Rend. Sem. Mat. Univ. Politec. Torino
Bruxelles-Torino Talks in PDE's –Turin, May 2–5, 2016
Vol. 74, 2 (2016), 81 – 93

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A POINTWISE FINITE-DIMENSIONAL REDUCTION METHOD FOR EINSTEIN-LICHNEROWCIZ TYPE SYSTEMS

Abstract. We explain the construction of non-compactness examples for the fully coupled Einstein-Lichnerowicz system in the focusing case recently obtained in [15]. The construction follows from a combination of pointwise a priori asymptotic analysis techniques with a finite-dimensional reduction and a fixed-point argument on the remainder part of the expected blow-up decomposition.

1. Introduction

1.1. Statement of the results

Let (M,g) be a closed Riemannian manifold of dimension $n \geqslant 6$. We investigate non-compactness issues in strong spaces for the set of positive solutions of the Einstein-Lichnerowicz system in M:

$$
\begin{cases} \triangle_g u + hu = fu^{2^*-1} + \dfrac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{u^{2^*+1}} \\ \overrightarrow{\triangle}_g T = u^{2^*} X + Y. \end{cases} \tag{1.1}
$$

The unknowns of (1.1) are u, a smooth positive function in M, and T, a smooth field of 1-forms in M. In (1.1) $\mathcal{L}_g T$ is the conformal Killing derivative of T and we have let, for any 1-form T: $\overrightarrow{\triangle}_g T = -\mathrm{div}_g\big(\mathcal{L}_g T\big)$. Also, in (1.1), $\triangle_g = -\mathrm{div}_g(\nabla\cdot)$ is the Laplace-Beltrami operator, h, f, π are smooth functions in M, σ is a smooth field of 2-forms with $\mathrm{tr}_g\sigma = 0$ and $\mathrm{div}_g\sigma = 0$ and X and Y are smooth fields of 1-forms in M. The exponent $2^* = \frac{2n}{n-2}$ is critical for the embedding of the Sobolev space $H^1(M)$ into Lebesgue spaces. We also assume that

$$
f > 0 \text{ in } M \quad \text{(focusing case)}, \tag{1.2}
$$

and that $\triangle_g + h$ is coercive (which is necessary in view of (1.2)). System (1.1) arises in the initial-value problem in Mathematical General Relativity as a conformal formulation of the constraint equations (see [2]). Assumption (1.2) covers the case of non-trivial non-gravitational physics data. Existence and multiplicity results for (1.1) in the focusing case (1.2) are in [9, 13, 14].

We are interested here in the stability features of system (1.1). Following [3] (see also [8]), we say that system (1.1) is *stable* if, for any sequence $(h_k, f_k, \pi_k, \sigma_k, X_k, Y_k)_k$ of coefficients converging towards $(h, f, \pi, \sigma, X, Y)$ as $k \to +\infty$ in some strong topology (to be precised), and for any sequence $(u_k, T_k)_k$ of solutions of

$$\begin{cases} \triangle_g u_k + h_k u_k = f_k {u_k}^{2^*-1} + \dfrac{|\mathcal{L}_g T_k + \sigma_k|_g^2 + \pi_k^2}{u_k^{2^*+1}} \\ \overrightarrow{\triangle}_g T_k = u_k^{2^*} X_k + Y_k, \end{cases} \tag{1.3}$$

with $u_k > 0$, there holds, up to a subsequence and up to elements in the kernel of $\mathcal{L}_g$, that $(u_k, T_k)_k$ converges to some positive solution (u_0, T_0) of (1.1) in $C^{1,\eta}(M)$ for all $0 < \eta < 1$. The *compactness* of (1.1) is defined analogously, for constant sequences of coefficients $(h_k, f_k, \pi_k, \sigma_k, X_k, Y_k)_k$. In the focusing case (1.2) stability results were first obtained in [4, 10, 14] for the decoupled system (when $X \equiv 0$). For the fully coupled case $X \not\equiv 0$, the stability of (1.1) has been investigated in [6] and [16] on locally conformally flat manifolds. In particular, system (1.1) is always stable in dimensions $3 \leqslant n \leqslant 5$ provided $\pi \not\equiv 0$. For higher dimensions, the picture is more nuanced: instability results can occur. In [17] a first *in*stability result for the physical case of (1.1) with $X \equiv 0$ was obtained. The instability behavior for the fully coupled case $X \not\equiv 0$ of system (1.1) was later addressed in [15], where the following non-compactness result was obtained.

THEOREM 1.1 (P., [15]). Let (M, g) be a closed Riemannian manifold of dimension $n \geqslant 6$ of positive Yamabe type and possessing no non-trivial conformal Killing fields. There exist regular coefficients $(h, f, \pi, \sigma, X, Y)$, with $\triangle_g + h$ coercive, $f > 0$, $\pi \not\equiv 0$ and $X \not\equiv 0$ such that the associated system of equations (1.1)

$$\begin{cases} \triangle_g u + hu = f u^{2^*-1} + \dfrac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{u^{2^*+1}} \\ \overrightarrow{\triangle}_g T = u^{2^*} X + Y \end{cases}$$

possesses a blowing-up sequence of solutions $(u_k, T_k)_k$, that is $\|u_k\|_{L^\infty(M)} \to +\infty$ and $\|\mathcal{L}_g T_k\|_{L^\infty(M)} \to +\infty$ as $k \to +\infty$. Also, the u_k are positive, possess a single blow-up point and blow-up with a non-zero limit profile.

A manifold (M, g) is said to be of positive Yamabe type if the operator $\triangle_g + \frac{n-2}{4(n-1)} S_g$ is coercive, where S_g is the scalar curvature of g. The assumption that (M, g) possesses no non-trivial conformal Killing fields is generic and implies that $\overrightarrow{\triangle}_g$ has no kernel. A striking consequence of Theorem 1.1 is the existence of an infinite number of solutions of (1.1), see [15]. This article is devoted to a presentation of the ideas of the proof of Theorem 1.1.

1.2. Strategy of the proof of Theorem 1.1.

In the fully coupled case $X \not\equiv 0$ treated here, because of the strong nonlinear coupling via the $(|\mathcal{L}_g T + \sigma|_g^2 + \pi^2) u^{-2^*-1}$ term, (1.1) does not possess a variational structure in $H^1(M)$. The only known existence results for (1.1) are therefore based on fixed-point methods in strong spaces. This is a serious obstacle to the application of the usual Lyapounov-Schmidt construction scheme (see [1, 20, 21] and the references therein) which proved to be a valuable tool in constructing instability examples for critical elliptic equations on manifolds ([7, 19, 18]). To prove Theorem 1.1 we therefore work in strong topologies. We construct a blowing-up sequence of solutions $(u_k, T_k)_k$ of (1.1) whose scalar component writes as

$$u_k = W_{k,t,p} + u + \varphi_{k,t,p}, \tag{1.4}$$

where $W_{k,t,p}$ denotes a positive bubbling profile depending on $(n+1)$ parameters (t, p) and u is a positive strictly stable function. But this time $\varphi_{k,t,p}$ is a *globally pointwise* small remainder, precisely

$$|\varphi_{k,t,p}| \leqslant \varepsilon_k \big(W_{k,t,p} + u\big) \quad \text{pointwise in } M, \tag{1.5}$$

for some $(\varepsilon_k)_k$, $\varepsilon_k \to 0$ as $k \to \infty$. The motivation for the choice of (1.4) comes from the a priori blow-up analysis of (1.1) performed in [6, 16] which shows that (1.5) holds, at least at a local scale, for blowing-up solutions of (1.1). See also [5] where a global control as (1.5) was proven to hold for sequences of solutions of critical stationary Schrödinger equations.

Since (1.1) is not variational, there is no canonical choice of a remainder $\varphi_{k,t,p}$ anymore. We construct it through an involved Banach-Picard fixed-point argument which goes through several steps: a semi-decoupling of (1.1) followed by a finite-dimensional reduction (Section 3), an accurate a priori pointwise description in strong spaces of the remainder constructed (Section 4), a Banach-Picard fixed-point argument in strong spaces for the remainders' mapping (Section 5) and a uniform expansion of the kernel coefficients (Section 6).

2. Setting of the problem and notations

In this article we only sketch the $n \geqslant 7$ case and refer to [15] for the six-dimensional one. Let $(\tau_k)_k$ be a sequence of positive real numbers such that $\sum_k \tau_k < +\infty$. We define a sequence $(\mu_k)_k$ as follows:

$$\mu_k = \begin{cases} \tau_k^{\frac{2}{n-6}} & \text{if } (M,g) \text{ is locally conformally flat or if } 7 \leqslant n \leqslant 10, \\ \tau_k^{\frac{1}{2}} & \text{if } n \geqslant 11 \text{ and } (M,g) \text{ is not locally conformally flat.} \end{cases} \tag{2.1}$$

Let $(\xi_k)_k$ be a sequence of points of M converging towards a given $\xi_0 \in M$ and satisfying $d_g(\xi_k, \xi_{k+1}) << \frac{1}{k^2}$ as $k \to +\infty$. Let $(\beta_k)_k$ be a sequence of positive numbers converging to zero as $k \to +\infty$ and satisfying

$$\beta_k >> \mu_k. \tag{2.2}$$

Let f be a smooth positive function, let σ be a smooth traceless and divergence-free $(2,0)$-tensor in M and let π be a smooth function in M with $\pi \not\equiv 0$. Let Y be a smooth field of 1-forms and denote by $\tilde{Y}$ the only solution of $\overrightarrow{\triangle}_g \tilde{Y} = Y$ in M. We let also H be a smooth nonnegative function in $\mathbb{R}^n$, compactly supported in $B_0(1)$ with $H(0) = 1$, and for which 0 is a *non-degenerate critical point*. We define

$$h = \frac{n-2}{4(n-1)} S_g + \sum_{k \geqslant 0} \tau_k H \left(\frac{1}{\beta_k} (\exp_{\xi_k}^{g_{\xi_k}})^{-1}(x) \right). \tag{2.3}$$

Here $g_\xi = \Lambda_\xi^{\frac{4}{n-2}}$ is a conformal modification of the original metric g. The factor Λ_ξ is chosen in light of the conformal normal coordinates result of [12] to achieve the highest precision in the expansion of the volume element of g_ξ around ξ, see [15]. Note, with (2.1) and (2.2), that for any $r \in \mathbb{N}^*$, one can always choose β_k as in (2.2) so that $h \in C^r(M)$. Let u_0 be a smooth, positive, strictly stable solution of the following Einstein-Lichnerowicz equation:

$$\triangle_g u_0 + h u_0 = f u_0^{2^*-1} + \frac{|\mathcal{L}_g \tilde{Y} + \sigma|_g^2 + \pi^2}{u_0{}^{2^*+1}}. \tag{2.4}$$

The coefficients f, π, σ and Y can always be chosen so that such a u_0 exists, see [14]. For every $n \geqslant 7$ the implicit function theorem shows that there exists a constant $\eta_0 = \eta_0(n, g, h, f, \pi, \sigma, Y)$ such that, for any X satisfying

$$\|X\|_{L_\infty(M)} = \eta \leqslant \eta_0, \tag{2.5}$$

the Einstein-Lichnerowicz system of equations

$$\begin{cases} \triangle_g u + h u = f u^{2^*-1} + \dfrac{|\mathcal{L}_g T + \sigma|^2 + \pi^2}{u^{2^*+1}} \\ \overrightarrow{\triangle}_g T = u^{2^*} X + Y, \end{cases} \tag{2.6}$$

with h given by (2.3), possesses a solution $(u(X), T(X))$ such that $u(X) \to u_0$ in $C^2(M)$ as η, defined in (2.5), goes to 0. Up to choosing η small enough, it is easily seen that $u(X)$ is again a strictly stable solution of the scalar equation of (2.6). In the following, for a given X, the solution $(u(X), T(X))$ will just be denoted by (u, T).

We endow $H^1(M)$ with the following scalar product

$$\langle u, v \rangle_h = \int_M (\langle \nabla u, \nabla v \rangle_g + h u v)\, dv_g, \quad \text{for any } u, v \in H^1(M), \tag{2.7}$$

where h is given by (2.3). Let $(r_k)_k$, $r_k > 0$, $r_k \to 0$ as $k \to \infty$, satisfying

$$\beta_k << r_k << d_g(\xi_k, \xi_{k+1}) \quad \text{and} \quad r_k^N >> \mu_k, \tag{2.8}$$

where β_k is given by (2.2), for some large enough integer N, as $k \to +\infty$. For $t > 0$ we define: $\delta_k(t) = \mu_k t$, where μ_k is as in (2.1). The defects of compactness investigated here are the following ones:

$$
(2.9) \qquad W_{k,t,\xi}(x) = \Lambda_\xi(x)\chi\left(\frac{d_{g_\xi}(\xi,x)}{r_k}\right)\delta_k^{\frac{n-2}{2}}\left(\delta_k^2 + \frac{f(\xi)}{n(n-2)}d_{g_\xi}(\xi,x)^2\right)^{1-\frac{n}{2}},
$$

where $\chi \in C^\infty(\mathbb{R})$ is a nonnegative, smooth compactly supported function in $[-2,2]$. We also define, for any $x \in M$, any $1 \leqslant i \leqslant n$ and any $\xi \in M$

$$
\begin{aligned}
Z_{0,k,t,\xi}(x) &= \Lambda_\xi(x)\chi\left(\frac{d_{g_\xi}(\xi,x)}{r_k}\right)\delta_k^{\frac{n-2}{2}}\left(\delta_k^2 + \frac{f(\xi)}{n(n-2)}d_{g_\xi}(\xi,x)^2\right)^{-\frac{n}{2}} \\
&\quad \times \left(\frac{f(\xi)}{n(n-2)}d_{g_\xi}(\xi,x)^2 - \delta_k^2\right) \\
Z_{i,k,t,\xi}(x) &= \Lambda_\xi(x)\chi\left(\frac{d_{g_\xi}(\xi,x)}{r_k}\right)\delta_k^{\frac{n}{2}}\left(\delta_k^2 + \frac{f(\xi)}{n(n-2)}d_{g_\xi}(\xi,x)^2\right)^{-\frac{n}{2}} \\
&\quad \times f(\xi)\left\langle \left(\exp_\xi^{g_\xi}\right)^{-1}(x), e_i(\xi)\right\rangle_{g_\xi(\xi)},
\end{aligned}
$$

where the $(e_i)_i$ are a local orthonormal basis for g_ξ around ξ_0. Finally, we let

$$
(2.10) \qquad K_{k,t,\xi} = \mathrm{Span}\left\{Z_{i,k,t,\xi}, i = 0\ldots n\right\}.
$$

Then $K_{k,t,\xi}$ is $(n+1)$-dimensional for k large enough and the $Z_{i,k,t,\xi}$ are "almost" orthogonal. We denote by $K_{k,t,\xi}^\perp$ its orthogonal in $H^1(M)$ for the scalar product given by (2.7).

We now define f and X. Let $f_0 > 0$ be a positive constant and define

$$
(2.11) \qquad f = f_0 + \sum_{k \geqslant 0} s_k \chi\left(\frac{1}{r_k}\left(\exp_{\xi_k}^{g_{\xi_k}}\right)^{-1}(x)\right),
$$

where $(s_k)_k$ satisfies $|s_k| = O(\mu_k^N)$ for a sufficiently large $N \in \mathbb{N}^*$. Let X_0 denote any smooth field of 1-forms in M which vanishes in a neighbourhood of ξ_0. Let Z be a fixed smooth 1-form in $\mathbb{R}^n$, compactly supported in $B_0(1)$, and with $|Z_0(0)| > 0$. Define then, for any $x \in M$

$$
(2.12) \qquad X(x) = X_0(x) + \sum_{k \geqslant 0} \mu_k^{\frac{n-1}{2}} Z\left(\frac{1}{r_k}\left(\exp_{\xi_k}^{g_{\xi_k}}\right)^{-1}(x)\right),
$$

where μ_k and r_k are as in (2.1) and (2.8). Up to reducing $\|X_0\|_\infty$ and the τ_k such an X always satisfies (2.5). Again, with (2.1) and (2.8), f and X can always be chosen to belong to $C^r(M)$ for $r \in \mathbb{N}^*$. Finally, let

$$
(2.13) \qquad \mathcal{E} = \left\{(\varepsilon_k)_{k\in\mathbb{N}}, \varepsilon_k > 0, \lim_{k\to\infty} \varepsilon_k = 0\right\}
$$

be the set of sequences of positive real numbers converging to 0. For $(\varepsilon_k)_k \in \mathcal{E}$ and for a given value of $(t,\xi) \in (0,+\infty) \times M$ we define the following sequence of subsets of $C^2(M)$:

$$F_k = F\left(\varepsilon_k, t, \xi\right) = \left\{ v \in C^0(M) \text{ such that } \left\| \frac{v}{u + W_{k,t,\xi}} \right\|_{C^0(M)} \leqslant \varepsilon_k \right\}, \tag{2.14}$$

where $u = u(X)$ is defined after (2.6) and $W_{k,t,\xi}$ is as in (2.9).

3. Semi-decoupling and H^1 reduction.

Let $(\varepsilon_k)_k \in \mathcal{E}$, $(t,\xi) \in (0,+\infty) \times M$ and $v_k \in F_k = F\left(\varepsilon_k, t, \xi\right)$, where $\mathcal{E}$ and F_k are defined in (2.13) and (2.14). Since $\overrightarrow{\triangle}_g$ has no kernel by assumption, there exists a unique 1-form $T_{k,t,\xi}$ in M satisfying

$$\overrightarrow{\triangle}_g T_{k,t,\xi} = \left(u + W_{k,t,\xi} + v_k\right)^{2^*} X + Y. \tag{3.1}$$

Pointwise bounds on $\mathcal{L}_g T_{k,t,\xi}$ follow from the assumption $v_k \in F_k$, see [15]. It turns out that $\mathcal{L}_g T_{k,t,\xi}$ blows up too fast for an H^1 finite-dimensional reduction to apply to the scalar equation of (1.1) with $\mathcal{L}_g T_{k,t,\xi}$ seen as a coefficient: even the very first step (the uniform inversion of the linearized operator) fails. We therefore artificially discard the $|\mathcal{L}_g T_{k,t,\xi} + \sigma|_g^2$ term into a source term and consider instead the equation

$$\triangle_g w + hw = f w^{2^*-1} + \frac{|\mathcal{L}_g T + \sigma|^2 + \pi^2}{\rho(w)^{2^*+1}} + \frac{|\mathcal{L}_g T_{k,t,\xi} + \sigma|_g^2 - |\mathcal{L}_g T + \sigma|_g^2}{\left(u + W_{k,t_k,\xi_k} + v_k\right)^{2^*+1}}, \tag{3.2}$$

where T satisfies $\overrightarrow{\triangle}_g T = u^{2^*} X + Y$ and where we have let $\rho = \rho_{\varepsilon_0}$ for some $\varepsilon_0 > 0$, where

$$\rho_\varepsilon(r) = \begin{cases} \varepsilon \text{ if } r < \varepsilon \\ r \text{ if } r \geqslant \varepsilon. \end{cases}$$

The first step of the proof of Theorem 1.1 is as follows.

Proposition 3.1. Let $D > 0$ and $(\varepsilon_k)_k \in \mathcal{E}$ and assume that $\varepsilon_k >> \mu_k^{\frac{3}{2}}$ as $k \to +\infty$, where μ_k is as in (2.1). Let $(t_k, \xi_k)_k$ be a sequence in $[1/D, D] \times M$ and, for any k, let $v_k \in F_k = F\left(\varepsilon_k, t_k, \xi_k\right)$. For k large enough, there exists a function $\phi_k = \phi_k\left(t_k, \xi_k, v_k\right) \in K_{k,t_k,\xi_k}^{\perp}$ that satisfies

(3.3)
$$\Pi_{K_{k,t_k,\xi_k}^{\perp}} \Bigg\{ u + W_{k,t_k,\xi_k} + \phi_k - (\triangle_g + h)^{-1} \Bigg(f\left(u + W_{k,t_k,\xi_k} + \phi_k\right)^{2^*-1} + \frac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{\rho\left(u + W_{k,t_k,\xi_k} + \phi_k\right)^{2^*+1}} \Bigg) - (\triangle_g + h)^{-1} \left(\frac{|\mathcal{L}_g T_{k,t,\xi} + \sigma|_g^2 - |\mathcal{L}_g T + \sigma|_g^2}{\left(u + W_{k,t_k,\xi_k} + v_k\right)^{2^*+1}} \right) \Bigg\} = 0.$$

This ϕ_k is the unique solution of (3.3) in $K^{\perp}_{k,t_k,\xi_k} \cap B_{H^1(M)}(0, C\eta\varepsilon_k)$, where C is independent of k, of the choice of $(t_k,\xi_k)_k$ and of η as in (2.5). Also, in (3.3), K_{k,t_k,ξ_k} is as in (2.10) and $T_{k,t,\xi}$ is as in (3.1).

As an obvious consequence of Proposition 3.1, the function ϕ_k constructed therein satisfies

$$\|\phi_k\|_{H^1(M)} \leqslant C\eta\varepsilon_k, \tag{3.4}$$

for some constant C which is independent of $(t_k,\xi_k)_k$, k and η. In (3.3), the truncation ρ is a technical shortcut required to handle the negative nonlinearity in $H^1(M)$. Lemma 2.1 in [17] shows however that ρ has no influence on the construction process provided ε_0 is small enough. Proposition 3.1 is proven by a Banach-Picard fixed-point method, and crucially relies on the pointwise estimates on $\mathcal{L}_g T_{k,t,\xi}$ directly induced from the a priori *pointwise* control (1.5) on v.

4. Asymptotic pointwise description of the remainder ϕ_k

4.1. Rough pointwise control

In view of an application of a fixed-point argument to the remainders mapping $v_k \mapsto \phi_k$ in F_k defined in (2.14) we need to choose $(\varepsilon_k)_k$ so that the remainder ϕ_k given by Proposition 3.1 belongs to F_k. Proving this is the core of the analysis of [15]. This is far from being obvious: first ϕ_k only comes with an H^1 bound by essence. Then, the criticality of (1.1) does not allow a simple bootstrap procedure to increase regularity. And finally, ϕ_k is only a solution up to some kernel elements. Precisely, letting $u_{k,t_k,\xi_k,v_k} = u + W_{k,t_k,\xi_k} + \phi_k(t_k,\xi_k,v_k)$, there holds

$$\begin{aligned}(\triangle_g + h)u_{k,t_k,\xi_k,v_k} - f u^{2^*-1}_{k,t_k,\xi_k,v_k} - \frac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{u^{2^*+1}_{k,t_k,\xi_k,v_k}} - \frac{|\mathcal{L}_g T_k + \sigma|_g^2 - |\mathcal{L}_g T + \sigma|_g^2}{\left(u + W_{k,t_k,\xi_k} + v_k\right)^{2^*+1}} \\ = \sum_{i=0}^{n} \lambda_k^i(t_k,\xi_k,v_k)\,(\triangle_g + h) Z_{i,k,t,\xi}\end{aligned} \tag{4.1}$$

for some numbers $(\lambda_k^i(t_k,\xi_k,v_k))_{0\leqslant i\leqslant n}$, where T_{k,t_k,ξ_k} is as in (3.1) and $Z_{i,k,t,\xi}$ as in (2.10). The first step towards a pointwise control on ϕ_k consists in showing that ϕ_k is globally small (in $C^0(M)$) with respect to $W_{k,t,\xi} + u$.

PROPOSITION 4.1. Let $D > 0$ and $(\varepsilon_k)_k \in \mathcal{E}$ and assume that $\varepsilon_k >> \mu_k^{\frac{3}{2}}$ as $k \to +\infty$, where μ_k is as in (2.1). Let $(t_k,\xi_k)_k$ be a sequence of points in $[1/D, D] \times M$, and let $v_k \in F_k = F\left(\varepsilon_k, t_k, \xi_k\right)$. There exists a sequence $(\nu_k)_k$ of positive numbers that goes to zero as $k \to +\infty$ such that

$$|\phi_k(x)| \leqslant \nu_k \Big(u(x) + W_{k,t_k,\xi_k}(x)\Big) \qquad \text{for any } x \in M. \tag{4.2}$$

In (4.2) we have let $\phi_k = \phi_k\big(t_k, \xi_k, v_k\big) \in K_{k,t_k,\xi_k}$ be the solution of (3.3) given by Proposition 3.1.

In the course of the proof the first thing one has to obtain is a control of the $|\lambda_k^i(t_k, \xi_k, v_k)|$ and a lower bound on ϕ_k so as to get rid of the truncation ρ. Then the proof of Proposition 4.1 consists in an adaptation of the methods developed in [5] (see also [8]) to take into account the source term in (3.2). One first obtains a global weak pointwise estimate together with a local rescaled convergence, later refined into a global uniform control by means of successive approximations. The methods of [5] a priori do not apply to nonlinear equations with a source term, but we manage to adapt them here because the source term is *pointwise* controlled.

4.2. Second-order estimates

The main challenge in the proof of Theorem 1.1 is to quantify precisely ν_k in (4.2). The first step towards this is to obtain a local improvement of (4.2) in the region where the bubbling profile is dominant.

PROPOSITION 4.2. Let $D > 0$, $(\varepsilon_k)_k \in \mathcal{E}$ and assume that $\varepsilon_k >> \mu_k^{\frac{3}{2}}$ as $k \to +\infty$, where μ_k is as in (2.1). Let $(t_k, \xi_k)_k$ be a sequence in $[1/D, D] \times M$, let $v_k \in F_k = F\big(\varepsilon_k, t_k, \xi_k\big)$ and let $\phi_k = \phi_k\big(t_k, \xi_k, v_k\big)$ be given by Proposition 3.1. Let $(x_k)_k$ be any sequence of points in $B_{\xi_k}(2\sqrt{\delta_k})$. There holds

(4.3)
$$\theta_k(x_k)\,|\nabla\phi_k(x_k)| + |\phi_k(x_k)| \lesssim \|\phi_k\|_{L^\infty(\Omega_k)} + \sqrt{\delta_k}\|\nabla\phi_k\|_{L^\infty(\Omega_k)} + \delta_k + \Bigg[\delta_k^{\frac{n}{2}} + \delta_k\|\nabla f\|_{L^\infty(2r_k)}$$
$$+ \|h - c_nS_g\|_{L^\infty(2r_k)}\delta_k^2\left|\ln\left(\frac{\theta_k(x_k)}{\delta_k}\right)\right| + \|h - c_nS_g\|_{L^\infty(2r_k)}\theta_k(x_k)^2 + \theta_k(x_k)^4\mathbb{1}_{nlcf}\Bigg]W_k(x_k)$$
$$+ \left(\frac{\delta_k}{\theta_k(x_k)}\right)^2,$$

where we have let: $\Omega_k = B_{\xi_k}(2r_k)\backslash B_{\xi_k}(\sqrt{\delta_k})$ and $\theta_k(x_k) = \delta_k + d_{g_{\xi_k}}(\xi_k, x_k)$.

Here the notation "$\lesssim$" stands for "$\cdot \leqslant C\cdot$" for a positive constant C independent of k. Estimate (4.3) is obtained by writing a representation formula for ϕ_k (with (4.1)) and estimating precisely every term which appears. Of course, many nonlinear terms to be estimated do depend on ϕ_k: we therefore first obtain a control of $\|\phi_k\|_{L^\infty(B_{\xi_k}(2r_k))}$ that we later iteratively improve into (4.3). The control is the following.

CLAIM 4. There holds

$$\|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))} \lesssim \max(1, M_k), \tag{4.4}$$

where we have let

$$M_k = \|\phi_k\|_{L^\infty(\Omega_k)} + \sqrt{\delta_k}\|\nabla\phi_k\|_{L^\infty(\Omega_k)} + \delta_k + \delta_k^{2-\frac{n}{2}}\|\nabla f\|_{L^\infty(2r_k)} + \delta_k^{3-\frac{n}{2}}\|h - c_n S_g\|_{L^\infty(2r_k)} + \delta_k^{5-\frac{n}{2}}\mathbb{1}_{nlcf}.$$

To prove (4.4) we go through an involved contradiction argument: assuming that (4.4) does not hold, we localize the maximum point of ϕ_k in M and show that a limiting equation for some suitable rescaling of ϕ_k – denoted $\tilde{\phi}_k$ – can be obtained. The limiting equation ensures that the limit of $\tilde{\phi}_k$ lies in the kernel for the linearized equation of the standard bubble equation in $\mathbb{R}^n$. The contradiction then follows from the nature of ϕ_k constructed in Proposition 3.1, which is by construction almost orthogonal to some approximate rescalings of these kernel elements. Of course these two assertions come at different heights, and the main challenge is to be sure that they can be related after rescaling. In the course of the proof of Proposition 4.2, to iteratively improve the estimates we also derive the following control on the λ_k^i:

$$\sum_{i=0}^{n}|\lambda_k^i| \lesssim \delta_k^{\frac{n-2}{2}}\Big(\|\phi_k\|_{L^\infty(\Omega_k)} + \sqrt{\delta_k}\|\nabla\phi_k\|_{L^\infty(\Omega_k)}\Big) + \delta_k^{\frac{n-2}{2}} + \delta_k\|\nabla f\|_{L^\infty(2r_k)} + \|h - c_n S_g\|_{L^\infty(2r_k)}\delta_k^2 + \delta_k^4\mathbb{1}_{nlcf}.$$

As a second step to quantify ν_k in (4.2) we derive *global* estimates for ϕ_k over M.

PROPOSITION 4.3. Let $D > 0$ and $(\varepsilon_k)_k \in \mathcal{E}$ and assume that $\varepsilon_k >> \mu_k^{\frac{3}{2}}$ as $k \to +\infty$, where μ_k is as in (2.1). Let $(t_k, \xi_k)_k$ be a sequence in $[1/D, D] \times M$, let $v_k \in F_k = F\big(\varepsilon_k, t_k, \xi_k\big)$ and let $\phi_k = \phi_k\big(t_k, \xi_k, v_k\big)$ be given by Proposition 3.1. Let $(x_k)_k$ be any sequence of points in M. There holds

$$|\phi_k(x_k)| \leqslant C(\delta_k + \eta\varepsilon_k)\Big(u(x_k) + W_k(x_k)\Big), \tag{4.5}$$

where η is as in (2.5), for some positive constant C independent of η and k.

The proof of Proposition 4.3 goes again through a global representation formula for ϕ_k. The terms to be estimated are integrals involving again ϕ_k. The contributions of these integrals in the region where $W_{k,t,\xi}$ is dominant – the most problematic – are handled thanks to Proposition 4.2. One of the main subtleties of the proof of Theorem 1.1 is to obtain estimates – as in Propositions 4.2 or 4.3 – which are uniform in the choice of $(\varepsilon_k)_k, (t_k)_k, (\xi_k)_k$ and $(v_k)_k$. Note that the statement of Proposition 4.2 is much more precise than what is required in the proof of Proposition 4.3. But this high precision will turn out to be crucial in section 6 to obtain precise asymptotic expansions of the $\lambda_k^i(t,\xi)$. The a priori analysis techniques used in our proof have been developed in the context of the C^0 theory in [5] Related techniques have independently been developed in the investigation of compactness phenomena for the Yamabe problem (see [11] and the references therein).

5. Global fixed-point argument and resolution of the reduced problem

In this section we explain how a solution of the reduced problem for (1.1) is obtained. By this we mean a function $\varphi_k(t,\xi)$ such that $(u_{k,t,\xi}, W_{k,t,\xi})_k$, with $u_{k,t,\xi} = W_{k,t,\xi} + u + \varphi_k(t,\xi)$, which solves

$$(5.1)\qquad \begin{cases} \triangle_g u_{k,t,\xi} + h u_{k,t,\xi} = f u_{k,t,\xi}^{2^*-1} + \dfrac{\pi^2 + |\sigma + \mathcal{L}_g W_{k,t,\xi}|_g^2}{u_{k,t,\xi}^{2^*+1}} + \displaystyle\sum_{j=0}^{n} \lambda_{k,j}(t,\xi) Z_{j,k,t,\xi}\,, \\ \overrightarrow{\triangle}_g W_{k,t,\xi} = u_{k,t,\xi}^{2^*} X + Y, \end{cases}$$

where the $Z_{j,k,t,\xi}$ are defined in (2.10). This amounts to showing that $v_k \mapsto \phi_k$ has a fixed-point in F_k (defined in (2.14)). With Proposition 4.3 we already see that, provided ε_k is suitably chosen and η is small enough, F_k is a stable set for $v_k \mapsto \phi_k$ for any k. Standard elliptic theory together with a Schauder fixed-point theorem would yield, for any k, a solution $\varphi_k(t,\xi)$ of (5.1). However, we need more than this. Schauder's fixed-point theorem comes with no uniqueness statement about the fixed-point it constructs and therefore does not allow to show that such a fixed-point continuously depends in strong spaces in (t,ξ). We therefore apply Banach-Picard's fixed-point theorem to $v_k \mapsto \phi_k$ in F_k.

PROPOSITION 5.1. Let $D > 0$. Assume that η defined in (2.5) is small enough. There exists $k_0 \in \mathbb{N}$ such that for any sequence $(t_k,\xi_k)_k \in [1/D, D] \times M$ and for any $k \geqslant k_0$, there exists a function $\varphi_k = \varphi_k\big(t_k,\xi_k\big) \in K^{\perp}_{k,t_k,\xi_k}$ that satisfies the following system of equations

$$(5.2)\qquad \begin{cases} \Pi_{K^{\perp}_{k,t_k,\xi_k}}\left[u_k - (\triangle_g + h)^{-1}\left(f u_k^{2^*-1} + \dfrac{|\mathcal{L}_g T_k + \sigma|_g^2 + \pi^2}{u_k^{2^*+1}}\right)\right] = 0, \\ \overrightarrow{\triangle}_g T_k = u_k^{2^*} X + Y, \end{cases}$$

where we have let $u_k = u + W_{k,t_k,\xi_k} + \varphi_k(t_k,\xi_k)$. Also, for any k, the mapping $(t,\xi) \mapsto \varphi_k(t,\xi) \in C^1(M)$ is continuous and there exists a positive constant C, independent of $(t_k,\xi_k)_k$ such that there holds

$$(5.3)\qquad \|\varphi_k(t_k,\xi_k)\|_{H^1(M)} \leqslant C\delta_k \text{ and } |\varphi_k(t_k,\xi_k)| \leqslant C\delta_k\Big(u + W_{k,t_k,\xi_k}\Big) \text{ in } M,$$

and such that $\varphi_k(t_k,\xi_k)$ is the unique solution of (5.2) in $K^{\perp}_{k,t_k,\xi_k}$ satisfying in addition (5.3).

Proposition 5.1 shows that the estimates on φ_k only depend on the data μ_k and ε_k.We prove it – and Theorem 1.1 – assuming that the L^∞ norm of the coupling field X is small (depending on n, g, h, f, π, σ) . In view of (4.5) this is required to have a stable set for the remainder's mapping, and this assumption is actually necessary since smallness conditions on X are necessary for solutions of (1.1) to exist: see [9, 13, 14]. To prove Proposition 5.1 we prove that $v_k \mapsto \phi_k$ is $\frac{1}{2}$ - contractible in F_k for the norm given by

(2.14). If ϕ_k^i are associated to v_k^i, $i = 1..2$, we estimate the maximum value of $\frac{\phi_k^1-\phi_k^2}{u+W_{k,t,\xi}}$ directly using again representation formulae for (4.1). If this maximum is achieved at a distance from ξ_k comparable to the parameter of the bubble δ_k, we proceed using similar techniques to those used in the proof of Proposition 4.2. Otherwise, it is the smallness of η in (2.5) that allows to conclude.

6. Expansion of the Kernel coefficients and conclusion

The final step in the proof of Theorem 1.1 consists in finding, for any k, a suitable (t_k, ξ_k) which annihilates the $\lambda_k^i(t_k, \xi_k)$ in (5.1). This is achieved through an asymptotic expansion of the $\lambda_{k,j}(t,\xi)$ in C^0 and a limiting degree argument. In standard cases where only H^1 estimates are involved, the precision of such an expansion only depends on the choice of the approximate solution $u + W_{k,t,\xi}$. Here, however, the lack of a variational structure and the strong nonlinear coupling of (1.1) do not give us a better precision than (5.3) on φ_k – no matter the precision of the *ansatz* $u + W_{k,t,\xi}$ –, which is way too rough. We again overcome this by relying on the asymptotic analysis results obtained in Sections 4 and 5. We write the scalar equation in (5.1) as

$$
\begin{aligned}
&\sum_{i=0}^{n}\lambda_k^i(t,\xi)\left(\triangle_g + h\right) Z_{i,k,t,\xi} = \left(\triangle_g + h\right) W_{k,t,\xi} - f(\xi) W_{k,t,\xi}^{2^*-1} + \left(f(\xi) - f\right) W_{k,t,\xi}^{2^*-1}\\
&- f\left[\left(u + W_{k,t,\xi} + \varphi_k(t,\xi)\right)^{2^*-1} - \left(u + W_{k,t,\xi}\right)^{2^*-1} - (2^*-1)\left(u + W_{k,t,\xi}\right)^{2^*-2}\phi_k(t,\xi)\right]\\
&- f\left[\left(u + W_{k,t,\xi}\right)^{2^*-1} - u^{2^*-1} - W_{k,t,\xi}^{2^*-1}\right] + \frac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{u^{2^*+1}} - \frac{|\mathcal{L}_g T_{k,t,\xi} + \sigma|_g^2 + \pi^2}{\left(u + W_{k,t,\xi} + \varphi_k(t,\varphi)\right)^{2^*+1}}\\
&+ \left(\triangle_g + h\right)\varphi_k(t,\xi) - (2^*-1) f(\xi) W_{k,t,\xi}^{2^*-2}\varphi_k(t,p) + (2^*-1)\left(f(\xi) - f\right) W_{k,t,\xi}^{2^*-2}\varphi_k(t,\xi)\\
&- (2^*-1) f\left[\left(u + W_{k,t,\xi}\right)^{2^*-2} - W_{k,t,\xi}^{2^*-2}\right]\varphi_k(t,\xi),
\end{aligned}
$$

multiply both sides by $Z_{j,k,t,\xi}$, for a given j, and estimate all the integrals in the right-hand side. At this point we also express ξ as $\xi = \exp_{\xi_k}^{g_{\xi_k}}(\beta_k p)$, with β_k as in (2.2) and $p \in B_0(1) \subset \mathbb{R}^n$. These integrals are directly computed using pointwise a priori estimates on φ_k obtained by our blow-up analysis, and the explicit expression of $W_{k,t,\xi}$ and $Z_{j,k,t,\xi}$ in (2.9) and (2.10). Different contributions in M are estimated differently: when the integration domain is the ball $B_{\xi_k}(\sqrt{\delta_k})$ we use Proposition 4.2, at finite distances from ξ_k we use (4.5) while in the intermediate region we prove that for any sequence $(R_k)_k$, $R_k \geqslant 1$ there holds

$$
\|\varphi_k\|_{L^\infty(M \setminus B_{\xi_k}(R_k\sqrt{\delta_k}))} \lesssim \frac{\delta_k}{R_k^2} + R_k^2\delta_k^2 + \delta_k^{\frac{n-2}{2}} r_k^{-n}. \tag{6.1}
$$

If for instance (M,g) is locally conformally flat or $7 \leqslant n \leqslant 10$, direct estimations give in the end
(6.2)

$$\Big(I_{n+1}+O(\delta_k)\Big)\begin{pmatrix}\lambda_k^0(t,p)\\ \vdots \\ \lambda_k^n(t,p)\end{pmatrix} = \begin{pmatrix} \mu_k^{\frac{n-2}{2}}\Bigg[C_1 f(\xi_0)^{-\frac{n}{2}}H(p)t^2 - C_2 f(\xi_0)^{1-\frac{n}{2}}u(\xi_0)t^{\frac{n-2}{2}} \\ \qquad -C_3 f(\xi_0)^{-4}K_{10}^{-10}|W_g(\xi)|_g^2 t^4 \mathbb{1}_{n=10} + R_k^0(t,p)\Bigg] \\ \frac{\mu_k^{\frac{n}{2}}}{\beta_k}\Bigg[C_4 f(\xi_0)^{-\frac{n}{2}}\nabla_i H(p)t^3 + R_k^i(t,p)\Bigg]\end{pmatrix},$$

where, for $0 \leqslant i \leqslant n$, $R_k^i(t,p)$ is a function which converges to zero in $C^0\big([1/D,D]\times\overline{B_0(1)}\big)$ as $k\to+\infty$ and $C_1,\ldots,C_4$ are positive constant only depending on n. With (6.2), we conclude with a degree argument (see [15]), and the remaining cases (when (M,g) is not locally conformally flat and $n\geqslant 11$) are treated in the same way.

Let us point out again that expansion (6.2) is computed by asymptotic analysis techniques and is not obtained *via* $H^1(M)$ estimates. Estimates (4.3) and (6.1) – which are much more precise than (4.5) – comes crucially into play to estimate the $\lambda_k^i(t,p)$ with a sufficiently high precision. The continuity of the remainders R_k^i – necessary for the concluding degree argument – is a direct consequence of the continuity of φ_k in (t,p) in strong spaces, as given by Proposition 5.1.

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AMS Subject Classification: 58J05, 35J20

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Lavoro pervenuto in redazione il 31.12.2016.