Rend. Sem. Mat. Univ. Politec. Torino
Bruxelles-Torino Talks in PDE's –Turin, May 2–5, 2016
Vol. 74, 2 (2016), 61 – 70

M. Nys

PROPERTIES OF GROUND STATES OF NONLINEAR SCHRÖDINGER EQUATIONS UNDER A WEAK CONSTANT MAGNETIC FIELD

Abstract. We study the qualitative properties of ground states of the time-independent magnetic semilinear Schrödinger equation

$$-(\nabla + iA)^2 u + u = |u|^{p-2}u \qquad \text{in } \mathbb{R}^N$$

where the magnetic potential A induces a constant magnetic field. When the latter magnetic field is small enough, we show that the ground state solution is unique up to magnetic translations and rotations in the complex phase space and that ground state solutions share the rotational invariance of the magnetic field. This is based on an article in collaboration with D. Bonheure and J. VanSchaftingen [3].

1. Introduction

We are interested in the *time-independent magnetic semilinear Schrödinger equation*

$$-\Delta_A u + u = |u|^{p-2}u \qquad \text{in } \mathbb{R}^N, N \geqslant 2, \tag{1}$$

with a *linear magnetic potential* $A \in \mathrm{Lin}(\mathbb{R}^N, \bigwedge^1 \mathbb{R}^N)$ which allows to define the *magnetic Laplacian*

$$-\Delta_A := -\Delta - 2iA\cdot\nabla - i\mathrm{div}A + |A|^2,$$

and a subcritical power p in the nonlinearity, i.e. $2 < p < \frac{2N}{N-2}$.

In this work we are interested in the qualitative properties of the *ground states* of (1), which can be obtained and characterized as minimizers of the variational problem

$$\inf\big\{ I_A(u) \ : \ u \in H^1_A(\mathbb{R}^N, \mathbb{C})\backslash\{0\} \text{ and } I'_A(u) = 0\big\}.$$

Here the *magnetic Sobolev space* $H^1_A(\mathbb{R}^N, \mathbb{C})$ is a *real* Hilbert space given by

$$H^1_A(\mathbb{R}^N, \mathbb{C}) := \big\{u \in L^2(\mathbb{R}^N, \mathbb{C}) \ : \ D_A u \in L^2(\mathbb{R}^N)\big\}$$

endowed with the norm

$$\|u\|^2_{H^1_A(\mathbb{R}^N,\mathbb{C})} = \int_{\mathbb{R}^N} |D_A u|^2 + |u|^2,$$

deriving from the *real scalar product*

$$(u|v)_{H^1_A(\mathbb{R}^N,\mathbb{C})} = \int_{\mathbb{R}^N} (D_A u|D_A v) + (u|v),$$

*

where $(\cdot|\cdot)$ denotes the canonical *real scalar product* of vectors in $\mathbb{C}$ and in $\mathrm{Lin}(\mathbb{R}^N,\mathbb{C})$. The *magnetic covariant derivative* is defined by

$$D_A u = Du + iAu,$$

and the functional $I_A : H^1_A(\mathbb{R}^N,\mathbb{C}) \to \mathbb{R}$ is defined for each function $u \in H^1_A(\mathbb{R}^N,\mathbb{C})$ by

$$I_A(u) := \frac{1}{2}\int_{\mathbb{R}^N} \big(|D_A u|^2 + |u|^2\big) - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p.$$

Critical points of the functional I_A correspond to weak solutions of the equation (1), that is solutions $u \in H^1_A(\mathbb{R}^N,\mathbb{C})$ such that for every $v \in H^1_A(\mathbb{R}^N,\mathbb{C})$,

$$\int_{\mathbb{R}^N} (D_A u|D_A v) + (u|v) = \int_{\mathbb{R}^N} |u|^{p-2}(u|v).$$

The aim of the present work is to understand the properties of the ground states of equation (1) and their dependence on the magnetic field $B = dA \in \bigwedge^2 \mathbb{R}^N$. In order to alleviate the statement of the results, we simply the problem by gauge fixing.

The *gauge invariance* means that if for some function $\psi \in C^1(\mathbb{R}^N)$, we set

$$\tilde{A} = A + d\psi \quad\text{and}\quad \tilde{u} = \mathrm{e}^{-i\psi}u, \tag{2}$$

then

$$D_{\tilde{A}}\tilde{u} = \mathrm{e}^{-i\psi}D_A u.$$

In particular, if $u \in H^1_A(\mathbb{R}^N,\mathbb{C})$, then $I_{\tilde{A}}(\tilde{u}) = I_A(u)$ and therefore solutions of equation (1) with A and $\tilde{A}$ can be related to each other using the relation (2). Since $d\tilde{A} = dA$ and $|\tilde{u}| = |u|$, the gauge invariance means that only the *magnetic field* dA plays a role in the physical behavior of the solutions of (1). When, as in the present work, the magnetic field dA is constant, one of the simplest gauge choice is to assume that A is linear and skew-symmetric. If A is linear, such a choice can be made by setting $\psi(x) = -A(x)[x]/2$ in (2). This choice is equivalent to the choice of the Coulomb gauge with a transversal boundary condition at infinity i.e.,

$$\begin{cases} \mathrm{div} A = 0 & \text{in } \mathbb{R}^N,\\ \dfrac{A(x)[x]}{|x|^2} \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{3}$$

In particular, if $B \in \bigwedge^2 \mathbb{R}^N$ is a constant skew-symmetric form, there exists a unique $A \in \mathrm{Lin}(\mathbb{R}^N, \bigwedge^1 \mathbb{R}^N)$ satisfying $dA = B$ and (3). As $I_{\tilde{A}}(\tilde{u}) = I_A(u)$ when (2) holds, the precise choice (3) allows to define the *ground-energy function* $\mathcal{E} : \bigwedge^2 \mathbb{R}^N \to \mathbb{R}$ by

$$\mathcal{E}(B) = \mathcal{E}(dA) := \inf\big\{ I_A(v) \;:\; v \in H^1_A(\mathbb{R}^N,\mathbb{C})\setminus\{0\} \text{ and } I'_A(v) = 0\big\}.$$

The function $u \in H^1_A(\mathbb{R}^N,\mathbb{C})$ is a *ground state* of (1) if u is a weak solution of (1) such that

$$I_A(u) = \mathcal{E}(dA).$$

Because of the presence of the magnetic potential, equation (1) is not invariant under translations in $\mathbb{R}^N$. However, it is still invariant under *magnetic translations with respect to the connection D_A*. For $a \in \mathbb{R}^N$ and $u \in H^1_A(\mathbb{R}^N, \mathbb{C})$, that magnetic translation is defined by

$$\tau^A_a u(x) = \mathrm{e}^{-iA(a)[x]} u(x-a).$$

This definition depends on the gauge fixing made above. This magnetic translation commutes with the covariant derivative D_A, i.e., $D_A \circ \tau^A_a = \tau^A_a \circ D_A$. We observe that in general, magnetic translations do not commute. Indeed, one has

$$\tau^A_b \circ \tau^A_a = \mathrm{e}^{iA(a)[b]} \tau^A_{a+b}. \tag{4}$$

Our first result establishes that the ground state of (1) is unique up to magnetic translations and multiplication by a complex phase for dA sufficiently small.

THEOREM 1. *For every $N \geqslant 2$ and $p \in (2, \frac{2N}{N-2})$, there exists $\varepsilon > 0$ such that if $A \in \mathrm{Lin}(\mathbb{R}^N, \bigwedge^1 \mathbb{R}^N)$ satisfies $|dA| \leqslant \varepsilon$, if u and v are solutions of (1) satisfying $I_A(u) \leqslant \mathcal{E}(0) + \varepsilon$ and if $I_A(v) \leqslant \mathcal{E}(0) + \varepsilon$, then $u = \mathrm{e}^{i\theta} \tau^A_a v$ for some $a \in \mathbb{R}^N$ and $\theta \in \mathbb{R}$.*

The main idea of the proof of Theorem 1 is to take advantage of the well-known uniqueness and non-degeneracy of the solutions of (1) under a vanishing magnetic field $A = 0$, see e.g. [5, 7], and to extend the uniqueness by an implicit function argument. The main difficulty in this proof consists in the fact that the natural function space $H^1_A(\mathbb{R}^N, \mathbb{C})$ for the functional I_A *depends* on the magnetic field: the norm and the elements of the space differ in general for different magnetic fields.

A direct consequence of Theorem 1 is that the solutions inherit the symmetries of the magnetic potential in a sense explained below.

THEOREM 2. *Let $N \geqslant 2$, $p \in (2, \frac{2N}{N-2})$ and $\varepsilon > 0$ be as in Theorem 1. If $A \in \mathrm{Lin}(\mathbb{R}^N, \bigwedge^1 \mathbb{R}^N)$ is skew-symmetric and satisfies $|dA| \leqslant \varepsilon$ and if u is a solution of (1) such that $I_A(u) \leqslant \mathcal{E}(0) + \varepsilon$, then there exists $a \in \mathbb{R}^N$ such that for every linear isometry R of $\mathbb{R}^N$ satisfying $|A \circ R|^2 = |A|^2$, one has*

$$u(R(x+a) - a) = \mathrm{e}^{iA(a)[R(x+a)-(x+a)]} u(x).$$

Moreover, the function u is nondecreasing along any ray starting from the point a.

Since equation (1) is invariant under magnetic translations Theorem 2 implies the existence of a unique ground state u such that its conclusion holds with $a = 0$, that is, for every linear isometry R of $\mathbb{R}^N$ such that $|A \circ R|^2 = |A|^2$, one has $u \circ R = u$. Alternatively, Theorem 2 states that a ground state can be translated in such a way that it only depends monotonically on the norms of the projections on the eigenspaces of the quadratic form $|A|^2$. Also, because of the antisymmetric structure of A, the group of isometries such that $|A \circ R|^2 = |A|^2$ can be written, up to an isometry of the Euclidean space, as a product of orthogonal groups $O(2n_1) \times O(2n_2) \times \cdots \times O(2n_k) \times$

$O(N-2n_1-2n_2-\cdots-2n_k)$, with $n_1,n_2,\ldots,n_k\in\mathbb{N}$; when $N=3$ and $A\neq 0$, it is always of the form $O(2)\times O(1)$, corresponding to a decomposition in the transversal and longitudinal directions with respect to the magnetic field.

2. Preliminaries

We first introduce a preliminary lemma about the convergence in L^p-spaces.

LEMMA 1 (Continuous Sobolev embedding across magnetic Sobolev spaces). *Assume that, for every $n\in\mathbb{N}$, $(A_n)_{n\in\mathbb{N}}$ is a sequence in $L^2_{\mathrm{loc}}(\mathbb{R}^N,\bigwedge^1\mathbb{R}^N)$ and $u_n\in H^1_{A_n}(\mathbb{R}^N,\mathbb{C})$. If $A_n\to A$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, and $u_n\to u$ strongly in $L^2(\mathbb{R}^N)$, $D_{A_n}u_n\to D_Au$ strongly in $L^2(\mathbb{R}^N)$, then $u_n\to u$ strongly in $L^p(\mathbb{R}^N)$ for $2\leqslant p\leqslant\frac{2N}{N-2}$, as $n\to\infty$.*

2.1. Ground states

Here we recall the known properties of the ground states of the nonlinear Schrödinger equation (1), with or without magnetic potential. The first lemma comes from [4, Theorem 3.1].

LEMMA 2 (Existence and characterization of ground states). *For every magnetic potential $A\in\mathrm{Lin}(\mathbb{R}^N,\bigwedge^1\mathbb{R}^N)$, there exists $u\in H^1_A(\mathbb{R}^N,\mathbb{C})\setminus\{0\}$ such that*

$$I_A(u)=\mathcal{E}(dA)\ \text{and}\ \ I_A'(u)=0.$$

Moreover,

$$\mathcal{E}(dA)=\left(\tfrac{1}{2}-\tfrac{1}{p}\right)\inf_{v\in H^1_A(\mathbb{R}^N,\mathbb{C})\setminus\{0\}} Q_A(v)^{\frac{p}{p-2}},$$

where the functional $Q_A:H^1_A(\mathbb{R}^N,\mathbb{C})\setminus\{0\}\to\mathbb{R}$ is defined by

$$Q_A(v):=\frac{\displaystyle\int_{\mathbb{R}^N}|D_Av|^2+|v|^2}{\left(\displaystyle\int_{\mathbb{R}^N}|v|^p\right)^{\frac{2}{p}}}.$$

We recall some well established results for the problem without a magnetic field

$$-\Delta u+u=|u|^{p-2}u,\qquad \text{in } \mathbb{R}^N. \tag{5}$$

This problem is a natural limit for (1) when $A\to 0$. The next result states that the ground state of (5) in $H^1(\mathbb{R}^N,\mathbb{C})$ is unique up to rotations in $\mathbb{C}$ and translations in $\mathbb{R}^N$.

PROPOSITION 1 (Uniqueness up to rotations in $\mathbb{C}$ and translations in $\mathbb{R}^N$). *If $u,v\in H^1(\mathbb{R}^N,\mathbb{C})$ satisfy $I_0(u)=I_0(v)=\mathcal{E}(0)$ and $I_0'(u)=I_0'(v)=0$, then there exist $\theta\in\mathbb{R}$ and $a\in\mathbb{R}^N$ such that $v=e^{i\theta}\tau^0_a u$.*

It clearly follows that there exists a unique real, positive and radially symmetric ground state of (5), that we denote by u_0. The next proposition states the non-degeneracy property due to M. I. Weinstein [7] and Y.-G. Oh [6].

PROPOSITION 2 (Non-degeneracy of the ground states in absence of magnetic field). *Assume that* $u \in H^1(\mathbb{R}^N, \mathbb{C})$ *satisfies* $I_0(u) = \mathcal{E}(0)$ *and* $I_0'(u) = 0$. *If* $w \in H^1(\mathbb{R}^N, \mathbb{C})$ *satisfies*

$$-\Delta w + w = |u|^{p-2} w + (p-2)|u|^{p-4}(u|w)u, \tag{6}$$

then there exist $y \in \mathbb{R}^N$ *and* $\lambda \in \mathbb{R}$ *such that*

$$w = Du[y] + \lambda i u. \tag{7}$$

In particular if u is a solution of the equation (5) and if w is a solution of its linearised problem (6) given by (7), then u and w are orthogonal in the space $H^1(\mathbb{R}^N, \mathbb{C})$, i.e.

$$\int_{\mathbb{R}^N} (Du|Dw) + (u|w) = 0. \tag{8}$$

For every ground state $u \in H^1(\mathbb{R}^N, \mathbb{C})$ of (1), we can rewrite equation (6) as an eigenvalue equation in the following way

$$L_u w = \lambda w, \qquad w \in H^1(\mathbb{R}^N, \mathbb{C}),$$

where the operator $L_u : H^1(\mathbb{R}^N, \mathbb{C}) \to H^1(\mathbb{R}^N, \mathbb{C})$ is given by

$$L_u w := \big(-\Delta + 1\big)^{-1} \big(|u|^{p-2} w + (p-2)|u|^{p-4}(u \otimes u)[w]\big),$$

with

$$(u \otimes u)[w] := (u|w)u. \tag{9}$$

It is standard that the operator L_u is compact. Indeed, the ground states u of (5) decays as $|x|^{-(N-1)/2} \exp(-|x|)$, see e.g. [2, p332], so that they are in $L^q(\mathbb{R}^N)$ for every $q \geqslant 1$. For completeness Lemma 3 below gives a more general result including the one above. It is also standard, see e.g. [1, Remark 4.2], to check directly that the ground state u is the first eigenfunction of eigenvalue $\lambda_1(L_u) = (p-1) > 1$, while the functions w given in (7) are the following eigenfunctions corresponding to the eigenvalues $\lambda_i(L_u) = 1$, $i = 2, \dots, N+2$. Finally, $\lambda_i(L_u) < 1$ for $i > N+2$.

PROPOSITION 3. *The ground energy function* $\mathcal{E} : \bigwedge^2 \mathbb{R}^N \to \mathbb{R}$ *is continuous. Moreover, if* $(A_n)_{n\in\mathbb{N}}$ *is a sequence in* $L^2_{\mathrm{loc}}(\mathbb{R}^N, \bigwedge^1 \mathbb{R}^N)$ *such that* $A_n \to A$ *in* $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ *as* $n \to \infty$ *and if the sequence* $(u_n)_{n\in\mathbb{N}}$ *in* $H^1_{A_n}(\mathbb{R}^N, \mathbb{C})$ *satisfies* $I'_{A_n}(u_n) = 0$ *and* $I_{A_n}(u_n) = \mathcal{E}(dA_n)$, *then there exist* $u \in H^1_A(\mathbb{R}^N, \mathbb{C})$ *with* $I_A(u) = \mathcal{E}(dA)$ *and* $I'_A(u) = 0$, *a sequence* $(a_n)_{n\in\mathbb{N}}$ *in* $\mathbb{R}^N$ *and a subsequence such that* $\tau^{A_{n_\ell}}_{a_{n_\ell}} u_{n_\ell} \to u$ *and* $D_{A_{n_\ell}}(\tau^{A_{n_\ell}}_{a_{n_\ell}} u_{n_\ell}) \to D_A u$ *strongly in* $L^2(\mathbb{R}^N)$ *as* $\ell \to \infty$.

We note that the convergence $A_n \to A$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ is equivalent to the convergence $dA_n \to dA$ in the finite-dimensional space $\bigwedge^2 \mathbb{R}^N$.

3. Uniqueness up to magnetic translations and rotations in $\mathbb{C}$ of the ground states

Our main tool to prove Theorem 1 is to prove the stability of the spectrum of a magnetic operator under perturbations of the potentials. We first study the spectrum of the sequence of linear operators

$$L_n : H^1_{A_n}(\mathbb{R}^N, \mathbb{C}) \to H^1_{A_n}(\mathbb{R}^N, \mathbb{C}) : v \mapsto L_n v := (-\Delta_{A_n} + 1)^{-1} W_n[v],$$

where

(H_1) $(A_n)_{n\in\mathbb{N}}$ is a sequence in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$,

(H_2) $(W_n)_{n\in\mathbb{N}}$ is a sequence in $L^q(\mathbb{R}^N, \mathrm{Lin}(\mathbb{C},\mathbb{C}))$ with $q \geqslant \frac{N}{2}$ and $q > 1$,

(H_3) W_n is self-adjoint and $W_n \geqslant 0$ on $\mathbb{R}^N$, that is $(z|W_n[z]) \geqslant 0$ for every $z \in \mathbb{C}$.

LEMMA 3. *Under (H_1)–(H_3) the operator L_n is self-adjoint and compact.*

Lemma 3 and the positivity of W_n imply that L_n has a nonincreasing sequence of positive eigenvalues converging to 0 and by Fischer's min-max principle one has

$$\lambda_k(L_n) = \sup_{\substack{E\subset H^1_{A_n}(\mathbb{R}^N,\mathbb{C})\\ \dim E = k}} \inf_{v\in E} \frac{\displaystyle\int_{\mathbb{R}^N} (v|W_n[v])}{\displaystyle\int_{\mathbb{R}^N} |D_{A_n} v|^2 + |v|^2}.$$

PROPOSITION 4. *Assume that $(H_1)-(H_3)$ hold. If W_n converges strongly to W in $L^q(\mathbb{R}^N)$ and A_n converges strongly to A in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ as $n\to\infty$, then*

$$\lambda_k(L_n) \to \lambda_k(L),$$

where $\lambda_k(L_n)$, $\lambda_k(L)$ are respectively the k-th eigenvalues of L_n, L, and $L : H^1_A(\mathbb{R}^N,\mathbb{C}) \to H^1_A(\mathbb{R}^N,\mathbb{C})$ is defined as

$$Lv = (-\Delta_A + 1)^{-1} W[v].$$

Moreover, if $u_n \in H^1_{A_n}(\mathbb{R}^N,\mathbb{C})$ is an eigenfunction of L_n satisfying

$$L_n u_n = \lambda_k(L_n) u_n, \quad \textit{and} \quad \int_{\mathbb{R}^N} |D_{A_n} u_n|^2 + |u_n|^2 = 1,$$

then there exist $u \in H^1_A(\mathbb{R}^N,\mathbb{C})$ and a subsequence $(n_\ell)_{\ell\in\mathbb{N}}$ such that $u_{n_\ell} \to u$ and $D_{A_{n_\ell}} u_{n_\ell} \to D_A u$ strongly in $L^2(\mathbb{R}^N)$.

With those ingredients we can now prove Theorem 1.

Proof of Theorem 1. We first assume that $\mathrm{d}A_n \to 0$ as $n \to +\infty$, that is $A_n \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ as $n \to +\infty$ since A_n is skew-symmetric, and that u_n and v_n are ground states solutions of (1) with A_n. Our aim is to show that there exist $\theta_n \in \mathbb{R}$ and $a_n \in \mathbb{R}^N$ such that $u_n = \mathrm{e}^{i\theta_n} \tau^{A_n}_{a_n} v_n$ for n large enough.

Let U be a solution of the limit problem (5). By Proposition 1, U is unique up to rotations in $\mathbb{C}$ and translations in $\mathbb{R}^N$.

CLAIM 1. There exist sequences $(\tilde{\theta}_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ and $(\tilde{a}_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$ such that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|D_{A_n}(e^{i\tilde{\theta}_n}\tau_{\tilde{a}_n}^{A_n}u_n)-DU|^2+|e^{i\tilde{\theta}_n}\tau_{\tilde{a}_n}^{A_n}u_n-U|^2=0.$$

Proof of the claim. By Proposition 3, there exist a sequence $(b_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$, a subsequence $(n_\ell)_{\ell\in\mathbb{N}}$ in $\mathbb{N}$ and a function $V\in H^1(\mathbb{R}^N,\mathbb{C})$ such that $\tau_{b_{n_\ell}}^{A_{n_\ell}}u_{n_\ell}\to V$ and $D_{A_{n_\ell}}(\tau_{b_{n_\ell}}^{A_{n_\ell}}u_{n_\ell})\to DV$ strongly in $L^2(\mathbb{R}^N)$. Because of the uniqueness up to translations and rotations in $\mathbb{C}$ of the solution of (5), there exist $b\in\mathbb{R}^N$ and $\omega\in\mathbb{R}$ such that $V=e^{i\omega}\tau_b^0U$. We can therefore write that

$$\int_{\mathbb{R}^N}|D_{A_{n_\ell}}(\tau_{b_{n_\ell}}^{A_{n_\ell}}u_{n_\ell})-DV|^2=\int_{\mathbb{R}^N}|e^{-i\omega}\tau_{-b}^{A_{n_\ell}}D_{A_{n_\ell}}(\tau_{b_{n_\ell}}^{A_{n_\ell}}u_{n_\ell})-\tau_{-b}^{A_{n_\ell}}\tau_b^0DU|^2=$$
$$\int_{\mathbb{R}^N}|e^{-i\omega}e^{-iA_{n_\ell}(b_{n_\ell})[b]}D_{A_{n_\ell}}(\tau_{b_{n_\ell}-b}^{A_{n_\ell}}u_{n_\ell})-DU+DU-\tau_{-b}^{A_{n_\ell}}\tau_b^0U|^2\to 0,$$

as $\ell\to+\infty$. Here, we used the commutation between the translation and the connexion and (4). Moreover, by using Lebesgue dominated convergence, we have that

$$\int_{\mathbb{R}^N}|DU-\tau_{-b}^{A_{n_\ell}}\tau_b^0DU|^2\to 0,\qquad\text{as }\ell\to+\infty.$$

By the triangle inequality, we infer that

$$\int_{\mathbb{R}^N}|e^{-i\omega}e^{-iA_{n_\ell}(b_{n_\ell})[b]}D_{A_{n_\ell}}(\tau_{b_{n_\ell}-b}^{A_{n_\ell}}u_{n_\ell})-DU|^2\to 0,\qquad\text{as }\ell\to+\infty,$$

and proceeding exactly in the same way, we obtain

$$\int_{\mathbb{R}^N}|e^{-i\omega}e^{-iA_{n_\ell}(b_{n_\ell})[b]}\tau_{b_{n_\ell}-b}^{A_{n_\ell}}u_{n_\ell}-U|^2\to 0,\qquad\text{as }\ell\to+\infty.$$

Setting $\tilde{\theta}_{n_\ell}=-\omega-A_{n_\ell}(b_{n_\ell})[b]$ and $\tilde{a}_{n_\ell}=b_{n_\ell}-b$, the conclusion of the claim follows for this subsequence. The claim is true for the whole sequence. Indeed, if not we would find a subsequence n_ℓ for which the Claim does not hold, leading to a contradiction. $\diamond$

CLAIM 2. There exist sequences $(\theta_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ and $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$ such that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|D_{A_n}(e^{i\theta_n}\tau_{a_n}^{A_n}u_n)-DU|^2+|e^{i\theta_n}\tau_{a_n}^{A_n}u_n-U|^2=0.$$

Moreover, when $n\in\mathbb{N}$ is large enough, for every $w\in\mathbb{R}^N$, we have the following orthogonality relations

$$\int_{\mathbb{R}^N}\big(D_{A_n}(e^{i\theta_n}\tau_{a_n}^{A_n}u_n)\big|D(DU[w])\big)+(e^{i\theta_n}\tau_{a_n}^{A_n}u_n|DU[w])=0,$$
$$\int_{\mathbb{R}^N}\big(D_{A_n}(e^{i\theta_n}\tau_{a_n}^{A_n}u_n)\big|DiU\big)+(e^{i\theta_n}\tau_{a_n}^{A_n}u_n|iU)=0.$$

Proof of the claim. We already proved Claim 1 with $\tilde{\theta}_n$ and $\tilde{a}_n$. Let us first prove the two orthogonality relations. For this, we define the map $\Phi_n \in C(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$ for each $(x,\tau) \in \mathbb{R}^{N+1}$ and $(w,s) \in \mathbb{R}^{N+1}$ by the following scalar product

$$\begin{aligned}\big((w,s)\big|\Phi_n(x,\tau)\big) &= \int_{\mathbb{R}^N} \big(D_{A_n}(\mathrm{e}^{i(\tilde{\theta}_n+\tau)}\tau_x^{A_n}\tau_{\tilde{a}_n}^{A_n}u_n)\big|s(DiU)\big) + \big(\mathrm{e}^{i(\tilde{\theta}_n+\tau)}\tau_x^{A_n}\tau_{\tilde{a}_n}^{A_n}u_n\big|siU\big)\\ &+ \int_{\mathbb{R}^N} \big(D_{A_n}(\mathrm{e}^{i(\tilde{\theta}_n+\tau)}\tau_x^{A_n}\tau_{\tilde{a}_n}^{A_n}u_n)\big|D(DU[w])\big) + \big(\mathrm{e}^{i(\tilde{\theta}_n+\tau)}\tau_x^{A_n}\tau_{\tilde{a}_n}^{A_n}u_n\big|DU[w]\big).\end{aligned}$$

Since $D_{A_n} \circ \tau_x^{A_n}\tau_{\tilde{a}_n}^{A_n} = \tau_x^{A_n}\tau_{\tilde{a}_n}^{A_n} \circ D_{A_n}$, and thanks to the convergence proved in Claim 1, the sequence $(\Phi_n)_{n\in\mathbb{N}}$ converges to Φ uniformly over compact subsets, where the function $\Phi \in C(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$ is defined for every $(x,\tau) \in \mathbb{R}^{N+1}$ and $(w,s) \in \mathbb{R}^{N+1}$ by

$$\begin{aligned}\big((w,s)\big|\Phi(x,\tau)\big) &= \int_{\mathbb{R}^N} \big(D(\mathrm{e}^{i\tau}\tau_x^0 U)\big|s(DiU)\big) + \big(\mathrm{e}^{i\tau}\tau_x^0 U\big|siU\big)\\ &+ \int_{\mathbb{R}^N} \big(D(\mathrm{e}^{i\tau}\tau_x^0 U)\big|D(DU[w])\big) + \big(\mathrm{e}^{i\tau}\tau_x^0 U\big|DU[w]\big).\end{aligned}$$

We first remark that $\Phi(0,0) = 0$. This is due to the fact that $DU[w] + siU$ belongs to the tangent space of U, see (8). Next, observe now that

$$\begin{aligned}\big((w,s)\big|D\Phi(0,0)[z,r]\big) &= \int_{\mathbb{R}^N} \big(D(Du[z])\big|D(DU[w])\big) + \big(DU[z]\big|DU[w]\big)\\ &+ \int_{\mathbb{R}^N} \big(r(DiU)\big|s(DiU)\big) + \big(riU\big|siU\big),\end{aligned}$$

meaning that $D\Phi(0,0) \geqslant 0$. Therefore, for every small $\rho > 0$, the Brouwer topological degree $\deg(\Phi, B_\rho, 0)$ of Φ on B_ρ with respect to 0 is well-defined, and $\deg(\Phi, B_\rho, 0) = 1$. Hence, since we have the uniform convergence on compacts of the continuous functions Φ_n, for n large enough, we obtain that $\deg(\Phi_n, B_\rho, 0) = 1$. We conclude to the existence of a sequence (x_n, τ_n) such that $\Phi_n(x_n, \tau_n) = 0$ for every n large enough, and $(x_n, \tau_n) \to (0,0)$ as $n \to \infty$. Finally, setting $a_n = x_n + \tilde{a}_n$ and $\theta_n = \tilde{\theta}_n + \tau_n + iA(\tilde{a}_n)[x_n]$, we reach the conclusion in view of the composition formula for magnetic translations (4), and using again the Lebesgue dominated convergence. $\diamond$

Applying the first two claims to the sequence $(v_n)_n$ and renaming $\tilde{u}_n = \mathrm{e}^{i\theta_n}\tau_{a_n}^{A_n}u_n$ and $\tilde{v}_n = \mathrm{e}^{i\varphi_n}\tau_{c_n}^{A_n}v_n$ (where the couple $(\varphi_n, c_n) \in \mathbb{R}^{N+1}$ is given by the claims), we can assume that $\tilde{u}_n$ satisfied

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |D_{A_n}\tilde{u}_n - DU|^2 + |\tilde{u}_n - U|^2 = 0,$$

and for every $w \in \mathbb{R}^N$,

$$\int_{\mathbb{R}^N} (D_{A_n}\tilde{u}_n|D(DU[w])) + (\tilde{u}_n|DU[w]) = 0, \quad \int_{\mathbb{R}^N} (D_{A_n}\tilde{u}_n|DiU) + (\tilde{u}_n|iU) = 0,$$

and the same for $\tilde{v}_n$.

CLAIM 3. There exists $W_n \in L^q(\mathbb{R}^N, \mathrm{Lin}(\mathbb{C},\mathbb{C}))$ such that

$$-\Delta_{A_n}(\tilde{u}_n - \tilde{v}_n) + (\tilde{u}_n - \tilde{v}_n) = W_n[\tilde{u}_n - \tilde{v}_n] \qquad \text{in } \mathbb{R}^N,$$

and

$$W_n \to |U|^{p-2} + (p-2)|U|^{p-4} U \otimes U$$

in $L^q(\mathbb{R}^N)$ for every $2 \leqslant q(p-2) \leqslant \frac{2N}{N-2}$, where $\otimes$ was defined in (9).

Proof of the claim. We define $W_n : \mathbb{R}^N \to \mathrm{Lin}(\mathbb{C},\mathbb{C})$ by

$$(w|W_n[z]) = \int_0^1 Df\big((1-t)\tilde{u}_n + t\tilde{v}_n\big)[w,z]\,\mathrm{d}t,$$

for $f(u) = |u|^{p-2}u$. Both claim 1 and Lemma 1 imply that $\tilde{u}_n \to U$ and $\tilde{v}_n \to U$ in $L^{q(p-2)}(\mathbb{R}^N)$, for $2 \leqslant q(p-2) \leqslant \frac{2N}{N-2}$. Then, it is clear that $W_n \in L^q(\mathbb{R}^N)$ and $W_n \to Df(U) = |U|^{p-2} + (p-2)|U|^{p-4} U \otimes U$ in $L^q(\mathbb{R}^N)$ as $n \to \infty$. $\diamond$

CONCLUSION The compact operator defined by $L_n = (-\Delta_{A_n} + 1)^{-1} W_n$ enters in the hypothesis of Proposition 4. We know that the spectrum converges, i.e., $\lambda_k(L_n) \to \lambda_k(L)$. Moreover, since the limit equation is (6), we also know that $\lambda_1(L) = p-1 > 1$, $\lambda_i(L) = 1$, for $i = 2, \ldots, N+2$, and $\lambda_i(L) < 1$, for $i \geqslant N+3$.

We define the orthogonal projection operator P_n^+ on the first eigenvector, P_n^0 the projection on the eigenspace E_n^0 made by the $N+1$ following eigenvectors, and $P_n^- = I - P_n^+ - P_n^0$. We observe that L_n commutes with P_n^- and P_n^+ and $L_n(\tilde{u}_n - \tilde{v}_n) = \tilde{u}_n - \tilde{v}_n$. Moreover,

$$\begin{aligned}
\big\|P_+^n(\tilde{u}_n - \tilde{v}_n)\big\|^2_{H^1_{A_n}(\mathbb{R}^N,\mathbb{C})} &= \big(P_n^+(\tilde{u}_n - \tilde{v}_n)\big|P_n^+ L_n(\tilde{u}_n - \tilde{v}_n)\big)_{H^1_{A_n}(\mathbb{R}^N,\mathbb{C})} \\
&= \big(P_n^+(\tilde{u}_n - \tilde{v}_n)\big|L_n P_n^+(\tilde{u}_n - \tilde{v}_n)\big)_{H^1_{A_n}(\mathbb{R}^N,\mathbb{C})} \\
&= \lambda_1(L_n)\big\|P_n^+(\tilde{u}_n - \tilde{v}_n)\big\|^2_{H^1_{A_n}(\mathbb{R}^N,\mathbb{C})}.
\end{aligned}$$

Then, since $\lim_{n\to+\infty} \lambda_1(L_n) > 1$, $P_n^+(\tilde{u}_n - \tilde{v}_n) = 0$ for n large enough. Similarly,

$$\begin{aligned}
\big\|P_n^-(\tilde{u}_n - \tilde{v}_n)\big\|^2_{H^1_{A_n}(\mathbb{R}^N,\mathbb{C})} &= \big(P_n^-(\tilde{u}_n - \tilde{v}_n)\big|P_n^- L_n(\tilde{u}_n - \tilde{v}_n)\big)_{H^1_{A_n}(\mathbb{R}^N,\mathbb{C})} \\
&= \big(P_n^-(\tilde{u}_n - \tilde{v}_n)\big|L_n P_n^-(\tilde{u}_n - \tilde{v}_n)\big)_{H^1_{A_n}(\mathbb{R}^N,\mathbb{C})} \\
&\leqslant \lambda_{N+3}(L_n)\,\big\|P_n^-(\tilde{u}_n - \tilde{v}_n)\big\|^2_{H^1_{A_n}(\mathbb{R}^N,\mathbb{C})}.
\end{aligned}$$

Thus, since $\lim_{n\to\infty} \lambda_{N+3}(L_n) < 1$, $P_n^-(\tilde{u}_n - \tilde{v}_n) = 0$ for n large enough. Assume now by contradiction that, for every $n \in \mathbb{N}$, $\tilde{u}_n \neq \tilde{v}_n$. Then, the function

$$z_n = \frac{\tilde{u}_n - \tilde{v}_n}{\|\tilde{u}_n - \tilde{v}_n\|_{H^1_{A_n}(\mathbb{R}^N,\mathbb{C})}}$$

is in the eigenspace E_n^0 and is a linear combination of eigenvectors. By Proposition 4, there exists $z = DU[w] + \lambda iU \in E^0$, where E^0 is the eigenspace of L corresponding the eigenvalue 1 (see §2.1), such that, up to a subsequence still denoted by n,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |D_{A_n} z_n - Dz|^2 + |z_n - z|^2 = 0.$$

By Claim 2, we also know that

$$\int_{\mathbb{R}^N} (D_{A_n} z_n | Dz) + (z_n|z) = 0,$$

for n large enough. Finally

$$\begin{aligned} \|\tilde{u}_n - \tilde{v}_n\|^2_{H^1_{A_n}(\mathbb{R}^N,\mathbb{C})} &= \int_{\mathbb{R}^N} (D_{A_n}(\tilde{u}_n - \tilde{v}_n)|D_{A_n}(\tilde{u}_n - \tilde{v}_n)) + (\tilde{u}_n - \tilde{v}_n|\tilde{u}_n - \tilde{v}_n) \\ &= \|\tilde{u}_n - \tilde{v}_n\|^2_{H^1_{A_n}(\mathbb{R}^N,\mathbb{C})} \int_{\mathbb{R}^N} (D_{A_n} z_n|D_{A_n} z_n) + (z_n|z_n) \\ &= \|\tilde{u}_n - \tilde{v}_n\|^2_{H^1_{A_n}(\mathbb{R}^N,\mathbb{C})} \int_{\mathbb{R}^N} (D_{A_n} z_n - Dz|D_{A_n} z_n) + (z_n - z|z_n) \\ &\leqslant \|\tilde{u}_n - \tilde{v}_n\|^2_{H^1_{A_n}(\mathbb{R}^N,\mathbb{C})} \int_{\mathbb{R}^N} |D_{A_n} z_n - Dz|^2 + |z_n - z|^2. \end{aligned}$$

This is impossible. Then, $\tilde{u}_n = \tilde{v}_n$ for n large. □

References

[1] AMBROSETTI A. AND MALCHIODI A., *Perturbation methods and semilinear elliptic problems on $\boldsymbol{R}^n$*, Progress in Mathematics **240**, Birkhäuser, Basel 2006.

[2] AMBROSETTI A., MALCHIODI A., RUIZ D., *Bound states of nonlinear Schrödinger equations with potentials vanishing at infinity*, J. Anal. Math. **98** (2006), 317–348.

[3] BONHEURE D., NYS M., VAN SCHAFTINGEN J., *Properties of ground states of nonlinear Schrödinger equations under a weak constant magnetic field*, arXiv:1607.00170 (2016), 1–44.

[4] ESTEBAN M. J., LIONS P.-L., *Stationary solutions of nonlinear Schrödinger equations with an external magnetic field*, Partial differential equations and the calculus of variations, Vol. I (1989), 401–449.

[5] KWONG M. K., *Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\boldsymbol{R}^n$*, Arch. Rational Mech. Anal. **105** 3 (1989), 243–266.

[6] OH Y.-G., *On positive multi-lump bound states of nonlinear Schrödinger equations under multiple well potential*, Comm. Math. Phys. **131** 2 (1990), 223–253.

[7] WEINSTEIN M. I., *Modulational stability of ground states of nonlinear Schrödinger equations*, SIAM J. Math. Anal. **16** 3 (1985), 472–491.

AMS Subject Classification: 35J61 (35B07, 35B40, 35J10, 35Q55)

Manon NYS,
Dipartimento di Matematica Giuseppe Peano, Università degli Studi di Torino
Via Carlo Alberto 10, 10123 Torino, Italy
e-mail: `manonys@gmail.com`

Lavoro pervenuto in redazione il 12.10.2016.