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# SOME REMARKS ON CONVEX COMBINATIONS OF LOW EIGENVALUES

**Abstract.** In this survey we deal with shape optimization problems involving convex combinations of the first two eigenvalues of the Dirichlet Laplacian, mainly recalling and explaining some recent results. More precisely, we discuss some geometric properties of minimizers, in particular when they are no longer convex and the optimality of balls. This leads us to deal with the "attainable set" of the first two eigenvalues, which is a great source of open problems.

## 1. Introduction

The aim of this note is to introduce the reader to some shape optimization problems involving the first two eigenvalues of the Dirichlet Laplacian. In particular we focus on the minimization of convex combinations of these first two eigenvalues, among open subsets of the euclidean space with a measure constraint. Although this can seem a rather easy topic, as it often happens in shape optimization, there are many hidden difficulties and a lot of open conjectures. This work is mostly based on the papers [19] and [23], to which we refer for more details.

The topic of spectral optimization has received a lot of attention in the last years, see the books [8, 16, 18] for a broader introduction. The first issue for this kind of problems concerns the *existence* of an optimal shape: a result proved in the 1990s by Buttazzo and Dal Maso [13] is even now a cornerstone of the matter, and, for a large class of functionals, it ensures the existence of a solution in the class of quasi-open sets of fixed measure (*a priori* contained into a given box, which provides the necessary compactness to prove existence). Moreover, the *regularity* of an optimal shape is a highly difficult problem and a general regularity theory is nowadays not available: even a proof which guarantees that an optimal shape is open, and not merely quasi-open, is far from being trivial, see [12]. Another important point consists in proving some *geometric* properties of optimal shapes, such as connectedness, convexity, symmetry with respect to some axis, and this is the main topic of this note. In fact, only for few special functionals optimal shapes are explicitly known: classical examples are the lowest eigenvalues of the Dirichlet-Laplacian. We recall that, for a given integer $N \geqslant 2$ and an open set $\Omega \subset \mathbb{R}^N$ with finite measure, the *first* and *second eigenvalues* of the Dirichlet-Laplacian can be defined as

$$\lambda_1(\Omega) := \min_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u(x)|^2 dx}{\int_\Omega |u(x)|^2 dx}, \qquad \lambda_2(\Omega) := \min_{\substack{u \in H_0^1(\Omega)\setminus\{0\} \\ \int_\Omega u u_1 = 0}} \frac{\int_\Omega |\nabla u(x)|^2 dx}{\int_\Omega |u(x)|^2 dx},$$

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where these minima are attained, respectively, by the *first* and *second eigenfunctions* $u_1$ and $u_2$ (which are unique, up to a multiplicative constant).

The interest in the minimization of the first eigenvalue goes back to a conjecture due to Lord Rayleigh in 1877, then proved by Faber and Krahn in the 1920s. The *Faber-Krahn inequality* asserts that of all open sets of fixed measure, the ball has the minimum first eigenvalue: in formula, for every open set $\Omega \subset \mathbb{R}^N$ with unit measure

$$\lambda_1(\Omega) \geqslant \lambda_1(B) = \omega_N^{2/N} j_{N/2-1}^2, \tag{1}$$

where $\omega_N$ denotes the measure of the ball in $\mathbb{R}^N$ with unit radius, $j_n$ the first positive zero of the Bessel function $J_n$, and $B$ the open ball of unit measure in $\mathbb{R}^N$. Equality in (1) holds if and only if $\Omega$ is that ball (up to sets of capacity zero). The same issue for the second eigenvalue is known as the *Krahn-Szegö inequality*, which asserts that two disjoint open balls of half measure each are the unique (up to sets of capacity zero) minimizer, namely for every open set $\Omega \subset \mathbb{R}^N$ with unit measure

$$\lambda_2(\Omega) \geqslant \lambda_2(B_- \cup B_+) = 2^{2/N}\lambda_1(B) = (2\omega_N)^{2/N} j_{N/2-1}^2, \tag{2}$$

where $B_- \cup B_+$ is the union of two equal and disjoint open balls of half measure each, and equality in (2) holds if and only if $\Omega = B_- \cup B_+$.

Up to our knowledge, the only other functionals of eigenvalues for which an explicit minimizer is known are $\lambda_1/\lambda_2$ and $\lambda_2/\lambda_3$, see [3].

Starting with the important work of Keller and Wolf [21], there was a strong interest for *convex combinations* of the first two eigenvalues of the Dirichlet Laplacian, namely the functional $F_t$ defined, for every $t \in (0,1)$, as

$$F_t(\Omega) := t\lambda_1(\Omega) + (1-t)\lambda_2(\Omega), \tag{3}$$

where $\Omega \subset \mathbb{R}^N$ is an open set of finite measure. Then, the corresponding spectral optimization problem writes as

$$\min\left\{F_t(\Omega) \;:\; \Omega \subset \mathbb{R}^N, \Omega \text{ open}, |\Omega| = 1\right\}. \tag{4}$$

The existence of a minimizer for this problem is now well understood and is guaranteed by a general theory recently developed in the works [7, 12, 22], all based on the above mentioned result [13], but with the new difficulty of working in the entire space $\mathbb{R}^N$. Notice that, all these results guarantee the existence of an optimal shape in the larger class of quasi-open sets, and only *a posteriori* one proves that a minimizer of problem (1.1) is in fact *open*, and so problem (1.1) is well-posed. Moreover, in [19] it was proved that, for every $t \in (0,1)$, minimizers of (1.1) are *connected* (more generally, this topological property was studied for minimizers of convex combinations of the first three eigenvalues). The main idea in order to prove connectedness of minimizers is to characterize the optimal *disconnected* configuration and then to find an explicit connected competitor, which is often either the ball or a perturbation of balls. In two dimensions ($N = 2$), some numerical computations on the shape of these minimizers appeared in [20]. We sum up all these results in the following theorem.

THEOREM 1. *For every $t \in (0,1)$ we consider the shape optimization problem:*

$$\min\{F_t(\Omega)\,:\,\Omega \subset \mathbb{R}^N, |\Omega| = 1\}. \tag{5}$$

*The following facts hold true for* (5).

1. *There exists an optimal shape in the class of quasi-open sets (Buttazzo-Dal Maso [13], Bucur [7], Mazzoleni-Pratelli [22]),*
2. *Every optimal set is open and each of its first k eigenfunctions can be extended in $\mathbb{R}^N$ to a Lipschitz continuous function (Bucur-Mazzoleni-Pratelli-Velichkov [12]),*
3. *Every optimal set is bounded and has finite perimeter (Bucur [7]),*
4. *Every optimal set is connected (Iversen-Mazzoleni [19]).*

Notice that, if $t = 1$ the convex combination (3) is minimized by the ball with unit measure (because of the Faber-Krahn inequality (1)), while if $t = 0$, by two equal balls of half measure each (because of the Krahn-Szegö inequality (2)). Therefore, as $t$ moves from 1 to 0, one expects the shape of a minimizer $\Omega_t$ deforming from a ball of unit measure to two balls of half measure each; in particular, it is natural to conjecture that at some value of $t$ the convexity of all the minimizers in (1.1) is lost (as was numerically observed in [20], in two dimensions, the critical value for $t$ is expected to be $1/2$). We give a first answer to this question, though non-optimal. All the results presented in this note, unless otherwise specified, will hold in every dimension $N \geqslant 2$.

THEOREM 2. *There exists a threshold $T > 0$ such that, for all $t \in (0,T)$, every minimizer in* (1.1) *is no longer convex.*

We provide a *quantitative* proof of this theorem, namely we explicitly construct the threshold $T$ *via* the eigenvalues of the Dirichlet-Laplacian. Moreover, in two dimensions, it is possible to find a numerical lower bound on $T$ using a quantitative Krahn-Szegö inequality involving the so-called Fraenkel 2-asymmetry, for this topic we refer the reader to [23], where it is studied an auxiliary purely geometrical problem: the minimization of the Fraenkel 2-asymmetry among *convex* sets of given area. It is possible to show that the *mobile*, i.e., the intersection of the convex hull of two tangent balls with a strip is the unique minimizer satisfying an isoperimetric inequality for the Fraenkel 2-asymmetry (16). An explicit value for the constant in the quantitative Krahn-Szegö inequality will be also needed. This opens a new area of application for quantitative inequalities, which can be found in [23, Appendix].

A second question to address is the optimality of a special convex set: the ball, and we present a generalization of a result from [21].

THEOREM 3. *For all $t \in (0,1)$ the ball $B$ is never a minimizer in* (1.1).

The proof of this result follows from a more general proposition, i.e., that the second eigenvalue of a minimizer in (1.1) has to be simple and, as a consequence of the multiplicity of the second eigenvalue over balls, so we immediately get the result in Theorem 3. The proof of the simplicity of the second eigenvalue relies on some ideas developed in [16] and [10], with the help of a classical symmetry result due to Serrin [25] (see also [15]).

As an application of these results, we show how to get informations on the shape of the *attainable set*, namely the subset of the plane described by the range of the first two eigenvalues of the Dirichlet-Laplacian

(6) $$\mathcal{E} := \left\{(x,y) \in \mathbb{R}^2 \,:\, x = \lambda_1(\Omega),\ y = \lambda_2(\Omega),\ \Omega \subset \mathbb{R}^N,\ \Omega \text{ open},\ |\Omega| = 1\right\}.$$

This set was introduced in [21], and then deeply studied in [9] (see also [1, 2, 5]), where several geometrical properties of $\mathcal{E}$ were discussed.

The link between problem (1.1) and the set $\mathcal{E}$ is the following: for a fixed $t \in (0,1)$ a minimizer $\Omega_t$ in (1.1) corresponds to the first point of $\mathcal{E}$ of coordinates $(\lambda_1(\Omega_t), \lambda_2(\Omega_t))$ that we reach with a line $tx + (1-t)y = a$ increasing the value $a$, that is $P_{\Omega_t} := (\lambda_1(\Omega_t), \lambda_2(\Omega_t))$ is one of the intersection points of the tangent line to $\mathcal{E}$ with slope $t/(t-1)$. In particular, if $t = 1$ the tangent line $x = \lambda_1(B)$ has a *unique* intersection point corresponding to the ball $B$ (because of the Faber-Krahn inequality (1)), while if $t = 0$, the tangent line $y = \lambda_2(\Theta)$ has a *unique* intersection point corresponding to the two balls $B_- \cup B_+$ (because of the Krahn-Szegö inequality (2)).

Therefore, in Theorem 4, we will present a new strategy for studying the asymptotic behavior of the boundary of $\mathcal{E}$ near the points corresponding to $B$ and $B_- \cup B_+$, extending to all dimensions a result proved in [21] only in two dimensions, and recovering the result proved in [5].

We suspect that to properly understand the boundary behavior of the attainable set, one has to carefully analyze problem (1.1). For this reason we restate the long-standing conjecture about the convexity of the attainable set in the language of the minimizers of convex combinations of the lowest Dirichlet eigenvalues.

The paper, which is mainly a review of a talk given by the author at the "BruTo PDE's Conference" held in Torino on May 2nd–5th, 2016, is organized as follows. In Section 2 we discuss Theorem 2, while in Section 3 we deal with Theorem 3 and with the attainable set (6).

## 2. Non-convexity of minimizers for problem (1.1)

In order to study the non-convexity of minimizers for problem (1.1) we first need to deal briefly with the study of optimal sets for $\lambda_2$ among convex bodies, which was the topic of an important paper by Henrot and Oudet [17]. Finding an explicit minimizer in this class seems a very difficult problem: a possible candidate to be the optimum is the stadium (i.e., the convex hull of two tangent balls), but this conjecture was refuted in [17]. Indeed any set which contains on the boundary some pieces of balls can not be a minimizer. Nevertheless, in [17] it was proved the existence of a convex minimizer $\Omega_{\text{HO}}$ so that, for every open and *convex* set $\Omega \subset \mathbb{R}^N$ with unit measure,

(7) $$\lambda_2(\Omega) \geqslant \lambda_2(\Omega_{\text{HO}}),$$

(cf. (7) with the Krahn-Szegö inequality, where no-convexity constraint is required). Notice that, since $\Omega_{\text{HO}}$ has no pieces of balls on its boundary, in particular $\Omega_{\text{HO}} \neq B$ and

$$\omega_N^{2/N} j_{N/2}^2 = \lambda_2(B) > \lambda_2(\Omega_{\text{HO}}).$$

In two dimensions, Oudet in [24] and, more recently, Antunes and Henrot in [2], made some numerical computations, showing the shape of the optimal set $\Omega_{\text{HO}}$ and highlighting that $\Omega_{\text{HO}}$ is very close to the stadium, both from a geometrical and a numerical point of view; in particular

$$(8) \quad \lambda_2(B_- \cup B_+) = 2\pi j_0^2 \approx 36.336, \quad \lambda_2(\Omega_{\text{HO}}) \approx 37.987, \quad \lambda_2(\Omega_{\text{stadium}}) \approx 38.002,$$

where $\Omega_{\text{stadium}}$ is the stadium with $|\Omega_{\text{stadium}}| = 1$, i.e., a contracted version of the set hull$(\Theta)$. Note that we approximate all the numerically computed values only to the third decimal digit, for sake of simplicity.

Before proving Theorem 2 we need to set some notation. We say that $\Omega_t$ is a minimizer in (1.1) if for every admissible competitor $\Omega$

$$(9) \qquad t\lambda_1(\Omega_t) + (1-t)\lambda_2(\Omega_t) \leqslant t\lambda_1(\Omega) + (1-t)\lambda_2(\Omega),$$

and equivalently, rearranging the terms

$$(10) \qquad \lambda_1(\Omega_t) - \lambda_1(\Omega) + \lambda_2(\Omega) - \lambda_2(\Omega_t) \leqslant \frac{1}{t}\Big(\lambda_2(\Omega) - \lambda_2(\Omega_t)\Big).$$

*Proof of Theorem 2.* From the Krahn-Szegö inequality (2) and the connectedness of $\Omega_t$ it follows that

$$(11) \qquad \lambda_2(B_- \cup B_+) < \lambda_2(\Omega_t),$$

which plugged into (9) with $\Omega = B_- \cup B_+$ yields

$$(12) \qquad \lambda_1(\Omega_t) < \lambda_1(B_- \cup B_+).$$

Taking $\Omega = B_- \cup B_+$ also in (10) and dividing therein by the negative quantity $\lambda_2(B_- \cup B_+) - \lambda_2(\Omega_t)$ (recall (11)) we get to

$$(13) \qquad \frac{\lambda_1(B_- \cup B_+) - \lambda_1(\Omega_t)}{\lambda_2(\Omega_t) - \lambda_2(B_- \cup B_+)} + 1 \geqslant \frac{1}{t}.$$

From (11) and (12), the ratio on the left-hand side of this inequality turns out to be a positive number; therefore, we can use the Faber-Krahn inequality (1) to estimate $\lambda_1(\Omega_t)$ at the numerator of this ratio. Moreover, if $\Omega_t$ would be a *convex* set, we could also use (7) to estimate $\lambda_2(\Omega_t)$ at the denominator of this ratio, obtaining the following uniform bound on $t$:

$$(14) \qquad t \geqslant \frac{1}{\frac{\lambda_1(B_- \cup B_+) - \lambda_1(B)}{\lambda_2(\Omega_{\text{HO}}) - \lambda_2(B_- \cup B_+)} + 1}.$$

Calling $T$ the quantity on the right-hand side of this inequality, the Krahn-Szegö inequality for convex sets gives $\lambda_2(\Omega_{\text{HO}}) - \lambda_2(B_- \cup B_+) > 0$, thus $T > 0$. Therefore, if $t < T$, $\Omega_t$ can not be convex. □

The proof of Theorem 2 is constructive and reveals an explicit expression for the threshold $T$ in terms of the eigenvalues of the Dirichlet-Laplacian. In particular, the threshold $T$ in Theorem 2 has the following expression:

$$T = 1 - \frac{(2^{2/N}-1)\lambda_1(B)}{\lambda_2(\Omega_{\rm HO}) - \lambda_1(B)}, \tag{15}$$

where $\Omega_{\rm HO}$ is a minimizer in (7). As it is often the case, in the two dimensional case one can expect to be able to provide some explicit estimate for the constant $T$. This was done in [23] and we recall here only the result, with the needed explanations. First of all we have to define the Fraenkel 2-asymmetry of an open set $\Omega \subset \mathbb{R}^N$ with unit measure, that is,

$$\mathcal{A}_2(\Omega) := \min\{|\Omega \triangle (B_- \cup B_+)| \ : B_-, B_+ \text{ disjoint open balls}, |B_-| = |B_+| = 1/2\}. \tag{16}$$

Then we can state the quantitative Krahn–Szegö inequality (see, for example [4, 6]):

$$\frac{\lambda_2(\Omega)}{\lambda_2(B_- \cup B_+)} - 1 \geqslant \beta_{\rm KS} \mathcal{A}_2(\Omega)^\alpha, \tag{17}$$

for some constant $\beta_{KS} > 0$ and an exponent $\alpha > 0$. At last, we consider the following shape optimization problem,

$$\inf\{\mathcal{A}_2(\Omega) : \ \Omega \subset \mathbb{R}^N, \Omega \text{ open and convex}, |\Omega| = 1\}. \tag{18}$$

In [23] it is showed that the set $M$ (see [23, Definition 2.2]) defined as the intersection of the convex hull of two tangent balls with a strip is the unique minimizer satisfying an isoperimetric inequality for the Fraenkel 2-asymmetry (18).

Then it is possible to see, in two dimensions, that

$$T \geqslant 1 - \frac{1}{1 + 2\beta_{\rm KS}\mathcal{A}_2(M)^{9/2}} \approx 1.192 \cdot 10^{-14}, \tag{19}$$

where the constants $\beta_{\rm KS}$ and $\mathcal{A}_2(M)$ are as in (17) and (16) respectively. What is important to highlight of this bound is that it does not depend on the eigenvalues of the Dirichlet Laplacian.

REMARK 1. The explicit value for the lower bound to the threshold $T$ is not very accurate, mostly due to the fact that the constant $\beta_{\rm KS}$ is not the optimal one, but we believe it is important to show that a numerical value can actually be provided. Moreover, if $N = 2$, plugging the numerical computation of $\lambda_2(\Omega_{\rm HO})$ recalled in (8) into (15) and using $\lambda_1(B) = \pi j_0^2 \approx 18.168$, reveals a numerical approximation for the threshold defined by (15):

$$T \approx 0.083.$$

## 3. Optimality of the ball and attainable set

In order to deal with Theorem 3, the key point is to show the following, stronger Proposition. The main idea of the proof is to make careful perturbations of the boundary of an optimal set (provided it is regular enough) and then reduce the problem to an overdetermined PDE, which can be treated with techniques first developed by Serrin [25].

PROPOSITION 1. *For a fixed $t \in (0,1)$, let $\Omega_t$ be a minimizer of problem (1.1). If the boundary of $\Omega_t$ is of class $C^2$ and connected, then $\lambda_2(\Omega_t)$ is simple, namely $\lambda_1(\Omega_t) < \lambda_2(\Omega_t) < \lambda_3(\Omega_t)$. Moreover, on the boundary of $\Omega_t$, the following optimality condition holds:*

$$t\,|\nabla u_1(x)|^2 + (1-t)\,|\nabla u_2(x)|^2 = \frac{2F_t(\Omega_t)}{N}, \qquad x \in \partial\Omega_t. \tag{20}$$

For a proof of the above Proposition we refer to [23], but then Theorem 3 follows easily.

*Proof of Theorem 3.* The proof is a straightforward consequence of Proposition 1: in every dimension, the second eigenvalue $\lambda_2(B)$ is not simple, therefore the ball $B$ can not be a minimizer for any $t \in (0,1)$. □

REMARK 2. In two dimensions, the fact that balls are never minimizers was implicitly contained in the work [21]. For an arbitrary $\varepsilon > 0$ small enough, in [21] the authors constructed a nearly spherical competitor $B_\varepsilon$, with $|B_\varepsilon| = 1$, such that

$$\lambda_1(B_\varepsilon) \leqslant \lambda_1(B) + d_1\varepsilon^2, \quad \text{while} \quad \lambda_2(B_\varepsilon) \leqslant \lambda_2(B) - d_2\varepsilon,$$

for some positive constants $d_1, d_2$. Therefore, for every $t \in (0,1)$, it is possible to find $\varepsilon > 0$ so small so that

$$t\lambda_1(B_\varepsilon) + (1-t)\lambda_2(B_\varepsilon) < t\lambda_1(B) + (1-t)\lambda_2(B).$$

The last thing that we want to treat is the relation between problem (1.1) and Theorem 3 with the so called attainable set, defined in (6). We start listing the most important properties that are known on the attainable set $\mathcal{E}$ defined in (6) (for figures representing the set $\mathcal{E}$ we refer to [21, 9, 23]):

1. lies above the bisector $y = x$ (since by definition $\lambda_2(\Omega) \geqslant \lambda_1(\Omega)$ for every $\Omega \subset \mathbb{R}^N$).
2. lies on the right of the line $x = \lambda_1(B)$ (for the Faber-Krahn inequality (1)).
3. lies above the line $y = \lambda_2(B_- \cup B_+)$ (for the Krahn-Szegö inequality (2)).
4. lies below the line $y = \frac{\lambda_2(B)}{\lambda_1(B)}x$ (for the Ashbaugh-Benguria inequality [3]).
5. is conical with respect to the origin.

The numerical picture provided by Keller and Wolff suggests the following conjecture, which is still unsolved up to our knowledge.

**Conjecture A.** The attainable set $\mathcal{E}$ is convex.

The most important result in the direction of this conjecture was proposed by Bucur, Buttazzo and Figuereido in [9]. These authors proved that the attainable set (6), constructed through quasi-open sets instead of open sets, is convex in the vertical and in the horizontal direction and, as a consequence, that it is closed. Nevertheless the vertical and horizontal convexity *do not* imply convexity (think, for example to an L-shaped set).

From the properties of the set $\mathcal{E}$ listed above it is clear that the unique unknown part of the boundary of $\mathcal{E}$ is the curve $\mathcal{C}$ connecting the points $P_B = (\lambda_1(B), \lambda_2(B))$ and $P_{B_- \cup B_+} = (\lambda_1(B_- \cup B_+), \lambda_2(B_- \cup B_+))$. The convexity of $\mathcal{E}$ is then guaranteed as soon as $\mathcal{C}$ can be parametrized by a convex function. For this reason it is important to have more informations on the curve $\mathcal{C}$. In two dimensions, Keller and Wolf in [21] showed that the tangent of $\mathcal{C}$ at the point $P_B$ corresponding to a ball $B$ is vertical. They constructed a nearly spherical perturbation of $B$, as recalled in Remark 2, and then they computed the slope of the tangent to $\mathcal{C}$ as $\varepsilon \to 0$. Moreover, in all dimensions, Brasco, Nitsch and Pratelli showed that the tangent of $\mathcal{C}$ at the point $P_{B_- \cup B_+}$ corresponding to two balls $B_- \cup B_+$ is horizontal. In this case the limit as $\varepsilon \to 0$ was computed by overlapping the two balls $B_-$ and $B_+$ of a quantity measured in terms of the parameter $\varepsilon$. In the following proposition we recover these limits relying on the minimality condition of the minimizers of convex combinations (9) without any explicit construction. Notice that the strategy that we adopt holds in all dimensions.

THEOREM 4. *For every dimension $N \geqslant 2$ and $t \in (0,1)$, let $\Omega_t$ be a minimizer of problem* (1.1)*. Then we have:*

*i) the tangent of $\mathcal{C}$ at the point $P_B$ corresponding to one ball is* vertical*, namely*

$$\lim_{t \to 1} \frac{\lambda_2(\Omega_t) - \lambda_2(B)}{\lambda_1(\Omega_t) - \lambda_1(B)} = -\infty \tag{21}$$

*ii) the tangent of $\mathcal{C}$ at the point $P_{B_- \cup B_+}$ corresponding to two identical balls is* horizontal*, namely*

$$\lim_{t \to 0} \frac{\lambda_2(\Omega_t) - \lambda_2(B_- \cup B_+)}{\lambda_1(\Omega_t) - \lambda_1(B_- \cup B_+)} = 0. \tag{22}$$

*Moreover, the following limits holds*

$$\lim_{t \to 0} \lambda_2(\Omega_t) = \lambda_2(B_- \cup B_+) \qquad \text{and} \qquad \lim_{t \to 1} \lambda_1(\Omega_t) = \lambda_1(B). \tag{23}$$

*Proof.* From the Faber-Krahn inequality (1) and Theorem 3 we find that

$$\lambda_1(B) < \lambda_1(\Omega_t), \tag{24}$$

which plugged into (9) with $\Omega = B$ yields

$$\lambda_2(\Omega_t) < \lambda_2(B). \tag{25}$$

Taking $\Omega = B$ in (10) and dividing therein by $\lambda_2(B) - \lambda_2(\Omega_t)$ (which from (25) is a strictly positive value) yields

$$\frac{\lambda_1(\Omega_t) - \lambda_1(B)}{\lambda_2(B) - \lambda_2(\Omega_t)} + 1 \leqslant \frac{1}{t}.$$

From (24) and (25) one can see that the ratio on the left-hand side of this inequality is a positive number, therefore, letting $t \uparrow 1$, necessarily, it holds the limit in (21). Moreover, repeating the computations made in the proof of Theorem 2 and letting $t \downarrow 0$ in (13), it follows the limit in (22).

Finally, the limits in (23) are a consequence of (21), (22) and of the boundedness of the denominator in (22) (because of (12)) and of the numerator in (21) (because of (25)). □

We finish this discussion formulating an *isospectral* conjecture on the minimizers of problem (1.1), which could be used to prove the convexity of the attainable set $\mathcal{E}$.

**Conjecture B.** Let $t \in (0,1)$ and assume $X, Y \subset \mathbb{R}^N$ to be minimizers of problem (1.1) with $F_t(X) = F_t(Y)$. Then, the lowest eigenvalues of $X$ and $Y$ coincide, namely

$$\lambda_1(X) = \lambda_1(Y) \qquad \text{and} \qquad \lambda_2(X) = \lambda_2(Y).$$

PROPOSITION 2. *The validity of Conjecture B implies that Conjecture A holds true.*

*Proof.* If $\mathcal{E}$ is not convex, then we can find two points $P_X, P_Y \in \mathcal{C}$, corresponding, respectively, to $X, Y$, and a straight line $l$ passing through these points such that the curve $\mathcal{C}$ lies above $l$. Therefore, it is clear that $l$ will be of the form $tx + (1-t)y = a$ for some fixed $t \in (0,1)$ and a real number $a$. Hence the sets $X, Y$ are minimizers in (1.1) for such a $t$, but $\lambda_1(X) \neq \lambda_1(Y)$ and $\lambda_2(X) \neq \lambda_2(Y)$, a contradiction with Conjecture A. □

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