)$ depend on σ and N. Thanks to these inequalities, we can prove a 2-parameters Gagliardo-Nirenberg interpolation type inequality involving the L^2 norms of $u, \nabla u$ and Δu.

LEMMA 2.1. Assume $0 < \sigma < 4/(N-4)$ if $N > 4$. Let $0 < \delta < \sigma < \tau$ and assume $\tau < 4/(N-4)$ and $\delta < 2/(N-2)$. Then, there exists $C > 0$ such that, for any $u \in H^2(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |u|^{2\sigma+2} dx \leqslant C \left(\int_{\mathbb{R}^N} u^2 dx\right)^p \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^q \left(\int_{\mathbb{R}^N} |\Delta u|^2 dx\right)^r, \tag{2.3}$$

where $p = 1 - \frac{\sigma(N-4)+N\delta(1-\lambda)}{4}$, $q = \frac{\delta N}{2}(1-\lambda)$, $r = \frac{\tau N}{4}\lambda$ and $\lambda = (\sigma-\delta)/(\tau-\delta)$. Moreover, we have $C \leqslant (B_N(\sigma))^{\lambda}(C_N(\sigma))^{1-\lambda}$.

As a direct consequence of this lemma, when $2/N < \sigma < 4/N$, there exists a constant $C_{\sigma,N} > 0$ such that

$$\int_{\mathbb{R}^N} |u|^{2\sigma+2} dx \leqslant C_{\sigma,N} \left(\int_{\mathbb{R}^N} |u|^2 dx\right)^{\sigma} \left(\int_{\mathbb{R}^N} |\Delta u|^2 dx\right)^{\frac{\sigma N}{2}-1} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{\frac{4-\sigma N}{2}}. \tag{2.4}$$

This inequality will be very useful in the following.

2.2. Estimates of the energy

This subsection is devoted to energy estimates on the functional E_γ. The main aim is to deduce the sign of $I_\gamma(\mu)$ as a function of μ. We begin by showing the coercivity of E_γ.

LEMMA 2.2. The energy E_γ is bounded from below and coercive over M_μ when $0 < \sigma < 4/N$. Moreover, for $\sigma \in (0, 4/N)$ the map $\mu \mapsto I_{\gamma,1}(\mu)$ is non-increasing, $I_{\gamma,1}(\mu) \leqslant 0$ for all $\mu > 0$. When $\sigma > 4/N$, we have $I_{\gamma,1}(\mu) = -\infty$ for every $\mu > 0$.

Proof. First, we infer from the Gagliardo-Nirenberg inequality (2.1) that

$$\begin{aligned}E_\gamma(u) &= \frac{\gamma}{2}\int_{\mathbb{R}^N}|\Delta u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,dx - \frac{1}{2\sigma+2}\int_{\mathbb{R}^N}|u|^{2\sigma+2}\,dx\\ &\geqslant \frac{\gamma}{2}\int_{\mathbb{R}^N}|\Delta u|^2\,dx - \frac{B_N(\sigma)\mu^{1+\sigma-\frac{\sigma N}{4}}}{2\sigma+2}\left(\int_{\mathbb{R}^N}|\Delta u|^2\,dx\right)^{\frac{\sigma N}{4}}.\end{aligned}$$

This shows that the functional E_γ is bounded from below and coercive over M_μ when $0<\sigma<4/N$. Now, let $u\in M_\mu$ and consider $u_\lambda(x)=\lambda^{\frac{N}{2}}u(\lambda x)$ for $\lambda>0$ so that $u_\lambda\in M_\mu$. Then,

$$I_\gamma(\mu)\leqslant E_\gamma(u_\lambda)=\frac{\gamma\lambda^4}{2}\int_{\mathbb{R}^N}|\Delta u|^2\,dx+\frac{\lambda^2}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,dx-\frac{\lambda^{\sigma N}}{2\sigma+2}\int_{\mathbb{R}^N}|u|^{2\sigma+2}\,dx \tag{2.5}$$

for all $\lambda>0$. Letting λ go to zero, we get $I_\gamma(\mu)\leqslant 0$. Note that the so-called *large inequalities*

$$I_\gamma(\mu)\leqslant I_\gamma(\theta)+I_\gamma(\mu-\theta),\qquad \text{for all } \theta\in]0,\mu[, \tag{2.6}$$

always hold true. Indeed, for any $\varepsilon>0$ we may choose test functions $u_\varepsilon\in M_\theta$ and $v_\varepsilon\in M_{\mu-\theta}$ with compact supports such that

$$I_\gamma(\theta)\leqslant E_\gamma(u_\varepsilon)\leqslant I_\gamma(\theta)+\varepsilon,\qquad I_\gamma(\mu-\theta)\leqslant E_\gamma(v_\varepsilon)\leqslant I_\gamma(\mu-\theta)+\varepsilon.$$

Then, if $e\in\mathbb{R}^N$ is a unit vector, we have that for k large enough the supports of u_ε and $v_\varepsilon(\cdot+ke)$ are disjoint. So, using the translation invariance of E_γ and M_μ we have $u_\varepsilon+v(\cdot+ke)\in M_\mu$ for k large and therefore

$$I_\gamma(\mu)\leqslant\limsup_{k\to\infty}E_\gamma(u_\varepsilon+v_\varepsilon(\cdot+ke))\leqslant I_\gamma(\theta)+I_\gamma(\mu-\theta)+2\varepsilon.$$

Hence, (2.6) holds and as a consequence we infer that $\mu\mapsto I_\gamma(\mu)$ is non-increasing since $I_\gamma(\mu)\leqslant 0$ for all μ. We finally observe that the last claim follows by letting $\lambda\to\infty$ in (2.5) when $\sigma>4/N$. □

Using scaling arguments, it is possible to show that $I_\gamma(\mu)$ is strictly negative when σ is H^1-subcritical or the mass is large.

LEMMA 2.3. Let $0<\sigma<2/N$. For any given $\mu>0$, we have $I_\gamma(\mu)<0$. Moreover, if $2/N<\sigma<4/N$, we have that $I_\gamma(\mu)<0$ for large enough μ.

Proof. Take $u\in M_\mu$ and set $u_\lambda(x)=\lambda^{\frac{N}{2}}u(\lambda x)$ for $\lambda>0$. We have

$$\frac{E_\gamma(u_\lambda)}{\lambda^{\sigma N}}=\frac{\gamma\lambda^{4-\sigma N}}{2}\int_{\mathbb{R}^N}|\Delta u|^2\,dx+\frac{\lambda^{2-\sigma N}}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,dx-\frac{1}{2\sigma+2}\int_{\mathbb{R}^N}|u|^{2\sigma+2}\,dx.$$

Taking λ small enough, we deduce that $I_\gamma(\mu)<0$ when $0<\sigma<2/N$. Next, we assume $2/N<\sigma<4/N$. Taking $u\in M_1$, it is easy to see that $E_\gamma(\sqrt{\mu}u)<0$ when μ is large enough. □

Using the extended Gagliardo-Nirenberg interpolation inequality (2.4), it is possible to deduce some refined estimates on the sign of $I_\gamma(\mu)$ when $2/N < \sigma < 4/N$.

LEMMA 2.4. Let $2/N < \sigma < 4/N$. There exists an (explicit) constant μ_c depending on $C_{\sigma,N}$ and γ such that $I_\gamma(\mu) = 0$ if and only if $\mu \leqslant \mu_c$.

2.3. Proof of Theorem 1.2

We begin by sketching the proof of point 3 when $\mu > \mu_c$ (we will not discuss the equality case). Recall that we always have the following inequality

$$I_\gamma(\mu) \leqslant I_\gamma(\theta) + I_\gamma(\mu-\theta), \qquad \text{for all } \theta \in]0,\mu[.$$

It is standard that the Concentration-Compactness method [16] yields that the minimizing sequences, up to translations, are relatively compact if and only if the *strict subaddivity condition* holds, namely

$$I_\gamma(\mu) < I_\gamma(\theta) + I_\gamma(\mu-\theta), \qquad \text{for all } \theta \in]0,\mu[. \tag{2.7}$$

In fact, arguing as in [16], the inequality (2.7) is easily obtained provided $I_\gamma(\mu) < 0$ which holds true thanks to the two previous lemma.

Next we sketch the proof of point 1. Assume by contradiction that there exists $\widetilde{\mu} \in (0,\mu_c)$ such that $I_\gamma(\widetilde{\mu})$ has a minimizer $u_{\widetilde{\mu}}$. From the definition of μ_c, we have that $I_\gamma(\mu_c) = 0$. It is easy to check that, for $t > 1$, we have

$$I_\gamma(t\widetilde{\mu}) \leqslant E_\gamma(\sqrt{t}u_{\widetilde{\mu}}) < tE_\gamma(u_{\widetilde{\mu}}) = tI_\gamma(\widetilde{\mu}),$$

which implies that $I_\gamma(\mu_1) < 0$ for any $\mu_1 > \widetilde{\mu}$. Hence, a contradiction with the definition of μ_c.

3. Qualitative properties

3.1. Existence of positive standing waves with a prescribed mass

In this section, we consider the slightly modified minimization problem

$$\widetilde{I}_\gamma(\mu) = \inf_{u \in M_\mu} \widetilde{E}_\gamma(u) \tag{3.1}$$

where M_μ is defined as in (1.9) and

$$\widetilde{E}_\gamma(u) = \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \frac{1}{2\sigma+2}\int_{\mathbb{R}^N} |u^+|^{2\sigma+2}\,dx. \tag{3.2}$$

The proof of Theorem 1.2 applies to problem (3.1) with straightforward modifications. Let us recall that if u is a solution of problem (3.1), then u solves

$$\gamma\Delta^2 u - \Delta u + \alpha(\mu)u = |u|^{2\sigma}u^+, \tag{3.3}$$

where

$$-\alpha(\mu)\mu = \int_{\mathbb{R}^N} (\gamma|\Delta u|^2 + |\nabla u|^2 - |u|^{2\sigma+2})\,dx = 2E_\gamma(u) - \frac{\sigma}{\sigma+1}\int_{\mathbb{R}^N} |u^+|^{2\sigma+2}\,dx. \tag{3.4}$$

It is immediate to see that $\alpha(\mu) \geqslant 0$. We establish the positivity of solutions to (3.1) provided that $\alpha(\mu)$ is small enough.

PROPOSITION 3.1. Suppose that $\alpha(\mu) \leqslant 1/(4\gamma)$. Then, any solution to (3.1) is strictly positive (or strictly negative).

Proof. Using that $\alpha(\mu) \leqslant 1/(4\gamma)$, we can rewrite the equation satisfied by u as

$$\begin{cases} -\sqrt{\gamma}\Delta u + \lambda_1 u &= v, \\ -\sqrt{\gamma}\Delta v + \lambda_2 v &= |u|^{2\sigma}u^+, \end{cases}$$

for some positive constants λ_i, $i = 1,2$ satisfying $\lambda_2\lambda_1 = \alpha(\mu)$ and $\lambda_1 + \lambda_2 = 1/\sqrt{\gamma}$. It is then standard to see that $u > 0$. □

We next estimate the Lagrange multiplier of problem (3.1) by the L^2 mass, namely we will prove

$$\alpha(\mu) \leqslant C\mu^{\frac{\sigma}{1-\sigma N/4}}$$

for some $C > 0$. This estimate enables us to apply the previous theorem when the mass is small enough.

COROLLARY 3.1. Assume $0 < \sigma < 4/N$. There exists $\mu_0 > 0$ such that for all $\mu \leqslant \mu_0$ then

$$\alpha(\mu) \leqslant 1/(4\gamma),$$

and therefore any solution to (3.1) is strictly positive (or strictly negative).

Proof. First, we recall that any solution to (3.1) satisfies the Pohozaev identity

$$2\gamma\int_{\mathbb{R}^N} |\Delta u|^2\,dx + \int_{\mathbb{R}^N} |\nabla u|^2\,dx = \frac{\sigma N}{2\sigma+2}\int_{\mathbb{R}^N} |u^+|^{2\sigma+2}\,dx.$$

Thanks to this equality, we deduce that

$$\alpha(\mu) \leqslant \frac{1}{\mu}\left(2 - \frac{\sigma N}{2\sigma+2}\right)\int_{\mathbb{R}^N} |u^+|^{2\sigma+2}\,dx.$$

The Gagliardo-Nirenberg inequality (2.1) then implies that

$$\int_{\mathbb{R}^N} |u^+|^{2\sigma+2}\,dx \leqslant B_N(\sigma)\left(\int_{\mathbb{R}^N} |u^+|^{2\sigma+2}\,dx\right)^{\frac{\sigma N}{4}}\left(\int_{\mathbb{R}^N} |u|^2\,dx\right)^{1+\sigma-\frac{\sigma N}{4}}.$$

Combining the two previous lines, we conclude that

$$\alpha(\mu) \leqslant B_N(\sigma)\left(2 - \frac{\sigma N}{2\sigma+2}\right)\mu^{\frac{\sigma}{1-\sigma N/4}}.$$

□

3.2. Radial symmetry of at least one minimal standing wave with prescribed mass

Using the method of [7], one can show that at least one solution of (1.8) is radially symmetric if $2\sigma \in \mathbb{N}_0$.

PROPOSITION 3.2. Suppose that problem (1.8) has a minimizer and assume $2\sigma \in \mathbb{N}_0$. Then there exists at least one radially symmetric minimizer for (1.8).

Proof. The proof is a direct adaptation of [7, Appendix A.2]. The main ingredient of their proof is the *Fourier rearrangement*. Namely, for $u \in L^2(\mathbb{R}^N)$ we set its Fourier rearrangement by $u^\sharp = \mathcal{F}^{-1}\{(\mathcal{F}u)^*\}$, where $\mathcal{F}$ stands for the Fourier transform and f^* denotes the symmetric-decreasing rearrangement of a measurable function $f : \mathbb{R}^N \to \mathbb{R}$ that vanishes at infinity. Observe that $\left\|u^\sharp\right\|_{L^2} = \|u\|_{L^2}$. Then, assuming that $u \in H^2$, we have [7]

$$\left\|\Delta u^\sharp\right\|_{L^2} \leqslant \|\Delta u\|_{L^2}\,,\ \left\|\nabla u^\sharp\right\|_{L^2} \leqslant \|\nabla u\|_{L^2}\ \textit{and}\ \|u\|_{L^{2m}} \leqslant \left\|u^\sharp\right\|_{L^{2m}}, \tag{3.5}$$

for any $m \in \mathbb{N}_0$. Therefore, if u is a minimizer for (1.8), then $u^\sharp$ is a minimizer as well. $\square$

It is an open problem to extend the previous proposition for $2\sigma \notin \mathbb{N}_0$. Observe also that we do not know whether or not all solutions of (1.8) are radially symmetric even if $2\sigma \in \mathbb{N}_0$. Indeed, Boulenger and Lenzmann proved that equality holds in (3.5) if and only if $|\mathcal{F}u| = |\mathcal{F}u|^*$.

3.3. Orbital stability

Finally, using the method of Cazenave and Lions [11], we prove the orbital stability of the set of solutions to (1.8). Let us begin with a definition.

DEFINITION 3.1. Let $\mathcal{G} := \{U \in H^2(\mathbb{R}^N) :\ U\ \textit{is a solution of}\ (1.8)\}$. We say that the set $\mathcal{G}$ is stable in $\mathbb{H}^2 = H^2(\mathbb{R}^N) \times H^2(\mathbb{R}^N)$ if, given $\varepsilon > 0$, there exists $\delta > 0$ such that if $\Psi_0 \in H^2(\mathbb{R}^N)$ satisfies $\|\Psi_0 - U\|_{H^2} < \delta$ for some $U \in \mathcal{G}$, then the solution $\Psi(t)$ of (Mixed 4NLS) with initial data Ψ_0 exists for all $t \geqslant 0$ and satisfies, for some $V \in \mathcal{G}$,

$$d(\Psi(t), V) < \varepsilon,\ \textit{for all}\ t \geqslant 0,$$

where, for any $f, g \in \mathbb{H}^2$, $d(f,g) := \inf\left\{\left\|f(.) - e^{i\theta}g(. - r)\right\|_{\mathbb{H}^2} : \theta, r \in \mathbb{R}\right\}$.

As a direct consequence of Theorem 1.2, we have:

THEOREM 3.1. The set $\mathcal{G}$ is stable.

Proof. Let $U \in \mathcal{G}$. Assume by contradiction that there exists a sequence of solutions $(\Psi_k)_k$ of (Mixed 4NLS) with $\Psi_k(0) = \varphi_k$, for some $(\varphi_k)_k$ such that $\lim_{k\to\infty}\|\varphi_k - U\|_{H^2} = 0$ and such that there exists $(t_k)_k \subset \mathbb{R}^+$ with $d(\Psi_k(t_k), U) \geqslant \varepsilon$, for some $\varepsilon > 0$ fixed. Using the conservation of the energy and the mass, it is easy to see that $\|\Psi_k(t_k)\|_{L^2} \to$

$\|U\|_{L^2}$ and $E_\gamma(\Psi_k(t_k)) \to E_\gamma(U) = I_\gamma$ as $k \to \infty$. Therefore, using Theorem 1.2, we get that $d(\Psi_k(t_k), V) \to 0$, for some $V \in \mathcal{G}$ which gives a contradiction. $\square$

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